UNIVERSIDADE DE ÉVORA

Combinatory Problems in Numerical Semigroups

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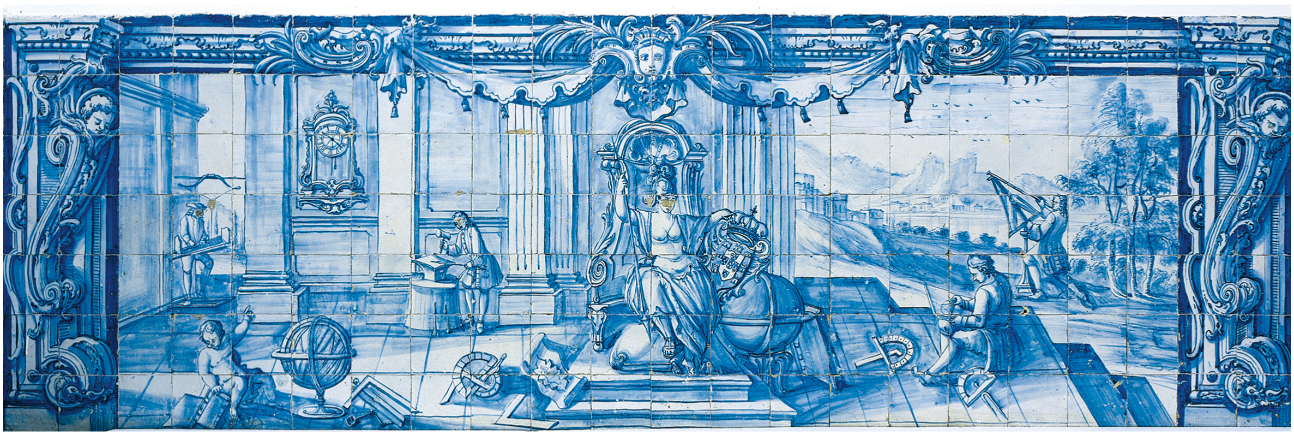

INSTITUTO DE INVESTIGAÇÃO E FORMAÇÃO AVANÇADA

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I also thank to my husband, Luís. Thank you for your listening, your patience, and your unconditional support and love.

To my parents that incited me to strive towards my goals.

To Alexandre.

Contents

Abstract

This thesis is devoted to the study of the theory of numerical semigroups. First, the focus is on saturated numerical semigroups. We will give algorithms that allows us to compute, for a given integer g (respectively integer F), the set of all saturated numerical semigroups with genus g (respectivaly with Frobenius number F). After that, we will solve the Frobenius problem for three particular classes of numerical semigroups: Mersenne, Thabit and Repunit numerical semigroups. Lastly, we will characterize and study the digital semigroups and the bracelet monoids.

Resumo

Problemas Combinatórios em Semigrupos Numéricos

Esta tese é dedicada ao estudo da teoria dos semigrupos numéricos. O primeiro foco é o estudo dos semigrupos numéricos saturados. Daremos algoritmos que nos irão permitir calcular, dado um inteiro g (repectivamente, um inteiro F), o conjunto de todos os semigrupos numéricos saturados com género g (respectivamente, com número de Frobenius F). Depois disso, iremos resolver o problema de Frobenius para três classes particulares de semigrupos numéricos: semigrupos numéricos de Mersenne, de Thabit e de Repunit. Por fim, iremos caracterizar e estudar os semigrupos digitais e os monóides braceletes.

Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition, contains the zero element and has finite complement in $\mathbb{N}$. The greatest integer that does not belong to S (respectively, the cardinal of $\mathbb{N}\backslash S$) is called the Frobenius number of S (respectively, genus of S), and it is denoted by $F(S)$ (respectively, $g(S)$).

In literature we can find a long list of works that study one dimensional analytically irreducible local domains via their value semigroups (see for instance [**2**] and the references given there). One important property studied for this kind of ring is of being saturated.

Saturated rings were introduced in three different ways in [**9**], [**23**] and in [**53**] and the three definitions coincide for algebraically closed fields of zero characteristic. From the characterization of a saturated ring through its value semigroups it arose the concept of saturated semigroup (see [**13**] and [**22**]).

Given a non empty subset A from $\mathbb{N}$ and $a \in A$ we denote by $d_A(a) = \gcd\{x \in A \mid x \leq a\}$. From [**9**] we say that a numerical semigroup S is saturated if $s + d_S(s) \in S$ for all $s \in S$.

Chapter 2 of this thesis is devoted to the study of saturated numerical semigroups. The results of Section 2 were published in [**31**] and the main result is an algorithm that allows us to compute, for a given integer g, the set of all saturated numerical semigroups of genus g. The methodology used in this algorithm is based in sorting the set of all saturated numerical semigroups in a tree rooted in $\mathbb{N}$ and describing the childs of the vertices of that tree.

The results of Section 3 were published in [**32**] and the main result is an algorithm that allows us to compute, for a given integer F, the set of all saturated numerical semigroups with Frobenius number F. The efficiency of this algorithm is fundamentally based in the description of a algorithmic method that allows us to calculate, for a given k-tuple of positive integers $(d_1,d_2,\dots,d_k)$ were $d_1 > d_2 > \cdots > d_k = 1$ and $d_{i+1} \mid d_i$ for all $i \in \{1,\dots,k-1\}$, the set of all nonnegative integer solutions from the equation $d_1x_1+\cdots+d_kx_k = c$ were c is a nonnegative integer.

During the early part of the last century, Ferdinand Georg Frobenius (1849-1917) raised, in his lectures the problem of giving a formula for the largest integer that is not representable as a linear combination with nonnegative integer coefficients of a given set of positive integers whose greater common divisor is one. He also raised the question of determining how many positive integers do not have such a representation. By using our terminology, the first problem is equivalent to give a formula, in terms of the elements in a minimal system of generators of a numerical semigroups S, for the greatest integer not in S known, as we have seen before, as the Frobenius number. The second problem consist on finding the cardinality of the set of gaps of that numerical semigroup, that is, the genus of S (see [**24**] for a nice state of art on this problem).

At first glance, the Frobenius Problem may look deceptively specialized. Nevertheless it crops up again and again in the most unexpected places. It turned out that the knowledge of Frobenius number has been extremely useful to investigate many different problems.

This problem was solved by Sylvester and Curran Sharp (see [**47**], [**48**] and [**49**]) for numerical semigroups with embedding dimension two. It was demonstrated that if $\{n_1,n_2\}$ is a minimal system of generators of S, then $F(S) = n_1n_2 - n_1 - n_2$ and $g(S) =$

$\frac{1}{2}(n_1-1)(n_2-1)$. The Frobenius problem remains open for numerical semigroups with embedding dimension greater than or equal to three.

In Chapter 3 we will solve the Frobenius problem for three particular classes of numerical semigroups: Mersenne, Thabit and Repunit numerical semigroups. The results of this chapter were published in [**34**], [**36**] and [**35**].

A positive integer x is a Mersenne number if $x = 2^n - 1$ for some $n \in \mathbb{N}\backslash\{0\}$. We say that a numerical semigroup S is a Mersenne numerical semigroup if there exist $n \in \mathbb{N}\backslash\{0\}$ such that $S = \left\langle \left\{2^{n+i} - 1 \mid i \in \mathbb{N}\right\}\right\rangle$. The main purpose of Section 1 is to study this class of numerical semigroups and will denoted by $S(n) = \left\langle \left\{2^{n+i} - 1 \mid i \in \mathbb{N}\right\}\right\rangle$. We give formulas for the embedding dimension, the Frobenius number, the type and the genus for a numerical semigroup generated by the Mersenne numbers greater than or equal to a given Mersenne number. We see that the minimal system of generators of $S(n)$ is equal to $\left\{2^n-1, 2^{n+1}-1, \ldots, 2^{2n-1}-1\right\}$ and thus $\mathrm{e}(S(n)) = n$. We will solve the Frobenius problem for the Mersenne numerical semigroups, in fact, we will prove that $\mathrm{F}(S(n)) = 2^{2n} - 2^n - 1$ and $\mathrm{g}(S(n)) = 2^{n-1}(2^n + n - 3)$.

Two numbers m and n are called amicable numbers if the sum of proper divisors (the divisors excluding the number itself) of one number equals the other. A positive integer x is a Thabit number if $x = 3.2^n - 1$ for some $n \in \mathbb{N}$ (named so in honor of the mathematician, physician, astronomer and translator Al-Sabi Thabit ibn Qurra al-Harrani 826- 901). These numbers expressed in binary representation are $n+2$ bits long being "10" by n $1's$. Thabit ibn Qurra was the first to study these numbers and their relation to amicable numbers. He discovered and proved that if $p = 3 \cdot 2^n - 1$, $q = 3 \cdot 2^{n-1} - 1$and $r = 9 \cdot 2^{n-1} - 1$ are prime numbers, then $M = 2^n pq$ and $N = 2^n r$ are a pair of amicable numbers. Thus, for $n = 2$, $n = 4$ and $n = 7$ we have the amicable pairs $(220, 284)$, $(17296, 18416)$ and $(9363584, 9437056)$, respectively, but no other such pairs are known. We say that a numerical semigroup S is a Thabit numerical semigroup if there exist $n \in \mathbb{N}$ such that $S = \left\langle \left\{ 3.2^{n+i} - 1 \mid i \in \mathbb{N} \right\} \right\rangle$, and will be denoted by $T(n)$. The main purpose of Section 2 is to study this class of numerical semigroups. In this setting, we will see that the minimal system of generators of $T(n)$ is equal to $\{3 \cdot 2n + i - 1 \mid i \in \{0, 1, \ldots, n+1\}\}$ and therefore $e(T(n)) = n+2$. If n is a positive integer, we will prove here that $F(T(n)) = 9 \cdot 2^{2n} - 3 \cdot 2^n - 1$ and $g(T(n)) = 9 \cdot 2^{2n-1} + (3n-5)2^{n-1}$.

In Section 3, we will study the Repunit numerical semigroups. In number theory, a Repunit is a number consisting of copies of the single digit 1. The numbers 1, 11, 111 or 1111, etc., are examples of Repunits. The term stands for repeated unit and was coined by Albert H. Beiler in [**3**]. In general, the set of Repunits in base b is $\left\{ \frac{b^n-1}{b-1} \mid n \in \mathbb{N} \backslash \{0\} \right\}$. In binary, these are known like Mersenne numbers. In the literature there are many problems related to this kind of numbers (see, for example, [**45**] and [**52**]). A numerical semigroup S is a Repunit numerical semigroup if there exist integers $b \in \mathbb{N} \backslash \{0,1\}$ and $n \in \mathbb{N} \backslash \{0\}$ such that $S = \left\langle \left\{ \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N} \right\} \right\rangle$ and it will denoted by $S(b,n)$. We will prove that $\left\{ \frac{b^{n+i}}{b-1} \mid i \in \{0, \ldots, n-1\} \right\}$ is the minimal system of generators of $S(b,n)$ and so $e(S(b,n)) = n$. We will solve Frobenius problem for the Repunit numerical semigroup, specifically, we will prove that $F(S(b,n)) = \dfrac{b^n-1}{b-1} b^n - 1$ and $g(S(b,n)) = \dfrac{b^n}{2} \left(\dfrac{b^n - b}{b-1} + n - 1 \right)$.

Chapter 4 is dedicated to the study of the digital semigroups (Section 1) and the bracelet monoids (Section 2). These results were published in [**33**] and [**30**], respectively. Given a positive integer n, we denote by $\ell(n)$ the number of digits of n writen in decimal expansion. For example $\ell(137) = 3$ and $\ell(2335) = 4$. Given A a subset of $\mathbb{N} \backslash \{0\}$, we also denote by $L(A) = \{\ell(a) \mid a \in A\}$. A digital semigroup D is a subsemigroup of $(\mathbb{N} \backslash \{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N} \backslash \{0\} \mid \ell(x) = \ell(d)\} \subseteq D$ and a numerical semigroup S is called **LD-semigroup** if there exist a digital semigroup D such that $S = L(D) \cup \{0\}$. Our main goal in Section 1 is to find the smallest digital semigroup containing a set of positive integers. We characterize the LD-semigroups in the following way: a numerical semigroup S is a LD-semigroup if and only if $a+b-1 \in S$ for all $a,b \in S\backslash\{0\}$. This fact allows us prove that the set of all LD-semigroups is a Frobenius variety.

In order to clarify a bit more the study of LD-semigroups, we refer two papers that motivate their study. By using the terminology of [**6**] a LD-semigroup is a numerical semigroup that fulfills a nonhomogeneous pattern $x_1 + x_2 - 1$. As a consequence of [[**6**], Example 6.4] LD-semigroups can be characterized by the fact that the minimum element in each interval of nongaps is a minimal generator.

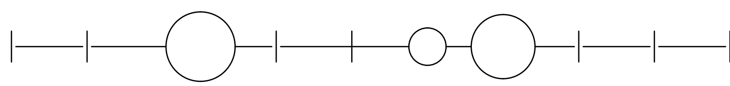
A (v,b,r,k)-configuration is a connected bipartite graph with v vertices on one side, each of them of degree r, and b vertices on the other side, each of them of degree k, and with no cycle of length 4. We say that the tuple (v,b,r,k) is configurable if a (v,b,r,k)-configuration exists. In [**7**] it is proved that if (v,b,r,k) is configurable then $vr = bk$ and consequently there exists d such that $v = d\frac{k}{\gcd(r,k)}$ and $b = d\frac{r}{\gcd(r,k)}$. The fundamental result in [**7**] states that if r and k are integers greater than or equal to two, then $S_{(r,k)} = \left\{d \in \mathbb{N} \mid \left(d\frac{k}{\gcd(r,k)}, d\frac{r}{\gcd(r,k)}, r, k\right) \text{is configurable}\right\}$ is a numerical semigroup. Moreover, in [**46**] it is shown that for balanced configurations, i.e. when $r = k$, it follows that $\{x+y-1, x+y+1\} \subseteq S_{(r,r)}$ for all $x, y \in S_{(r,r)}\backslash\{0\}$, and thus $S_{(r,r)}$ is a LD-semigroup.

Suppose that a plumber has an unlimited number of pipes with lengths $l_1, \ldots, l_q$. To join two pipes he can solder them or he cans use pipe joints $J_1, \ldots, J_p$. In the first case the total length is equal to the sum of the lengths of the used pipes and if he uses a pipe joint J_i the total length is the sum of lengths of pipes plus n_i (where n_i is the positive length of J_i). The main purpose of Section 2 is to study the set of lengths of pipes that the plumber can make.

The previous situation leads us to the following definition. Let S be a set of segments and let C be a set of circles. A (S,C)-bracelet is a finite sequence b of the elements in the set $S \cup C$ fulfilling the following conditions:

(1) b begins and ends with a segment;

(2) in b there are no two consecutive circles.



The length of a (S,C)-bracelet b is equal to the sum of all lengths of its segments and all diameters of its circles, and it is denoted $\ell(b)$.

Let $B(S,C) = \{b \mid b \text{ is a } (S,C) - \text{bracelet}\}$ and let $LB(S,C) = \{\ell(b) \mid b \in B(S,C)\}$. Suppose that $\emptyset$ is a (S,C)-bracelet and $\ell(\emptyset) = 0$.

If S is a set of segments and C is a set of circles, where their lengths and diameters are positive integers, then it is easy to prove that $LB(S,C)$ is a submonoid of $(\mathbb{N},+)$. Note that if $c \in C$ then diameter(c) may not be in $LB(S,C)$. But if $\ell_1, \ell_2 \in LB(S,C) \setminus \{0\}$ then $\ell_1 + \ell_2 + \text{diameter}(c) \in LB(S,C)$. From here the following definition comes naturally. Let $n_1, \ldots, n_p$ be positive integers and let M be a submonoid of $(\mathbb{N},+)$. We say that M is a $(n_1, \ldots, n_p)$**-bracelet** if $a + b + \{n_1, \ldots, n_p\} \subseteq M$ for every $a, b \in M \setminus \{0\}$. From this we obtain that the set of lengths of pipes that the plumber can make is the smallest (with respect to the set inclusion order) $(n_1, \ldots, n_p)$-bracelet containing a set $\{l_1, \ldots, l_q\}$ of positive integers (it is the smallest $(n_1, \ldots, n_p)$-bracelet that contains a finite subset X of $\mathbb{N}$).

Recall that a numerical semigroup is a submonoid S of $(\mathbb{N},+)$ such that $gcd(S) = 1$. This fact motivates the following definition. A numerical $(n_1, \ldots, n_p)$-bracelet is a $(n_1, \ldots, n_p)$-bracelet M such that $gcd(M) = 1$. Therefore, following the notation introduced in [**6**], a numerical $(n_1, \ldots, n_p)$-bracelet is a numerical semigroup fulfilling nonhomogeneous patterns $x_1+x_2+n_1, x_1+x_2+n_2, \ldots, x_1+x_2+n_p$. And thus by using again [Example 6.4 [**6**]] (1)-bracelets can be characterized by the numerical semigroups fulfilling that the maximum element in each interval of non-gaps is one of its minimal generators. The notion of pattern for numerical semigroups was introduced in [**5**]. Recently, the study of (1)-bracelets has been done in [**25**] and also suggested in [**7**] and [**46**].

Introdução

Seja $\mathbb{N}$ o conjunto dos inteiros não negativos. Um semigrupo numérico é um subconjunto S de $\mathbb{N}$ que é fechado para a adição, contém o elemento zero e tem complemento finito em $\mathbb{N}$. O maior inteiro que não pertence a S (respectivamente, o cardinal de $\mathbb{N}\backslash S$) é chamado o número de Frobenius de S (respectivamente, o género de S), e denota-se por $F(S)$ (respectivamente, $g(S)$).

Na literatura podemos encontrar uma longa lista de trabalhos dedicados ao estudo de domínios locais analiticamente irredutíveis de dimensão 1 via um semigrupo de valores (ver por exemplo [**2**] e as referências aí dadas). Uma propriedade importante estudada para este tipo de anéis é a de ser saturado.

Os anéis saturados foram introduzidos de três formas distintas em [**9**], [**23**] e em [**53**] e as três definições dadas coincidem para corpos algebricamente fechados de característica 0 (zero). Desta caracterização de um anel saturado via um semigrupo de valores surgiu o conceito de semigrupo saturado (ver [**13**] and [**22**]).

Dado um subconjunto não vazio A de $\mathbb{N}$ e $a \in A$ denotamos por $d_A(a) = \gcd\{x \in A \mid x \leq a\}$. De [**9**] dizemos que um semigrupo numérico S é saturado se $s + d_S(s) \in S$ para todo $s \in S$.

O Capítulo 2 desta tese é dedicado ao estudo dos semigrupos numéricos. Os resultados da Secção 2 foram publicados em [**31**] e o seu principal resultado é um algoritmo que nos permite calcular, para um dado inteiro g, o conjunto de todos os semigrupos numéricos saturados com género g. A metodologia usada neste algoritmo é baseada na ordenação do conjunto destes semigrupos numéricos saturados numa árvore com raíz em $\mathbb{N}$ e na descrição dos filhos dos vértices dessa árvore.

Os resultados da Secção 3 foram publicados em [**32**] e o seu principal resultado é um algoritmo que nos permite calcular, para um dado inteiro F, o conjunto de todos os semigrupos numéricos saturados com número de Frobenius F. A eficiência deste algoritmo é fundamentalmente baseada na descrição de um método algorítmico que nos permite calcular, para uma dada k-tupla de inteiros positivos $(d_1, d_2, \ldots, d_k)$ onde $d_1 > d_2 > \cdots > d_k = 1$ e $d_{i+1}|d_i$ para todo $i \in \{1, \ldots, k-1\}$, o conjunto de todas as soluções inteiras não negativas da equação $d_1x_1 + \cdots + d_kx_k = c$ onde c é um inteiro não negativo.

Durante a primeira parte do século passado, Ferdinand Georg Frobenius (1849-1917) levantou, nas suas palestras, o problema de dar uma fórmula para o maior inteiro que não pode ser representado como a combinação linear de um dado conjunto de inteiros positivos cujo máximo divisor comum é igual a 1 e em que os coeficientes sejam inteiros não negativos. Ele também levantou a questão de determinar quantos inteiros positivos não têm tal representação. Usando a nossa terminologia, o primeiro problema é equivalente a dar uma fórmula, em termos dos elementos do sistema minimal de geradores de um semigrupo numérico S, para o maior inteiro que não está em S, conhecido, como já vimos anteriormente, por número de Frobenius. O segundo problema consiste em determinar a cardinalidade do conjunto dos buracos desse semigrupo numérico, ou seja, o género de S (ver [**24**] para uma boa referência do estado de arte deste problema).

À primeira vista,o problema de Frobenius pode parecer especializado. No entanto ele surge-nos nos lugares mais inesperados. O conhecimento do número de Frobenius é-nos extremamente útil para investigar diversos problemas.

Este problema foi resolvido por Sylvester e Curran Sharp (ver [**47**], [**48**] e [**49**]) para semigrupos númericos com dimensão de imersão dois. Foi demonstrado que se $\{n_1, n_2\}$ é um sistema minimal de geradores de S, então $F(S) = n_1 n_2 - n_1 - n_2$ e $g(S) = \frac{1}{2}(n_1 - 1)(n_2 - 1)$. O problema de Frobenius continua em aberto para semigrupos numéricos com dimensão de imersão maior ou igual que três.

No Capítulo 3 iremos resolver o problema de Frobenius para três classes particulares de semigrupos numéricos: os semigrupos numéricos de Mersenne, de Thabit e de Repunit. Os resultados deste capítulo foram publicados em [**34**], [**36**] and [**35**].

Um inteiro positivo x é um número de Mersenne se $x = 2^n - 1$ para algum $n \in \mathbb{N} \backslash \{0\}$. Dizemos que um semigrupo numérico S é um semigrupo numérico de Mersenne se existir um $n \in \mathbb{N} \backslash \{0\}$ tal que $S = \langle \{2^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$. O objectivo principal da Secção 1 é estudar esta classe de semigrupos numéricos que denotaremos por $S(n) = \langle \{2^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$. Daremos fórmulas para a dimensão de imersão, o número de Frobenius, o tipo e o géneron de um semigrupo numérico gerado por números de Mersenne maiores ou iguais a um dado número de Mersenne. Veremos que o sistema minimal de geradores de $S(n)$ é igual a $\{2^n - 1, 2^{n+1} - 1, \ldots, 2^{2n-1} - 1\}$ e portanto $\mathrm{e}(S(n)) = n$. Iremos resolver o problema de Frobenius para os semigrupos numéricos de Mersenne, de facto, provamos que $\mathrm{F}(S(n)) = 2^{2n} - 2^n - 1$ e $\mathrm{g}(S(n)) = 2^{n-1}(2^n + n - 3)$.

Dois números m e n dizem-se amigáveis se a soma dos divisores próprios (os divisores à excepção do próprio número) de um dos números for igual à do outro. Um inteiro positivo x é um número de Thabit se $x = 3.2^n - 1$ para algum $n \in \mathbb{N}$ (chamado assim em honra ao matemático, físico, astrónomo e tradutor Al-Sabi Thabit ibn Qurra al-Harrani 826- 901). Estes números expressos em representação binária têm $n + 2$ bits de comprimento sendo compostos por "10" seguidos de n $1's$. Thabit ibn Qurra foi o primeiro a estudar esses números e a sua relação com os números amigáveis. Ele descobriu e provou que se $p = 3 \cdot 2^n - 1$, $q = 3 \cdot 2^{n-1} - 1$ e $r = 9 \cdot 2^{n-1} - 1$ são números primos, então $M = 2^n pq$ e $N = 2^n r$ são um par de números amigáveis. Portanto, para $n = 2$, $n = 4$ e $n = 7$ temos os pares de amigáveis $(220, 284)$, $(17296, 18416)$ e $(9363584, 9437056)$, respectivamente, mas não são conhecidos mais nenhuns pares. Diremos que um semigrupo numérico S é um semigrupo numérico de Thabit se existir um $n \in \mathbb{N}$ tal que $S = \langle \{3.2^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$, e iremos denotá-los por $T(n)$. O objectivo principal da Secção 2 é estudar esta classe de semigrupos numéricos. Assim, iremos ver que o sistema minimal de geradores de $T(n)$ é igual a $\{3 \cdot 2n + i - 1 \mid i \in \{0, 1, \ldots, n+1\}\}$ e portanto $e(T(n)) = n + 2$. Seja n é um inteiro positivo, iremos provar que $F(T(n)) = 9 \cdot 2^{2n} - 3 \cdot 2^n - 1$ e $g(T(n)) = 9 \cdot 2^{2n-1} + (3n-5)2^{n-1}$.

Na Secção 3, iremos estudar os semigrupos numéricos Repunit. Na teoria dos números, um Repunit é um número composto pela repetição do dígito 1. Os números 1, 11, 111 ou 1111, etc., são exemplos de Repunits. O termo significa a repetição da unidade e foi introduzido por Albert H. Beiler em [**3**]. Em geral, o conjunto dos Repunit na base b é $\left\{ \frac{b^n - 1}{b-1} \mid n \in \mathbb{N} \backslash \{0\} \right\}$. Em linguagem binária, estes são conhecidos como os números de Mersenne. Na literatura existem diversos problemas relacionados com este tipo de números (ver, por exemplo, [**45**] e [**52**]). Um semigrupo numérico S é um semigrupo numérico Repunit se existirem inteiros $b \in \mathbb{N} \backslash \{0, 1\}$ e $n \in \mathbb{N} \backslash \{0\}$ tais que $S = \left\langle \left\{ \frac{b^{n+i} - 1}{b-1} \mid i \in \mathbb{N} \right\} \right\rangle$ e serão denotados por $S(b, n)$. Iremos provar que $\left\{ \frac{b^{n+i}}{b-1} \mid i \in \{0, \ldots, n-1\} \right\}$ é o sistema minimal de geradores de $S(b, n)$ e assim $e(S(b, n)) = n$. Iremos resolver o problema de Frobenius para os semigrupos numéricos Repunit, mais concretamente, iremos provar que $F(S(b, n)) = \dfrac{b^n - 1}{b - 1} b^n - 1$ e $g(S(b, n)) = \dfrac{b^n}{2} \left(\dfrac{b^n - b}{b - 1} + n - 1 \right)$.

O Capítulo 4 é dedicado ao estudo dos semigrupos digitais (Secção 1) e aos monóides braceletes (Secção 2). Estes resultados foram publicados em [**33**] e [**30**], respectivamente. Dado um inteiro positivo n, denotamos por $\ell(n)$ o número de dígitos de n escrito em representação decimal. Por exemplo $\ell(137)=3$ e $\ell(2335)=4$. Dado um subconjunto A de $\mathbb{N}\backslash\{0\}$, vamos denotar por $L(A)=\{\ell(a)\mid a\in A\}$. Um semigrupo digital D é um subsemigrupo de $(\mathbb{N}\backslash\{0\},\cdot)$ tal que se $d\in D$ então $\{x\in\mathbb{N}\backslash\{0\}\mid \ell(x)=\ell(d)\}\subseteq D$. Um semigrupo numérico S é chamado semigrupo-LD se existir um semigrupo digital D tal que $S=L(D)\cup\{0\}$. O nosso objetivo principal na Secção 1 é determinar o menor semigrupo digital que contém um conjunto de inteiros positivos. Caracterizamos os semigrupos-LD da seguinte forma: um semigrupo numérico S é um semigrupo-LD se e só se $a+b-1\in S$ para todos os $a,b\in S\backslash\{0\}$. Este facto permite-nos provar que o conjunto de todos os semigrupos-LD são uma variedade de Frobenius.

Com o intuito de clarificar um pouco mais o estudo dos semigrupos-LD, referimos dois trabalhos que motivam o seu estudo. Usando a terminologia de [**6**], um semigrupo-LD é um semigrupo numérico que verifica um padrão não homogéneo x_1+x_2-1. Como consequência de [[**6**], Exemplo 6.4] os semigrupos-LD podem ser caracterizados pelo facto de que o menor elemento em cada intervalo de elementos não-buracos é um gerador minimal.

Uma configuração-(v,b,r,k) é um grafo bipartido conectado, com v vértices de um lado, cada um deles de grau r, e b vértices no outro lado, cada um deles de grau k, e sem nenhum ciclo de comprimento 4. Dizemos que a tupla (v,b,r,k) é configurável se existir uma configuração-(v,b,r,k). Em [**7**] é provado que se (v,b,r,k) é configurável então $vr=bk$ e consequentemente existe um d tal que $v=d\frac{k}{\gcd(r,k)}$ e $b=d\frac{r}{\gcd(r,k)}$. O resultado principal em [**7**] afirma que se r e k são inteiros maiores ou iguais a dois, então $S_{(r,k)}=\left\{d\in\mathbb{N}\mid \left(d\frac{k}{\gcd(r,k)},d\frac{r}{\gcd(r,k)},r,k\right) \text{ é configurável}\right\}$ é um semigrupo numérico.

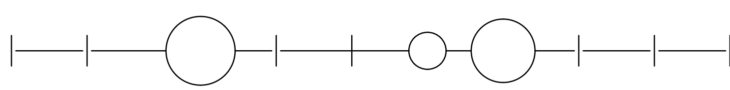
Mais, em [**46**] é mostrado que para configurações equilibradas, i.e. quando $r = k$, tem-se que $\{x+y-1, x+y+1\} \subseteq S_{(r,r)}$ para todos os $x, y \in S_{(r,r)} \backslash \{0\}$, e portanto $S_{(r,r)}$ é um semigrupo-LD.

Suponhamos que um canalizador tem um número ilimitado de tubos de comprimentos $l_1, \ldots, l_q$. Para unir dois tubos ele pode soldá-los ou pode usar juntas de tubos $J_1, \ldots, J_p$. No primeiro caso, o comprimento total é igual à soma dos comprimentos dos tubos que ele usa e se ele usar uma junta de tubos J_i o comprimento é a soma dos comprimentos dos tubos mais n_i (onde n_i é o comprimento de de J_i). O principal objectivo da Secção 2 é estudar o conjunto dos comprimentos dos tubos que o canalizador pode fazer.

A situação anterior conduz-nos à seguinte definição. Seja S um conjunto de segmentos e seja C um conjunto de círculos. Uma bracelete-(S,C) é uma sequência finita b dos elementos no conjunto $S \cup C$ que verifica as seguintes condições:

(1) b começa e acaba com um segmento;

(2) em b não há dois círculos consecutivos.



O comprimento de uma bracelete-(S,C) b é igual à soma de todos os comprimentos dos seus segmentos e todos os diâmetros dos seus círculos, e denota-se por $\ell(b)$.

Seja $B(S,C) = \{b \mid b \text{ é uma } (S,C)-\text{bracelet}\}$ e seja $LB(S,C) = \{\ell(b) \mid b \in B(S,C)\}$. Suponhamos que $\emptyset$ é uma bracelete-(S,C) e $\ell(\emptyset) = 0$.

Se S é um conjunto de segmentos e C é um conjunto de círculos, onde os seus comprimentos e diâmetros são inteiros positivos, então é fácil de provar que $LB(S,C)$ é um submonóide de $(\mathbb{N}, +)$. Note que se $c \in C$ então diâmetro(c) pode não estar em $LB(S,C)$. Mas se $\ell_1, \ell_2 \in LB(S,C) \backslash \{0\}$ então $\ell_1 + \ell_2 + \text{diâmetro}(c) \in LB(S,C)$.

Daqui, a seguinte definição vem naturalmente. Sejam $n_1,\ldots,n_p$ inteiros positivos e seja M um submonóide de $(\mathbb{N},+)$. Dizemos que M é uma bracelete-$(n_1,\ldots,n_p)$ se $a+b+\{n_1,\ldots,n_p\}\subseteq M$ para todos os $a,b\in M\backslash\{0\}$. De onde obtemos que o conjunto dos comprimentos dos tubosque o canalizador pode fazer é a menor (no que diz respeito à ordem de inclusão de conjuntos) bracelete-$(n_1,\ldots,n_p)$ que contém um conjunto $\{l_1,\ldots,l_q\}$ de inteiros positivos (é a menor bracelete-$(n_1,\ldots,n_p)$ que contém um subconjunto finito X de $\mathbb{N}$).

Recordemos que um semigrupo numérico é um submonóide S de $(\mathbb{N},+)$ tal que $gcd(S)=1$. Este facto motiva a seguinte definição. Uma bracelete-$(n_1,\ldots,n_p)$ numérica é uma bracelete-$(n_1,\ldots,n_p)$ M tal que $gcd(M)=1$. Assim, seguindo a notação introduzida em [**6**], uma bracelete-$(n_1,\ldots,n_p)$ numérica é um semigrupo numérico que satisfaz o padrão não homogéneo $x_1+x_2+n_1, x_1+x_2+n_2,\ldots,x_1+x_2+n_p$. E portanto, usando novamente o [Exemplo 6.4 [**6**]] as braceletes-(1) podem ser caracterizadas pelos semigrupos numéricos que verificam que o elemento máximo em cada intervalo de elementos não-buracos é um dos seus geradores minimais. A noção de padrão para semigrupos numéricos foi introduzida em [**5**]. Recentemente, o estudo de braceletes-(1) foi feito em [**25**] e também sugerida em [**7**] e [**46**].

CHAPTER 1

Preliminaries

In this chapter we present some basic definitions and known results, needed later in this work, related to the numerical semigroups. Some more specific definitions and known results may be presented locally when needed.

1. Notable elements

We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the set of nonnegative integers and the set of the integers, respectively.

A **semigroup** is a pair $(S,+)$, where S is a nonempty set and $+$ is a binary operation defined on S verifying the associative law, that is, for all $a,b,c \in S$ we have $a+(b+c) = (a+b)+c$. If there exists an element $t \in S$ such that $t+s = s+t = s$ for all $s \in S$ we say that $(S,+)$ is a **monoid**. This element is usually denoted by 0. In addition, S is a **commutative monoid** if for all $a,b \in S$, $a+b = b+a$. An example of a commutative monoid is $(\mathbb{N},+)$. All semigroups and monoids considered in this work are commutative. A **submonoid** of a monoid S is a subset A of S such that $0 \in A$ and for every $a,b \in A$ we have that $a+b \in A$.

Given a nonempty subset A of a monoid S, the **monoid generated** by A is the least (with respect to set inclusion) submonoid of S containing A, which turns out to be the intersection of all submonoids of S containing A. It follows easily that

$$\langle A\rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}\setminus\{0\}, x_1,\ldots,x_n \in A \text{ and } \lambda_1,\ldots,\lambda_n \in \mathbb{N}\}.$$

The set A is a system of generators of S if $< A >= S$, and we will say that S is generated by A. A monoid S is finitely generated if there exists a system of generators of S with finitely many elements. Moreover, we say that A is a **minimal system of generators** of S if no proper subset of A generates S.

Given two monoids X and Y, a map $f : X \to Y$ is a **monoid homomorphism** if $f(a+b) = f(a) + f(b)$ for all $a, b \in X$ and $f(0) = 0$. We say that f is a **monoid isomorphism** if f is bijective.

A **numerical semigroup** is a submonoid of $(\mathbb{N}, +)$ such that the greatest common divisor of its elements is equal to one, that is, $< A >$ is a numerical semigroup if and only if $gcd(A) = 1$.

PROPOSITION 1. *Every nontrivial submonoid of* $\mathbb{N}$ *is isomorphic to a numerical semigroup.*

The following result gives us alternative ways of defining a numerical semigroup.

PROPOSITION 2. *Let S a submonoid of* $\mathbb{N}$*. The following conditions are equivalent:*

(1) S is a numerical semigroup,
(2) the group spanned by S is $\mathbb{Z}$*,*
(3) $\mathbb{N} \backslash S$ *is finite.*

If $a_1 < a_2 < \cdots < a_k$ are integers, we denote by $\{a_1, a_2, \ldots, a_k, \rightarrow\}$ the set $\{a_1, a_2, \ldots, a_k\} \cup \{z \in \mathbb{Z} \mid z > a_k\}$. The submonoid $< 3, 7 >= \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$ is an example of a numerical semigroup.

Let A and B be subsets of integer numbers. To denote the set $\{a + b \ : \ a \in A, \ b \in B\}$ we use $A + B$.

LEMMA 3. *Let S be a numerical semigroup. Then* $(S \backslash \{0\}) \backslash (S \backslash \{0\} + S \backslash \{0\})$ *is a system of generators of S. Furthermore, every system of generators of S contains* $(S \backslash \{0\}) \backslash (S \backslash \{0\} + S \backslash \{0\})$.

Taking in account Proposition 2 it makes sense to consider the greatest integer not belonging to S. We call this element the **Frobenius number** of S and it is denoted by

$F(S)$. The cardinality of the set $\mathbb{N}\backslash S$ is called the **genus** of S (or **gender** of S) and is denoted by $g(S)$. The elements in this set are called the **gaps** of a numerical semigroup. The next lemma appears in [**38**] and is easy to prove.

LEMMA 4. *Let S and T be numerical semigroups. Then*

(1) $S\cap T$ is a numerical semigroup;

(2) if $S\neq\mathbb{N}$ then $S\cup\{\mathrm{F}(S)\}$ is a numerical semigroup.

Given $n\in S\backslash\{0\}$, the **Apéry set** (named so in honour of [**1**]) of S with respect to n is defined by

$$Ap(S,n)=\{s\in S\mid s-n\notin S\}.$$

It is easy to prove (see for instance [**38**]) the following result.

LEMMA 5. *Let S be a numerical semigroup and let n be a nonzero element of S. Then, $Ap(S,n)=\{0=w(0),w(1),\dots,w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n, for all $i\in\{0,\dots,n-1\}$.*

Observe that the above lemma in particular implies that the cardinality of $Ap(S,n)$ is n. With this result, we easily deduce the following.

LEMMA 6. *Let S be a numerical semigroup and let $n\in S\backslash\{0\}$. Then for all $s\in S$, there exists a unique $(k,w)\in\mathbb{N}\times Ap(S,n)$ such that*

$$s=kn+w.$$

The set $Ap(S,n)$ determines completely the semigroup S, since $S=<Ap(S,n)\cup\{n\}>$. Moreover, $Ap(S,n)$ contains in general more information that an arbitrary set of generators of S.

REMARK 7. If S is a numerical semigroup, $x\in S\backslash\{0\}$ then $\mathrm{Ap}(S,x)=\{w(0)=0,w(1),\dots,w(x-1)\}$. From Lemma 6 we have that an integer z is in S if and only if $z\geq w(z \bmod x)$.

As Lemma 3 states that $(S\backslash\{0\})\backslash(S\backslash\{0\}+S\backslash\{0\})$ is the minimal system of generators and as $S =< Ap(S,n)\cup\{n\} >$, for any $n \in S\backslash\{0\}$, we have the following result.

THEOREM 8. *Every numerical semigroup admits a unique minimal system of generators. This minimal system of generators is finite.*

From Proposition 1 and Theorem 8 we obtain the following consequence.

COROLLARY 9. *Let S be a submonoid of* $(\mathbb{N},+)$. *Then S has a unique minimal system of generators, which in addiction is finite.*

Let S be a numerical semigroup. The cardinality of the minimal system of generators of S is called **embedding dimension** of S, and is denoted by $e(S)$. The smallest nonzero element of S is called the **multiplicity** of S and is denoted by m(S).

The next result is due to Selmer [**44**] and can be used to compute $F(S)$ and $g(S)$, from one of the Apéry sets of the numerical semigroup S.

PROPOSITION 10. *Let S be a numerical semigroup and let n be a nonzero element of S. Then*

(1) $F(S) = max(Ap(S,n)) - n$;
(2) $g(S) = \frac{1}{n}(\sum_{w\in Ap(S,n)} w) - \frac{n-1}{2}$.

We say that a numerical semigroup S has a **monotonic** Apéry set if $w(1) < w(2) < \ldots < w(m(S)-1)$, with $\{0,w(1),\ldots,w(m(S)-1)\} = Ap(S,m(S))$.

Let S be a numerical semigroup. Following the notation introduced in [**29**], we say that the **pseudo-Frobenius numbers** of S are the elements of the set

$$PF(S) = \{x \in \mathbb{Z}\backslash S \mid x+s \in S \text{ for every } s \in S\backslash\{0\}\}.$$

The cardinality of the previous set is an important invariant of S called the **type** of S denoted by t(S). From the definition it easily follows that $F(S) \in PF(S)$, in fact, it is the maximum of this set.

Let S be a numerical semigroup. We define in $\mathbb{Z}$ the following relation:

$$a \leq_S b \text{ if } b - a \in S.$$

As noticed in [**38**], $\leq_S$ is an order relation (i.e. reflexive, antisymmetric and transitive). From the definition of $PF(S)$, it easily follows that the elements are those maximal gaps with respect to $\leq_S$. A characterization in terms of the Apéry set already appears in [[**15**], Proposition 7].

LEMMA 11. *Let S be a numerical semigroup and let x be a nonzero element of S. Then*

$$\mathrm{PF}(S) = \{w - x \mid w \in \max_{\leq_S} \mathrm{Ap}(S,x)\}$$

From previous lemma we obtain an upper bound from the type of a numerical semigroup, we have that $t(S) \leq m(S) - 1$.

2. Irreducible numerical semigroups

One type of numerical semigroups which are among the most studied are the irreducible numerical semigroups for their relevance in ring theory. A numerical semigroup is **irreducible** if it cannot be expressed as an intersection of two numerical semigroups properly containing it. The next result shows that the irreducible numerical semigroups are maximal in the set of numerical semigroups with fixed Frobenius number.

THEOREM 12. *[[**28**], Theorem 1] The following conditions are equivalent:*

(1) S is irreducible;
(2) S is maximal in the set of all numerical semigroups with Frobenius number $F(S)$;
(3) S is maximal in the set of all numerical semigroups that do not contain $F(S)$.

A numerical semigroup S is **symmetric** (respectively, **pseudo-symmetric**) if it is irreducible and $F(S)$ is odd (respectively, even).

PROPOSITION 13. *[[**38**], Proposition 4.4] Let S be a numerical semigroup.*

(1) S is symmetric if and only if $F(S)$ is odd and $x \in \mathbb{Z}\backslash S$ implies $F(S) - x \in S$;

(2) S is pseudo-symmetric if and only if $F(S)$ is even and $x \in \mathbb{Z}\backslash S$ implies that either $F(S) - x \in S$ or $x = \frac{F(S)}{2}$.

Sometimes the previous proposition is used as definition of the concepts of symmetric and pseudo-symmetric numerical semigroups.

The next result gives us a relation between the genus and the Frobenius number of irreducible numerical semigroups.

PROPOSITION 14. *[[**38**], Corollary 4.5] Let S be a numerical semigroup.*

(1) S is symmetric if and only if $g(S) = \dfrac{F(S)+1}{2}$.

(2) S is pseudo-symmetric if and only if $g(S) = \dfrac{F(S)+2}{2}$.

3. Families of numerical semigroups closed under finite intersections and for the Frobenius number

The results presented in this section can be found in [**27**].

A **Frobenius variety** is a nonempty set $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

(1) if S and T are in $\mathcal{V}$, then so is $S \cap T$;

(2) if S is in $\mathcal{V}$ and it is not equal to $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is in $\mathcal{V}$.

Clearly the set of all numerical semigroups is a Frobenius variety.

From 2) of Lemma 4, given a numerical semigroup S, we define recursively the following sequence of numerical semigroups:

- $S_0 = S$,
- If $S_i \neq \mathbb{N}$, then $S_{i+1} = S_i \cup \{F(S_i)\}$.

Since $\mathbb{N}\backslash S$ is finite, we obtain a finite chain of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq S_2, \cdots, \subsetneq S_n = \mathbb{N}$. Denote by $\mathcal{C}(S)$ the set $\{S_0, S_1, \ldots, S_n\}$.

The following results can be deduced from the definition of Frobenius variety.

LEMMA 15. *If $\mathcal{V}$ is a Frobenius variety and $S \in \mathcal{V}$, then $\mathcal{C}(S) \subseteq \mathcal{V}$.*

As a consequence of the above lemma we deduce that $\mathbb{N}$ belongs to every Frobenius variety and therefore the intersection of Frobenius varieties is always a nonempty family of numerical semigroups.

PROPOSITION 16. *The intersection of Frobenius varieties is a Frobenius variety.*

From 1) of Lemma 4 is easy to prove that a finite intersection of numerical semigroups is also a numerical semigroup. Note that nonfinite intersections of numerical semigroups are not in general a numerical semigroup as it is shown in the following example. Nevertheless, they are always submonoids of $\mathbb{N}$.

EXAMPLE 17. For every $n \in \mathbb{N}$, we have that $\{0, n, \rightarrow\}$ is a numerical semigroup. It is also easy to prove that $\cap_{n\in\mathbb{N}} \{0, n, \rightarrow\} = \{0\}$.

Let $\mathcal{V}$ be a Frobenius variety, we will say that a submonoid M of $\mathbb{N}$ is a **$\mathcal{V}$-monoid** if it can be expressed as an intersection of elements of $\mathcal{V}$.

The following result is easy to prove.

LEMMA 18. *The intersection of $\mathcal{V}$-monoids is also a $\mathcal{V}$-monoid.*

From this result we have the following definition. Let A be a subset of $\mathbb{N}$. The **$\mathcal{V}$-monoid generated by** A is the intersection of all the $\mathcal{V}$-monoids containing A. Denote such a $\mathcal{V}$-monoid by $\mathcal{V}(A)$. If $M = \mathcal{V}(A)$, then we say that A is a $\mathcal{V}$-system of generators of M. As every submonoid of $\mathbb{N}$ is finitely generated, we obtain the following result.

PROPOSITION 19. *Every $\mathcal{V}$-monoid has a finite $\mathcal{V}$-system of generators.*

If no proper subset of A is a $\mathcal{V}$-system of generators of M, then we say that A is a minimal $\mathcal{V}$-system of generators of M.

THEOREM 20. *Every $\mathcal{V}$-monoid has a unique minimal $\mathcal{V}$-system of generators and this set is finite.*

PROPOSITION 21. *Let M be a $\mathcal{V}$-monoid and $x \in M$. The set $M \backslash \{x\}$ is a $\mathcal{V}$-monoid if and only if x belongs to the minimal $\mathcal{V}$-system of generators of M.*

COROLLARY 22. *Let S be a numerical semigroup. The following statements are equivalent:*

(1) $S = S^{'} \cup \left\{ F(S^{'}) \right\}$, for some $S^{'} \in \mathcal{V}$;

(2) $S \in \mathcal{V}$ and the minimal $\mathcal{V}$-system of generators of S contains an element greater than $F(S)$.

A **graph** G is a pair (V,E), where V is a nonempty set whose elements are called vertices, and E is a subset of $\{(v,w) \in V \times V \mid v \neq w\}$. The elements of E are called edges of G. A path of length n connecting the vertices x and y of G is a sequence of distinct edges of the form (v_0,v_1), (v_1,v_2),$\ldots$,(v_{n-1},v_n) with $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (known as the root of G) such that for every other vertex x of G, there exist a unique path connecting x and r. If (x,y) is an edge of a tree, then we say that x is a child of y. A binary tree is a rooted tree in which every vertex has 0, 1 or 2 childs. A vertex with no childs is a leaf.

Given a Frobenius variety $\mathcal{V}$, define $G(\mathcal{V})$ the associated graph to $\mathcal{V}$ in the following way: the set of vertices of $G(\mathcal{V})$ is $\mathcal{V}$ and $(S^{'},S) \in \mathcal{V} \times \mathcal{V}$ is an edge of $G(\mathcal{V})$ if and only if $S = S^{'} \cup \left\{ F(S^{'}) \right\}$. From Corollary 22 we have the following result.

THEOREM 23. *Let $\mathcal{V}$ be a Frobenius variety. The graph $G(\mathcal{V})$ is a tree with root equal to $\mathbb{N}$. Furthermore, the children of a vertex $S \in \mathcal{V}$ are $S \backslash \{x_1\}, \ldots, S \backslash \{x_r\}$ where $x_1, \ldots, x_r$ are the elements of the minimal $\mathcal{V}$-system of generators of S which are greater that $F(S)$.*

CHAPTER 2

Saturated numerical semigroups

We start this chapter by recalling some results that appear in [**41**]. This allows us to introduce the concept of SAT system of generators for a saturated numerical semigroup and we will show that the set of saturated numerical semigroups is a Frobenius variety. This fact with the results from [**27**] enable us to arrange the set of all saturated numerical semigroups in a tree rooted in $\mathbb{N}$.

In Section 2, and from previous results, we present an algorithm for computing the set of saturated numerical semigroups of a given genus. The results from this section appear in [**31**].

Finally, in Section 3, we collect the results presented in [**32**]. In particular we give an efficient algorithmic method that, for a positive integer F, computes the whole set of saturated numerical semigroups with Frobenius number F. This is achieved by means of F-saturated sequences, associating to each one a saturated numerical semigroup.

1. Characterization of saturated numerical semigroups

In this section we give a characterization of saturated numerical semigroups, then we point out that the intersection of two saturated numerical semigroups is again saturated. This allows us to introduce the concept of a SAT system of generators of a saturated numerical semigroup. Then we will show that every saturated numerical semigroup has a unique minimal SAT system of generators. This will support the concept of SAT rank of a saturated numerical semigroup. Finally, we present a recursive method for computing the set of all saturated numerical semigroups, and arrange it in a binary tree rooted in $\mathbb{N}$ with no leaves.

1.1. A characterization. A numerical semigroup S is **saturated** if the following condition holds: if $s, s_1, \dots, s_r \in S$ are such that $s_i \leq s$ for all $i \in \{1, \dots, r\}$ and $z_1, \dots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then $s + z_1 s_1 + \cdots + z_r s_r \in S$.

For $A \subseteq \mathbb{N}$ and $a \in A$, set

$$d_A(a) = gcd\{x \in A | x \leq a\}$$

THEOREM 24. *[[**41**], Lemma 4] Let A be a nonempty subset of $\mathbb{N}$ such that $0 \in A$ and $gcd(A) = 1$. The following conditions are equivalent:*

1 *) A is a saturated numerical semigroup.*

2 *) $a + d_A(a) \in A$ for all $a \in A$.*

3 *) $a + kd_A(a) \in A$ for all $a \in A$ and $k \in \mathbb{N}$.*

1.2. SAT system of generators. Our next aim is to introduce the concept of a SAT system of generators for a saturated numerical semigroup. In order to do this we first need to prove that for a given $X \subseteq \mathbb{N}$ with $gcd(X) = 1$, there exists a least (with respect to inclusion) saturated numerical semigroup that contains X. The best candidate is the intersection of all saturated numerical semigroups that contain X.

The next result is easy to prove.

PROPOSITION 25. *[[**41**], Proposition 5] Let S_1 and S_2 be two saturated numerical semigroups. Then $S = S_1 \cap S_2$ is a saturated numerical semigroup.*

Let X be a subset of $\mathbb{N}$ such that $gcd(X) = 1$. Then every saturated numerical semigroup containing X must also contain $< X >$, and thus there are finitely many of them. We denote by $Sat(X)$ the intersection of all saturated numerical semigroups containing X, and call it the **saturated closure of** X. Observe that $Sat(X) = Sat(< X >)$. Clearly, we have that $Sat(X)$ is the smallest saturated semigroup containing X.

If S is a saturated numerical semigroup and X is a subset of $\mathbb{N}$ such that $gcd(X) = 1$ and $Sat(X) = S$, then we will say that X is a **SAT system of generators of** S. We say that X is a **minimal SAT system of generators** if in addition no proper subset of X is a SAT system of generators of S. It is well known that every numerical semigroup is finitely generated (as a semigroup). Hence for a given numerical semigroup S, there exists $\{n_1,\ldots,n_p\} \subset \mathbb{N}$ such that $S = < n_1,\ldots,n_p >$. If S is a saturated numerical semigroup, then clearly $Sat(n_1,\ldots,n_p) = Sat(S) = S$, and thus every saturated numerical semigroup admits a finite SAT system of generators. Note that if X is a SAT system of generators of S, then $< X >$ does not have to be equal to $S = Sat(X)$.

THEOREM 26. *[[**41**], Theorem 6] Let $n_1 < n_2 < \cdots < n_p$ be positive integers such that $gcd(n_1,\ldots,n_p) = 1$. For every $i \in \{1,\ldots,p\}$, set $d_i = gcd(n_1,\ldots,n_i)$ and for all $j \in \{1,\ldots,p-1\}$ define $k_j = max\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\}$. Then $Sat(n_1,\ldots,n_p) = \{0,n_1,n_1+d_1,\ldots,n_1+k_1d_1,n_2,n_2+d_2,\ldots,n_2+k_2d_2,\ldots,n_{p-1},n_{p-1}+d_{p-1},\ldots,n_{p-1}+k_{p-1}d_{p-1},n_p,n_p+1,\rightarrow\}$.*

EXAMPLE 27. Let $\{n_1,n_2,n_3\} = \{4,10,23\}$. Then $d_1 = 4$, $d_2 = 2$, $d_3 = 1$, $k_1 = 1$ and $k_2 = 6$. Hence $Sat(4,10,23) = \{0,4,8,10,12,14,16,18,20,22,23,24,\rightarrow\}$.

It may happen that one is interested in the minimal system of generators (as a semigroup) of $Sat(X)$. From [**26**] one can deduce that if $m = \min(X\backslash\{0\})$ (= min(Sat(X)$\backslash\{0\}$)), then the minimal system of generators of $S = Sat(X)$ is $A = \{m\} \cup (\{s \in S \mid s-m \notin S\}\backslash\{0\})$.

Observe that the cardinality of A is m. Theorem 26 allows us to compute $Sat(X)$. Therefore if we want to find out which are the elements of A, is suffices to look at the first m elements in $Sat(X)$ such that subtracting m from them, the resulting integers are no longer in $Sat(X)$. In the preceding example, $S = Sat(4,10,23)$, $m = 4$ and $\{s \in S \mid s-m \notin S\} = \{0,10,23,25\}$. Thus $Sat(4,10,23) = < 4,10,23,25 >$.

1.3. The rank of a saturated numerical semigroup. We start by showing that every saturated numerical semigroup has a unique minimal SAT system of generators.

THEOREM 28. *[[**41**], Theorem 11] Let S be a saturated numerical semigroup. Then $\{s_1,\ldots,s_r\} = \{s \in S \backslash \{0\} \mid d_S(s) \neq d_S(s')$ for all $s' < s$, $s' \in S\}$ is the unique minimal SAT system of generators of S.*

EXAMPLE 29. Let S be the saturated numerical semigroup

$$S = \{0,4,8,10,12,14,16,18,20,22,23,24,\rightarrow\}.$$

It follows that $d_S(4) = 4 = d_S(8)$, $d_S(10) = \ldots = d_S(22) = 2$ and $d_S(23) = 1 = d_S(23+n)$ for all $n \in \mathbb{N}$. By Theorem 28 the minimal SAT system of generators is $\{4,10,23\}$.

Using Theorem 28 it makes sense to define the **SAT rank** of a saturated numerical semigroup S by the cardinality of its minimal SAT system of generators, which we will denote by SAT-rank(S).

Using Theorem 26 for the description of $Sat(n_1,\ldots,n_p)$ and Theorem 28 we have the following result.

COROLLARY 30. *[[**41**], Corollary 14] Let $n_1 < n_2 < \cdots < n_p$ be positive integers with greatest common divisor one. Then $\{n_1,\ldots,n_p\}$ is the minimal SAT systems of generators of $Sat(n_1,\ldots,n_p)$ if and only if $gcd(n_1,\ldots,n_i) \neq gcd(n_1,\ldots,n_i,n_{i+1})$ for all $i \in \{1,\ldots,p-1\}$.*

The following result is a reformulation of Theorem 26 and will be useful in the next sections.

LEMMA 31. *Let $n_1 < n_2 < \cdots < n_p$ be positive integers such that* $\gcd\{n_1,\ldots,n_p\} = 1$. *For every $i \in \{1,...,p\}$ let $d_i = \gcd\{n_1,\ldots,n_i\}$. Then*

$\mathrm{Sat}\big(\{n_1,...,n_p\}\big) = \{0\} \cup (n_1 + \langle d_1\rangle) \cup \cdots \cup (n_p + \langle d_p\rangle).$

1.4. The tree of saturated numerical semigroups. Now we are going to show that the set of saturated numerical semigroups may be viewed as a binary tree rooted in $\mathbb{N}$ with no leaves. First we show how to construct the father of any non-root vertex. Repeating the process yields the path connecting the given vertex to the root.

The next result is easy to prove.

PROPOSITION 32. *[[**41**], Proposition 17] Let $S \neq \mathbb{N}$ be a saturated numerical semigroup. Then $\overline{S} = S \cup \{F(S)\}$ is also saturated.*

For a given numerical semigroup S, recall that S_n was defined recursively by

- $S_0 = S$,
- If $S_i \neq \mathbb{N}$, then $S_{i+1} = S_i \cup \{F(S_i)\}$.

Clearly, there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$. If in addiction S is a saturated numerical semigroup, Proposition 32 states that $S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = \mathbb{N}$ is a chain of saturated numerical semigroups. Moreover, $S_i = S_{i+1} \backslash \{a\}$ for some $a \in S_{i+1}$ (a becomes the Frobenius number of S_i). This idea motivates the next result, which explains how the childs of a vertex in the tree are constructed.

PROPOSITION 33. *[[**41**], Proposition 18] Let S be a saturated numerical semigroup. The following conditions are equivalent.*

1) $S = S' \cup \{F(S')\}$ with S' a saturated numerical semigroup,

2) the minimal SAT system of generators of S contains an element greater than $F(S)$.

The previous proposition allows us to construct recursively the tree containing the set $\mathcal{L}$ of all saturated numerical semigroups. As we have seen before, the graph $G(\mathcal{L})$ is defined in the following way: the set of vertices of $G(\mathcal{L})$ is $\mathcal{L}$ and $(T,S) \in \mathcal{L} \times \mathcal{L}$ is a edge of $G(\mathcal{L})$ if and only if $T \cup \{F(T)\} = S$.

With all this information the following property is easy to prove.

LEMMA 34. *The graph $G(\mathcal{L})$ is a tree with root equal to $\mathbb{N}$. Furthermore, the childs of a vertex $S \in \mathcal{L}$ are $S\backslash\{x_1\},\ldots,S\backslash\{x_r\}$ where $x_1,\ldots,x_r$ are the elements of the minimal SAT-system of generators of S which are greater that* $\mathrm{F}(S)$.

From Lemma 31 (see also Theorem 26) we easily deduce the following.

LEMMA 35. *Let S be a saturated numerical semigroup with minimal SAT-system of generators $A = \{n_1 < \cdots < n_p\}$ and let $X = \{n_i \in A \mid n_i > \mathrm{F}(S)\}$. Then $\{n_p\} \subseteq X \subseteq \{n_{p-1}, n_p\}$. Furthermore, $n_{p-1} \in X$ if and only if $n_{p-1} = n_p - 1$.*

REMARK 36. Note that as an immediate consequence of Lemmas 34 and 35, we have that if S is an element of $\mathcal{L}$ then S has 1 or 2 childs and thus $G(\mathcal{L})$ is a binary tree with no leaves.

EXAMPLE 37. Let S=Sat$(\{4,10,23\})$. Then 23 is the unique element in minimal SAT-system of generators of S greater that $\mathrm{F}(S)$. Hence $S \in \mathcal{L}$ has a unique child, that is, $S\backslash\{23\}$.

EXAMPLE 38. Let S=Sat$(\{8,12,14,15\})$. Then 14 and 15 are the elements in minimal SAT-system of generators of S greater that $\mathrm{F}(S)$. Therefore, the childs of $S \in \mathcal{L}$ are $S\backslash\{14\}$ and $S\backslash\{15\}$.

2. The set of saturated numerical semigroups of a given genus

Our goal in this section is to find a way to compute the set of all saturated numerical semigroups with a given genus. We will use Proposition 33 to build recursively a tree rooted in $\mathbb{N}$ of the saturated numerical semigroups. The results presented can be found in [**31**].

2.1. A method for computing the set of all saturated numerical semigroups of a given genus. Let g be a positive integer and $\mathcal{L}(g)$ be the set of all saturated numerical semigroups with a genus g. It is clear that $\mathcal{L}(g+1) =$

$\{S' \mid S'$ is child of an element of $\mathcal{L}(g)\}$. This fact allows us to recursively construct $\mathcal{L}(g)$, starting in $\mathcal{L}(0) = \{\mathbb{N}\}$.

$$\begin{tikzcd}[column sep=small]
 & & & & \mathrm{Sat}(1)=\mathbb{N} & & & & \\
 & & & & \mathrm{Sat}(2,3) \arrow[u] & & & & \\
 & & & \mathrm{Sat}(3,4) \arrow[ur] & & \mathrm{Sat}(2,5) \arrow[ul] & & & \\
 & & \mathrm{Sat}(4,5) \arrow[ur] & & \mathrm{Sat}(3,5) \arrow[ul] & & \mathrm{Sat}(2,7) \arrow[ul] & & \\
 & \mathrm{Sat}(5,6) \arrow[ur] & & \mathrm{Sat}(4,6,7) \arrow[ul] & & \mathrm{Sat}(3,7) \arrow[ul] & & \mathrm{Sat}(2,9) \arrow[ul] & \\
\cdots \arrow[ur] & & \cdots\ \cdots \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] & & \cdots \arrow[ul] & & \cdots \arrow[ul]
\end{tikzcd}$$

Our next aim is to describe the minimal SAT-systems of generators of the childs of a given saturated numerical semigroup from its minimal SAT-system of generators.

PROPOSITION 39. *Let S be a saturated numerical semigroup with minimal SAT-system of generators $\{n_1 < \cdots < n_p\}$ and let $d_{p-1} = \gcd\{n_1, \ldots, n_{p-1}\}$. Then the minimal SAT-system of generators of $S \setminus \{n_p\}$ is equal to:*

1) $\{n_1 < \cdots < n_{p-1} < n_p + 2\}$ *if* $d_{p-1} | n_p + 1$*;*
2) $\{n_1 < \cdots < n_{p-1} < n_p + 1\}$ *if* $\gcd\{d_{p-1}, n_p + 1\} = 1$*;*
3) $\{n_1 < \cdots < n_{p-1} < n_p + 1 < n_p + 2\}$ *in the other cases.*

PROOF. As a consequence of Lemma 31 we have that $S \setminus \{n_p\} = \mathrm{Sat}(\{n_1, \ldots, n_{p-1}, n_p + 1, n_p + 2\})$.

1) If $d_{p-1} | n_p + 1$ then $\gcd\{n_1, \ldots, n_{p-1}\} = d_{p-1} = \gcd\{n_1 \ldots, n_{p-1}, n_p + 1\}$. By applying Lemma 31, we get that $S \setminus \{n_p\} = \mathrm{Sat}(\{n_1, \ldots, n_{p-1}, n_p + 2\})$ and, from Corollary 30, we deduce that $\{n_1, \ldots, n_{p-1}, n_p + 2\}$ is the minimal SAT-system of generators of $S \setminus \{n_p\}$.

2) If $\gcd\{d_{p-1}, n_p+1\} = 1$ then, in view of Lemma 31, we obtain that $S \backslash \{n_p\} = \mathrm{Sat}(\{n_1, \ldots, n_{p-1}, n_p+1\})$. By applying Corollary 30 this implies that $\{n_1, \ldots, n_{p-1}, n_p+1\}$ is the minimal SAT-system of generators of $S \backslash \{n_p\}$;
3) From Corollary 30 it follows that $\{n_1, \ldots, n_{p-1}, n_p+1, n_p+2\}$ is the minimal SAT-system of generators of $S \backslash \{n_p\}$.

□

REMARK 40. Note that as a consequence of the previous proposition we have that SAT-rank$(S) \leq$SAT-rank$(S \backslash \{n_p\}) \leq$SAT-rank$(S)+1$.

EXAMPLE 41.

1) If $S = \mathrm{Sat}(\{8, 12, 15\})$ then, by applying Proposition 39, we have that $S \backslash \{15\} = \mathrm{Sat}(\{8, 12, 17\})$.
2) If $S = \mathrm{Sat}(\{6, 9, 19\})$ then, in view of Proposition 39, we obtain that $S \backslash \{19\} = \mathrm{Sat}(\{6, 9, 20\})$.
3) If $S = \mathrm{Sat}(\{8, 12, 17\})$ then , using again Proposition 39, we deduce that $S \backslash \{17\} = \mathrm{Sat}(\{8, 12, 18, 19\})$.

Recall that, as a consequence of Lemmas 34 and 35, we deduce that, if S is a saturated numerical semigroup with minimal SAT-system of generators $\{n_1 < \cdots < n_p\}$ then $S \backslash \{n_p\}$ is always child of S. Besides, $S \backslash \{n_{p-1}\}$ is another child of S if and only if $n_{p-1} = n_p - 1$.

PROPOSITION 42. *Let S be a saturated numerical semigroup with minimal SAT-system of generators $\{n_1 < \cdots < n_p\}$ such that $n_{p-1} = n_p - 1$. Then the minimal SAT-system of generators of $S \backslash \{n_{p-1}\}$ is equal to:*

a) $\{n_1+1, n_1+2\}$ *if $p = 2$;*
b) *If $p \geq 3$ and $d_{p-2} = \gcd\{n_1, \ldots, n_{p-2}\}$ then:*
   b.1) $\{n_1 < \cdots < n_{p-2} < n_p\}$ *if* $\gcd\{d_{p-2}, n_p\} = 1$*;*
   b.2) $\{n_1 < \cdots < n_{p-2} < n_p < n_p+1\}$ *in the other cases.*

PROOF. a) Since $S = \mathrm{Sat}\big(\{n_1, n_1+1\}\big)$, from Lemma 31, it follows that $S \backslash \{n_1\} = \mathrm{Sat}\big(\{n_1+1, n_1+2\}\big)$.

b) In view of Lemma 31, we get that $S \backslash \{n_{p-1}\} = \mathrm{Sat}\big(\{n_1, \dots, n_{p-2}, n_p, n_p+1\}\big)$.

b.1) If $\gcd\{d_{p-2}, n_p\} = 1$ then, by applying Lemma 31, we have $S = \mathrm{Sat}\big(\{n_1, \dots, n_{p-2}, n_p\}\big)$ and, from Corollary 30, we obtain that $\{n_1, \dots, n_{p-2}, n_p\}$ is the minimal SAT-system of generators of $S \backslash \{n_{p-1}\}$.

b.2) Observe that in this setting $d_{p-2} \neq \gcd\{n_1, \dots, n_{p-2}, n_p\}$, since otherwise $1 = \gcd\{n_1, \dots, n_{p-2}, n_{p-1}, n_p\} = \gcd\{d_{p-2}, n_{p-1}\} = \gcd\{n_1, \dots, n_{p-2}, n_{p-1}\}$, which is absurd. Therefore, if $\gcd\{d_{p-2}, n_p\} \neq 1$ then, using Corollary 30 once more, we have that $\{n_1, \dots, n_{p-2}, n_p, n_p+1\}$ is the minimal SAT-system of generators of $S \backslash \{n_{p-1}\}$.

$\square$

REMARK 43. Observe that as a consequence of the previous proposition we have that SAT-rank$(S) - 1 \leq$SAT-rank$(S \backslash \{n_{p-1}\}) \leq$SAT-rank$(S)$.

EXAMPLE 44. 1) If $S = \mathrm{Sat}\big(\{5,6\}\big)$ then, by applying Proposition 42, we have that $S \backslash \{5\} = \mathrm{Sat}\big(\{6,7\}\big)$.

2) If $S = \mathrm{Sat}\big(\{4,6,7\}\big)$ then, by using Proposition 42, we get that $S \backslash \{6\} = \mathrm{Sat}\big(\{4,7\}\big)$.

3) If $S = \mathrm{Sat}\big(\{6,8,9\}\big)$ then, using again Proposition 42, we obtain that $S \backslash \{8\} = \mathrm{Sat}\big(\{6,9,10\}\big)$.

2.2. An algorithm to compute $\mathcal{L}(g)$. Our next goal is to describe an algorithmic procedure to compute all the elements in $\mathcal{L}(g)$. Clearly $\mathbb{N} = \mathrm{Sat}\big(\{1\}\big)$ has a unique child, which is $\mathrm{Sat}\big(\{2,3\}\big) = \{0, 2, \rightarrow\}$. Furthermore, if $S \in \mathcal{L}$ and $S \neq \mathbb{N}$ then SAT-rank$(S) \geq 2$. As we have mentioned before, if we know $\mathcal{L}(g-1)$ then we can compute $\mathcal{L}(g)$, simply computing all childs of $\mathcal{L}(g-1)$. From Propositions 39 and 42, we can conclude that, we need to know d_{p-1} and in some cases d_{p-2} to compute the childs of a saturated numerical semigroups S with minimal SAT-system of generators $\{n_1 < \cdots < n_p\}$. To avoid having to make this calculation continuously and to maximize the efficiency of computation, we introduce the concept of α-representation of a saturated numerical semigroup.

Let $S \neq \mathbb{N}$ be a saturated numerical semigroup, an **α-representation** of S is $[(n_1, n_2, \ldots, n_p), (x_1, x_2 \ldots, x_{p-1})]$ such that $\{n_1 < n_2 < \cdots < n_p\}$ is the minimal SAT-system of generators of S and $x_i = \gcd\{n_1, \ldots, n_{p-i}\}$ for all $i \in \{1, \ldots, p-1\}$. Note that $x_1 = \gcd\{n_1, \ldots, n_{p-1}\} = d_{p-1}$ and $x_2 = \gcd\{n_1, \ldots, n_{p-2}\} = d_{p-2}$.

Now we give a method that, from an α-representation of a saturated numerical semigroup, allows to calculate the α-representations of its childs.

As an immediate consequence of Proposition 39, we have the following.

LEMMA 45. *Let $[(n_1, \ldots, n_p), (x_1, \ldots, x_{p-1})]$ be an α-representation of a saturated numerical semigroup $S \neq \mathbb{N}$. Then the α-representation of $(S \backslash \{n_p\})$ is equal to:*

1) $[(n_1, \ldots, n_{p-1}, n_p + 2), (x_1, \ldots, x_{p-1})]$ *if* $x_1 | n_p + 1$*;*
2) $[(n_1, \ldots, n_{p-1}, n_p + 1), (x_1, \ldots, x_{p-1})]$ *if* $\gcd\{x_1, n_p + 1\} = 1$*;*
3) $[(n_1, \ldots, n_{p-1}, n_p + 1, n_p + 2), (\gcd\{x_1, n_p + 1\}, x_1, \ldots, x_{p-1})]$ *in the other cases.*

EXAMPLE 46.
1) If $S = \mathrm{Sat}(\{8, 12, 15\})$ then the α-representation of S is $[(8, 12, 15), (4, 8)]$. By Applying Lemma 45, we have that the α-representation of $S \backslash \{15\}$ is $[(8, 12, 17), (4, 8)]$.
2) If $S = \mathrm{Sat}(\{6, 9, 19\})$ then the α-representation of S is $[(6, 9, 19), (3, 6)]$. From Lemma 45, we obtain that the α-representation of $S \backslash \{19\}$ is $[(6, 9, 20), (3, 6)]$

3) If $S=\mathrm{Sat}\big(\{8,12,17\}\big)$ then the α-representation of S is $\big[(8,12,17),(4,8)\big]$. By Lemma 45 again, we get that the α-representation of $S\backslash\{17\}$ is $\big[(8,12,18,19),(2,4,8)\big]$.

As a consequence of Proposition 42, we easily deduce the next result.

LEMMA 47. *Let* $\big[(n_1,\dots,n_p),(x_1,\dots,x_{p-1})\big]$ *be an* α*-representation of a saturated numerical semigroup* $S\neq\mathbb{N}$ *such that* $n_{p-1}=n_p-1$. *Then the* α*-representation of* $(S\backslash\{n_{p-1}\})$ *is equal to:*

a) $\big[(n_1+1,n_1+2),(n_1+1)\big]$ *if* $p=2$*;*
b) $\big[(n_1,\dots,n_{p-2},n_p),(x_2,\dots,x_{p-1})\big]$ *if* $p\geq 3$ *and* $\gcd\{x_2,n_p\}=1$*;*
c) $\big[(n_1,\dots,n_{p-2},n_p,n_p+1),(\gcd\{x_2,n_p\},x_2,\dots,x_{p-1})\big]$ *in the other cases.*

EXAMPLE 48. 1) If $S=\mathrm{Sat}\big(\{5,6\}\big)$ then the α-representation of S is $\big[(5,6),(5)\big]$. Applying Lemma 47, we have that the α-representation of $S\backslash\{5\}$ is $\big[(6,7),(6)\big]$.

2) If $S=\mathrm{Sat}\big(\{4,6,7\}\big)$ then the α-representation of S is $\big[(4,6,7),(2,4)\big]$. By Lemma 47, we obtain that the α-representation of $S\backslash\{6\}$ is $\big[(4,7),(4)\big]$.

3) If $S=\mathrm{Sat}\big(\{6,8,9\}\big)$ then the α-representation of S is $\big[(6,8,9),(2,6)\big]$. Using again Lemma 47, we get that the α-representation of $S\backslash\{8\}$ is $\big[(6,9,10),(3,6)\big]$.

We are ready to give the announced algorithm which shows how to compute $\mathcal{L}(g)$.

ALGORITHM 49. Input: g a positive integer.

Output: $\mathcal{L}(g)$.

1) $A=\{[(2,3),(2)]\}$, $i=1$, $B=\emptyset$.
2) If $i=g$ then return A.
3) For each $[(n_1,\dots,n_p),(x_1,\dots,x_{p-1})]\in A$ do
   3.1) If $x_1|n_p+1$ then
   $B=B\cup\big\{[(n_1,\dots,n_{p-1},n_p+2),(x_1,\dots,x_{p-1})]\big\}$ and go to Step 3.4).

3.2) If $\gcd\left\{x_1, n_p+1\right\} = 1$ then

$B = B \cup \left\{[(n_1,\ldots,n_{p-1},n_p+1),(x_1,\ldots,x_{p-1})]\right\}$ and go to Step 3.4).

3.3) $B = B \cup \left\{[(n_1,\ldots,n_{p-1},n_p+1,n_p+2),(\gcd\left\{x_1,n_p+1\right\},x_1,\ldots,x_{p-1})]\right\}$.

3.4) If $n_{p-1} \neq n_p - 1$ go to Step 4).

3.5) If $p = 2$ then $B = B \cup \{[(n_1+1,n_1+2),(n_1+1)]\}$ and go to Step 4).

3.6) If $\gcd\left\{x_2,n_p\right\} = 1$ then

$B = B \cup \left\{[(n_1,\ldots,n_{p-2},n_p),(x_2,\ldots,x_{p-1})]\right\}$ and go to Step 4).

3.7) $B = B \cup \left\{[(n_1,\ldots,n_{p-2},n_p,n_p+1),(\gcd\left\{x_2,n_p\right\},x_2,\ldots,x_{p-1})]\right\}$.

4) $A = B$, $i = i+1$, $B = \emptyset$ and go to Step 2).

EXAMPLE 50. Let us compute all saturated numerical semigroups with genus 10.

First, and using Algorithm 49, we compute the α-representation of all saturated numerical semigroups with genus less than 10 (denoted here by A_i);

. for $i = 1$ then $A_1 = \{[(2,3),(2)]\}$;

. for $i = 2$ then $A_2 = \{[(2,5),(2)],[(3,4),(3)]\}$;

. for $i = 3$ then $A_3 = \{[(2,7),(2)],[(3,5),(3)],[(4,5),(4)]\}$;

. for $i = 4$ then

$A_4 = \{[(2,9),(2)],[(3,7),(3)],[(4,6,7),(2,4)],[(5,6),(5)]\}$;

. for $i = 5$ then

$A_5 = \{[(2,11),(2)],[(3,8),(3)],[(4,6,9),(2,4)],[(4,7),(4)],$
$[(5,7),(5)],[(6,7),(6)]\}$;

. for $i = 6$ then

$A_6 = \{[(2,13),(2)],[(3,10),(3)],[(4,6,11),(2,4)],[(4,9),(4)],$
$[(5,8),(5)],[(6,8,9),(2,6)],[(7,8),(7)]\}$;

. for $i = 7$ then

$A_7 = \{[(2,15),(2)],[(3,11),(3)],[(4,6,13),(2,4)],[(4,10,11),(2,4)],$
$[(5,9),(5)],[(6,8,11),(2,6)],[(6,9,10),(3,6)][(7,9),(7)],[(8,9),(8)]\}$;

. for $i = 8$ then

$A_8 = \{[(2,17),(2)],[(3,13),(3)],[(4,6,15),(2,4)],[(4,10,13),(2,4)],$

$[(4,11),(4)],[(5,11),(5)],[(6,8,13),(2,6)],[(7,10),(7)],[(8,10,11),$
$(2,8)],[(9,10),(9)],[(6,9,11),(3,6)],[(6,10,11),(2,6)]\};$

. for $i = 9$ then
$A_9 = \{[(2,19),(2)],[(3,14),(3)],[(4,6,17),(2,4)],[(4,10,15),(2,4)],$
$[(4,13),(4)],[(5,12),(5)],[(6,8,15),(2,6)],[(7,11),(7)],[(8,10,13)(2,8),$
$[(8,11),(8)],[(9,11),(9)],[(10,11),(10)],[(6,11),(6)],[(6,10,13),(2,6)],$
$[(6,9,13),(3,6)]\}.$

And from this we get the minimal SAT-system of generators of the set of saturated numerical semigroups with genus 10,

$$\{\{2,21\},\{3,16\},\{4,6,19\},\{4,10,17\},\{4,14,15\},\{5,13\},\{6,8,17\},\{7,12\},\{8,10,15\},$$
$$\{8,12,13\},\{9,12,13\},\{10,12,13\},\{11,12\}\{6,13\},\{6,10,15\},\{6,9,14\}\},$$

which are the childs of elements in A_9.

Finally, we present the results of some computational experiments performed to analyze the apllicability of the algorithm previously proposed. These functions were implemented in GAP [**12**] and [**17**] and compute all saturated numerical semigroups with a given genus.

For Genus 10,

gap> Length(SaturatedNumericalSemigroupsWithFixedGenus(10)); 16

takes 0 ms, while computing the set of all saturated numerical semigroups with genus and then filtering those that are saturated takes 31 ms.

gap> Length(Filtered(NumericalSemigroupsWithGenus(10),IsSaturatedNumericalSemigroup));
16

As for 15 we get also 0 ms for

gap> Length(SaturatedNumericalSemigroupsWithFixedGenus(15)); 40

while it takes 390 ms for

gap> Length(Filtered(NumericalSemigroupsWithGenus(15),IsSaturatedNumericalSemigroup));
40

For 25 we still get 0 ms with

gap> Length(SaturatedNumericalSemigroupsWithFixedGenus(25)); 130

while it takes 100735 ms with

gap> Length(Filtered(NumericalSemigroupsWithGenus(25),IsSaturatedNumericalSemigroup)); 130

For genus 30 the time with this algorithm is also 0 ms while with the filtering was not possible to calculate because it gets an error message

”Error, exceeded the permitted memory”.

In the following table there are the results obtained for genus up to 150. For each positive integer g we wrote the number of saturated numerical semigroups (n_g) of the given genus (g).

g	n_g	g	n_g	g	n_g	g	n_g	g	n_g	g	n_g	g	n_g	g	n_g	g	n_g	g	n_g
1	1	16	43	31	228	46	701	61	1717	76	3634	91	6900	106	12057	121	20106	136	31790
2	2	17	51	32	251	47	757	62	1815	77	3805	92	7175	107	12503	122	20749	137	32758
3	3	18	56	33	272	48	805	63	1915	78	3970	93	7444	108	12939	123	21404	138	33730
4	4	19	67	34	295	49	864	64	2021	79	4163	94	7732	109	13411	124	22086	139	34755
5	6	20	78	35	324	50	918	65	2135	80	4348	95	8038	110	13886	125	22787	140	35751
6	7	21	85	36	346	51	973	66	2239	81	4532	96	8336	111	14382	126	23485	141	36764
7	9	22	91	37	373	52	1030	67	2365	82	4729	97	8669	112	14898	127	24239	142	37836
8	12	23	106	38	401	53	1103	68	2482	83	4952	98	9004	113	15441	128	24990	143	38951
9	15	24	117	39	432	54	1172	69	2599	84	5156	99	9348	114	15969	129	25753	144	40040
10	16	25	130	40	460	55	1248	70	2722	85	5373	100	9705	115	16524	130	26546	145	41170
11	21	26	143	41	500	56	1320	71	2868	86	5592	101	10083	116	17080	131	27379	146	42311
12	24	27	158	42	535	57	1385	72	3006	87	5822	102	10457	117	17634	132	28214	147	43477
13	29	28	170	43	581	58	1457	73	3158	88	6070	103	10866	118	18232	133	29081	148	44698
14	35	29	190	44	626	59	1548	74	3314	89	6345	104	11262	119	18857	134	29968	149	45956
15	40	30	205	45	662	60	1626	75	3470	90	6616	105	11643	120	19460	135	30859	150	47220

3. The set of saturated numerical semigroups with fixed Frobenius number

The main aim of this section is to give an algorithmic method that, given a positive integer F, computes all saturated numerical semigroups with a Frobenius number F. The results presented in this section can be found in [**32**]. We already saw that giving a saturated numerical semigroup S is equivalent to give a sequence of positive integers $n_1 < n_2 < \cdots < n_p$ with greatest common divisor one and $\gcd\{n_1, n_2, \ldots, n_i\} \neq \gcd\{n_1, n_2, \ldots, n_i, n_{i+1}\}$ for all $i \in \{1, \ldots, p-1\}$. In this case, we say that $\{n_1, n_2, \ldots, n_p\}$ is a minimal SAT-system of generators of S. Furthermore, if $d_i = \gcd\{n_1, \ldots, n_i\}$ for each $i \in \{1, \ldots, p\}$, then we say that S is a $(d_1, d_2, \ldots, d_p)$**-semigroup**.

A **saturated sequence of length** k, is a k-tuple of positive integers $(d_1, d_2, \ldots, d_k)$ such that $d_1 > d_2 > \cdots > d_k = 1$ and $d_{i+1} | d_i$ for all $i \in \{1, \ldots, k-1\}$.

Let F be positive integer. An F**-saturated sequence** is a saturated sequence $(d_1, d_2, \ldots, d_k)$ such that there exists at least one $(d_1, d_2, \ldots, d_k)$-semigroup with Frobenius number F.

Let $L(F) = \{l \mid l \text{ is an } F\text{-saturated sequence}\}$. For each $l \in L(F)$ define $\mathcal{L}(l) = \{S \mid S \text{ is a } l-\text{semigroup with Frobenius number } F\ \}$. Then $\bigcup_{l \in L(F)} \mathcal{L}(l)$ is the set of all saturated numerical semigroups with Frobenius number F. Therefore, to construct the set of all saturated numerical semigroups with Frobenius number F, it suffices to give an algorithmic procedure to compute all F-saturated sequences and given an F-saturated sequence l another algorithm that allows to determine the set $\mathcal{L}(l)$.

3.1. Minimal SAT-system of generators. The following theorem is the key to the development of this part of this work and describes a method that allows to obtain the minimal SAT-system of generators of saturated numerical semigroups with a given SAT-rank.

THEOREM 51. *Let $d_1 > d_2 > \cdots > d_p = 1$ be integers such that $d_{i+1} | d_i$ and let $t_1, t_2, \ldots, t_p$ be positive integers such that $\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ for all $i \in \{1, \ldots, p-1\}$. Then $\{d_1, t_1 d_1 + t_2 d_2, \ldots, t_1 d_1 + \cdots + t_p d_p\}$ is the minimal SAT-system of generators of a saturated numerical semigroup with SAT-rank p. Furthermore every minimal SAT-system of generators of a saturated numerical semigroup with SAT-rank p is of this form.*

PROOF. Taking into account Corollary 30, to prove that $\{d_1, t_1d_1+t_2d_2, \ldots, t_1d_1+\cdots+t_pd_p\}$ is the minimal SAT-system of generators of a saturated numerical semigroup it suffices to show that $\gcd\{d_1, t_1d_1+t_2d_2, \ldots, t_1d_1+\cdots+t_id_i\} = d_i$ for all $i \in \{1, \ldots, p\}$. We proceed by induction on i. For $i = 1$, the result follows easily from the fact that $\gcd\{d_1\} = d_1$. Assume that the statement is true for i and let us show it for $i+1$. In fact,

$$\begin{aligned}
&\gcd\{d_1, \ldots, t_1d_1+\cdots+t_id_i, t_1d_1+\cdots+t_{i+1}d_{i+1}\} = \\
&\qquad = \gcd\{\gcd\{d_1, \ldots, t_1d_1+\cdots+t_id_i\}, t_1d_1+\cdots+t_{i+1}d_{i+1}\} = \\
&\qquad = \gcd\{d_i, t_1d_1+\cdots+t_{i+1}d_{i+1}\} = \gcd\{d_i, t_{i+1}d_{i+1}\} = \\
&\qquad = d_{i+1}.\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = d_{i+1}.
\end{aligned}$$

Reciprocally, let $\{n_1 < n_2 < \cdots < n_p\}$ be a minimal SAT-system of generators of a saturated numerical semigroup. Let $d_i = \gcd\{n_1, \ldots, n_i\}$ for all $i \in \{1, \ldots, p\}$. It is clear that $d_{i+1} \mid d_i$ and, by Corollary 30, that $d_1 > d_2 > \cdots > d_p = 1$. We will see that there exist positive integers $t_1, \ldots, t_p$ such that $n_1 = d_1$, $n_2 = t_1d_1+t_2d_2, \ldots, n_p = t_1d_1+\cdots+t_pd_p$ and $\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ for all $i \in \{1, \ldots, p-1\}$. Let $t_1 = 1$ and $t_{i+1} = \frac{n_{i+1}-n_i}{d_{i+1}}$ for all $i \in \{1, \ldots, p-1\}$. To this end we prove by induction on i that $n_i = t_1d_1+\cdots+t_id_i$, for all $i \in \{2, \ldots, p\}$. For $i = 2$ the result is clear, since $t_1d_1 + t_2d_2 = 1n_1 + \frac{n_2-n_1}{d_2}d_2 = n_2$. Assume that the result holds for i and let us prove it for $i+1$. As $n_{i+1} = n_i + t_{i+1}d_{i+1}$, by applying the induction hypothesis, we obtain that $n_{i+1} = t_1d_1+\cdots+t_id_i+t_{i+1}d_{i+1}$. In order to conclude the proof, it is enough to see that $\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$ for all $i \in \{1, \ldots, p-1\}$. In fact,

$$\begin{aligned}
&d_{i+1} = \gcd\{n_1, \ldots, n_{i+1}\} = \gcd\{\gcd\{n_1, \ldots, n_i\}, n_{i+1}\} = \\
&\qquad = \gcd\{d_i, t_1d_1+\cdots+t_id_i+t_{i+1}d_{i+1}\} = \\
&\qquad = \gcd\{d_i, t_{i+1}d_{i+1}\} = d_{i+1}\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\}.
\end{aligned}$$

Therefore, $\gcd\left\{\frac{d_i}{d_{i+1}}, t_{i+1}\right\} = 1$. □

3.2. The Frobenius number. With previous results we are able to find a formula for the Frobenius number of a saturated numerical semigroup in terms of its minimal SAT-system of generators.

Given integers a, b and c we denote by $a \equiv b \bmod c$ if $a-b$ is a multiple of c. We also write $a|b$ to denote that a divides b.

PROPOSITION 52. *Let S be a saturated numerical semigroup with minimal SAT-system of generators $\{n_1 < n_2 < \cdots < n_p\}$. Let $d_i = \gcd\{n_1, \ldots, n_i\}$ for all $i \in \{1, \ldots, p\}$. Then*

$$F(S) = \begin{cases} n_p - 1, & \text{if } n_p \not\equiv 1 \bmod d_{p-1}, \\ n_p - 2, & \text{if } n_p \equiv 1 \bmod d_{p-1}. \end{cases}$$

PROOF. If $n_p \not\equiv 1 \bmod d_{p-1}$ then $n_p - 1 \not\equiv 0 \bmod d_{p-1}$. By applying Theorem 26, we have that $n_p - 1 \notin S$ and $\{n_p, \rightarrow\} \subseteq S$. Hence $\mathrm{F}(S) = n_p - 1$.

If $n_p \equiv 1 \bmod d_{p-1}$ then $n_p - 1 \equiv 0 \bmod d_{p-1}$, and by using again Theorem 26, we have that $n_p - 1 \in S$. From Corollary 30, we know that $d_{p-1} \geq 2$ and thus $n_p - 2 \not\equiv 0 \bmod d_{p-1}$. In addition, by Theorem 26, we deduce that $n_p - 2 \notin S$ and that $\{n_p - 1, n_p, \rightarrow\} \subseteq S$. Hence $F(S) = n_p - 2$. □

As a consequence of the above proposition, we obtain the following result.

COROLLARY 53. *Let S be a saturated numerical semigroup with minimal SAT-system of generators $\{d_1, t_1d_1 + t_2d_2, \ldots, t_1d_1 + \cdots + t_pd_p\}$ fulfilling the conditions of Theorem 51. Then*

$$F(S) = \begin{cases} t_1d_1 + \cdots + t_pd_p - 1, & \text{if } t_p \not\equiv 1 \bmod d_{p-1}, \\ t_1d_1 + \cdots + t_pd_p - 2, & \text{if } t_p \equiv 1 \bmod d_{p-1}. \end{cases}$$

A $(d_1, d_2, \ldots, d_p)$-semigroup is a saturated numerical semigroup such that if $\{n_1 < n_2 < \cdots < n_p\}$ is its minimal SAT-system of generators, then $d_i = \gcd\{n_1, \ldots, n_i\}$ for all $i \in \{1, \ldots, p\}$.

COROLLARY 54. *Let S be a $(d_1, d_2, \ldots, d_p)$-semigroup with Frobenius number F. Then*

1) $d_1 + \cdots + d_p \leq F + 2$;
2) $2^p \leq F + 3$;
3) *the SAT-rank of S is less than or equal to* $\log_2(F+3)$.

PROOF.

1) By using Theorem 51 and Corollary 53, we deduce that there exist positive integers $t_1, \ldots, t_p$ such that $t_1 d_1 + \cdots + t_p d_p \in \{F+1, F+2\}$. Hence $d_1 + \cdots + d_p \leq \mathrm{F} + 2$.
2) By Corollary 30, we have that $d_1 > d_2 > \cdots > d_p = 1$ and $d_{i+1} | d_i$ for all $i \in \{1, \ldots, p-1\}$. Then $d_i \geq 2d_{i+1}$ and thus $d_i \geq 2^{p-i}$ for all $i \in \{1, \ldots, p-1\}$. Applying 1), we deduce that $2^{p-1} + \cdots + 2 + 1 \leq F + 2$. By induction, it easily follows that $2^{p-1} + \cdots + 2 + 1 = 2^p - 1$, whence $2^p \leq F + 3$.
3) It is easily deduced from 2), since SAT-rank of S is equal to p.

□

Our next goal is to see which condition has to verify a saturated sequence $(d_1, \ldots, d_p)$ so that there exists at least one $(d_1, \ldots, d_p)$-semigroup with Frobenius number F.

Suppose that $\{d_1, t_1 d_1 + t_2 d_2, \ldots, t_1 d_1 + \cdots + t_p d_p\}$ is the minimal SAT-system of generators of a saturated numerical semigroup with Frobenius number F, fulfilling the conditions of Theorem 51. We distinguish two cases:

1) If $t_p \not\equiv 1 \mod d_{p-1}$ then, by applying Corollary 53 , we get that $F + 1 = t_1 d_1 + \cdots + t_p d_p$. Whence $F + 1 \geq d_1 + \cdots + d_p$. Moreover, as $\gcd\{t_1 d_1 + \cdots + t_p d_p, d_{p-1}\} = \gcd\{t_p d_p, d_{p-1}\} = \gcd\left\{t_p, \frac{d_{p-1}}{d_p}\right\} = 1$, then $\gcd\{F+1, d_{p-1}\} = 1$. Since $F + 1 = t_1 d_1 + \cdots + t_p d_p$ and $d_p = 1$, we deduce that $F + 1 \equiv t_p \mod d_{p-1}$ and thus $F + 1 \not\equiv 1 \mod d_{p-1}$.
2) If $t_p \equiv 1 \mod d_{p-1}$ then, by applying Corollary 53, we have that $F + 2 = t_1 d_1 + \cdots + t_p d_p$. Therefore $F + 2 \geq d_1 + \cdots + d_p$ and $F + 2 \equiv 1 \mod d_{p-1}$.

THEOREM 55. *Let F be a positive integer and let $(d_1,\ldots,d_p)$ be a saturated sequence. Then there exists a $(d_1,\ldots,d_p)$-semigroup with Frobenius number F if and only if it fulfills one of the conditions:*

1) $F+1 \geq d_1+\cdots+d_p$, $\gcd\{F+1,d_{p-1}\}=1$ *and* $F+1 \not\equiv 1 \bmod d_{p-1}$;
2) $F+2 \geq d_1+\cdots+d_p$ *and* $F+2 \equiv 1 \bmod d_{p-1}$.

PROOF. The necessary condition is a consequence of the comment preceding the theorem.

Let us see the sufficient condition. Assume that 1) is verified. Let $t_1=\cdots=t_{p-1}=1$ and $t_p=(F+1)-(d_1+\cdots+d_{p-1})>0$. Since $\gcd\left\{t_p,\frac{d_{p-1}}{d_p}\right\}=\gcd\{t_p,d_{p-1}\}=\gcd\{F+1,d_{p-1}\}=1$, we have that $\gcd\left\{t_{i+1},\frac{d_i}{d_{i+1}}\right\}=1$ for all $i\in\{1,\ldots,p-1\}$. By applying Theorem 51, we deduce that $\{d_1,d_1+d_2,\ldots,d_1+\cdots+d_{p-1},d_1+\cdots+d_{p-1}+\big((F+1)-(d_1+\cdots+d_{p-1})\big)d_p\}$ is a minimal SAT-system of generators of a saturated numerical semigroup S. As $F+1\equiv t_p \bmod d_{p-1}$, we have $t_p\not\equiv 1 \bmod d_{p-1}$. From Corollary 53, we obtain that $\mathrm{F}(S)=F$.

Assume now that 2) is true. Take $t_1=\cdots=t_{p-1}=1$ and $t_p=(F+2)-(d_1+\cdots+d_{p-1})>0$. As $F+2\equiv 1 \bmod d_{p-1}$ this implies that $t_p\equiv 1 \bmod d_{p-1}$, and consequently $\gcd\{F+2,d_{p-1}\}=1$. Then $\gcd\left\{t_p,\frac{d_{p-1}}{d_p}\right\}=1$, and thus $\gcd\left\{t_{i+1},\frac{d_i}{d_{i+1}}\right\}=1$ for all $i\in\{1,\ldots,p-1\}$. By applying again Corollary 53, we get $\mathrm{F}(S)=F$. □

REMARK 56. Observe that the conditions 1) and 2) of previous theorem can not happen simultaneously. In fact, if $F+2\equiv 1 \bmod d_{p-1}$ then $F+1\equiv 0 \bmod d_{p-1}$, whence $\gcd\{F+1,d_{p-1}\}=d_{p-1}\neq 1$.

The previous theorem gives a criterium to check if for a saturated sequence $(d_1,\ldots,d_p)$ there exists a $(d_1,\ldots,d_p)$-semigroup with Frobenius number F. We illustrate it with some examples.

EXAMPLE 57. 1) Does not exist $(12,6,1)$-semigroups with Frobenius number 25, because $\gcd\{25+1,6\} \neq 1$ and $25+2 \not\equiv 1 \bmod 6$. Consequently, the conditions 1) and 2) of Theorem 55 are not verified.

2) By applying 2) of Theorem 55, we deduce that there exists at least one $(4,2,1)$-semigroup with Frobenius number 9.

3) From 1) of Theorem 55, we have that there exists at least one $(6,3,1)$-semigroup with Frobenius number 13.

3.3. An algorithm for computing all $(d_1,\ldots,d_p)$-semigroups with a given Frobenius number. Assume from now on that $(d_1,\ldots,d_p)$ is a saturated sequence and F denotes a positive integer. Now we give an algorithmic procedure that allows to calculate all $(d_1,\ldots,d_p)$-semigroups with Frobenius number F.

PROPOSITION 58. *Let S be a $(d_1,\ldots,d_p)$-semigroup with Frobenius number F and let $\left\{d_1, t_1d_1+t_2d_2,\ldots,t_1d_1+\cdots+t_pd_p\right\}$ be its minimal SAT-system of generators fulfilling the conditions of Theorem 51. Then:*

1) $t_1d_1+\cdots+t_pd_p \in \{F+1,F+2\}$;
2) $t_1d_1+\cdots+t_pd_p = F+2$ *if only if* $F+2 \equiv 1 \bmod d_{p-1}$.

PROOF. 1) It is a consequence of Corollary 53.

2) (Necessity) If $F+2 = t_1d_1+\cdots+t_pd_p$, then by Corollary 53, we have that $t_p \equiv 1 \bmod d_{p-1}$ and consequently $F+2 \equiv 1 \bmod d_{p-1}$.

(Sufficiency) From 1) we know that $t_1d_1+\cdots+t_pd_p \in \{F+1,F+2\}$. If $t_1d_1+\cdots+t_pd_p = F+1$, then since $F+2 \equiv 1 \bmod d_{p-1}$, we have that $F+1 \equiv 0 \bmod d_{p-1}$ and thus $t_pd_p \equiv 0 \bmod d_{p-1}$. As $d_p = 1$, we obtain that $t_p \equiv 0 \bmod d_{p-1}$ and so we deduce that $\gcd\left\{t_p, \frac{d_{p-1}}{d_p}\right\} = d_{p-1} \neq 1$, which is impossible. Therefore $F+2 = t_1d_1+\cdots+t_pd_p$.

$\square$

If F does not verify neither Condition 1) nor Condition 2) of Theorem 55, we can state that there is no $(d_1,\ldots,d_p)$-semigroup with Frobenius number F.

If F verifies Condition 2) of Theorem 55, then by applying Theorem 51 and Proposition 58, we have that in order to get all $(d_1,\dots,d_p)$-semigroups with Frobenius number F, it suffices to calculate the positive integer solutions $(t_1,\dots,t_p)$ of the equation $d_1x_1+\cdots+d_px_p = F+2$ such that $\gcd\left\{t_{i+1},\frac{d_i}{d_{i+1}}\right\} = 1$ for all $i\in\{1,\dots,p-1\}$.

Analogously, if F verifies Condition 1) of Theorem 55, then from Theorem 51 and Proposition 58, we deduce that in order to obtain all $(d_1,\dots,d_p)$-semigroups with Frobenius number F, it suffices to calculate the positive integer solutions $(t_1,\dots,t_p)$ of the equation $d_1x_1+\cdots+d_px_p = F+1$ such that $\gcd\left\{t_{i+1},\frac{d_i}{d_{i+1}}\right\} = 1$ for all $i\in\{1,\dots,p-1\}$.

Observe that, if b is a positive integer greater than or equal to $d_1+\cdots+d_p$, then to calculate the positive integer solutions of the equation $d_1x_1+\cdots+d_px_p = b$, this is equivalent to calculate the nonnegative integer solutions to the equation $d_1y_1+\cdots+d_py_p = b-(d_1+\cdots+d_p)$. This is because $(y_1,\dots,y_p)$ is solution to the second equation if $(y_1+1,\dots,y_p+1)$ is solution to the first equation.

Our next goal is to give an algorithmic procedure that determines the nonnegative integer solutions of the equation

$$d_1x_1+\cdots+d_px_p = c \qquad (1)$$

with c a nonnegative integer.

Observe that the set of solutions of (1) corresponds with the set of integer partitions of c in which the parts belong to $\{d_1,\dots,d_p\}$. We use an argument similar to the ones used in [**54**] and [**20**] to find all restricted partitions. Note that (1) has a finite number of solutions and that $(0,\dots,0,c)$ is the smallest solution of (1) with respect to the lexicographic order. Therefore, if given a solution of (1), we are able to obtain the next solution of (1) with respect to the lexicographic order, then after a finite number of steps we obtain the set of solutions of (1).

The next result is the key to the above question. If $(x_1,\dots,x_p)$ is a solution of (1), we denote by $\mathrm{Next}(x_1,\dots,x_p)$ the next solution of (1) with respect to the lexicographic order.

PROPOSITION 59. *Let* $(x_1,\dots,x_p)$ *be a solution of (1).*

1) *If* $(x_1,\dots,x_p)$ *is not a maximal solution of (1) with respect to the lexicographic order, then there exists* $i\in\{1,\dots,p-1\}$ *such that* $d_{i+1}x_{i+1}+\cdots+d_px_p\geq d_i$*;*
2) *If* $j=\max\{i\in\{1,\dots,p-1\}\mid d_{i+1}x_{i+1}+\cdots+d_px_p\geq d_i\}$, *then* $\mathrm{Next}(x_1,\dots,x_p)=(x_1,\dots,x_{j-1},x_j+1,0,\dots,0,d_{j+1}x_{j+1}+\cdots+d_px_p-d_j)$.

PROOF. 1) Let $(x'_1,\dots,x'_p)$ be a solution of (1) such that $(x_1,\dots,x_p)<_{lex}(x'_1,\dots,x'_p)$. Then there exists $i\in\{1,\dots,p\}$ such that $x'_1=x_1,\dots,x'_{i-1}=x_{i-1}$ and $x'_i>x_i$. Let us see that $i\neq p$. If $x'_1=x_1,\dots,x'_{p-1}=x_{p-1}$, since $(x_1,\dots,x_p)$ and $(x'_1,\dots,x'_p)$ are solutions of (1), then we deduce that $x'_p=x_p$. Hence $(x_1,\dots,x_p)=(x'_1,\dots,x'_p)$ which is absurd. As $d_1x_1+\cdots+d_px_p=d_1x'_1+\cdots+d_px'_p$, we have that $d_ix_i+\cdots+d_px_p=d_ix'_i+\cdots+d_px'_p$ and so $d_{i+1}x_{i+1}+\cdots+d_px_p-d_i=d_i(x'_i-x_i-1)+\cdots+d_px_p\geq 0$.

2) Let $(\bar{x}_1,\dots,\bar{x}_p)=(x_1,\dots,x_{j-1},x_j+1,0,\dots,0,d_{j+1}x_{j+1}+\cdots+d_px_p-d_j)$. Clearly $(\bar{x}_1,\dots,\bar{x}_p)$ is a solution of (1) and $(x_1,\dots,x_p)<_{lex}(\bar{x}_1,\dots,\bar{x}_p)$. In order to conclude the proof, it suffices to prove that if $(x'_1,\dots,x'_p)$ is a solution of (1) such that $(x_1,\dots,x_p)<_{lex}(x'_1,\dots,x'_p)\leq_{lex}(\bar{x}_1,\dots,\bar{x}_p)$, then $(x'_1,\dots,x'_p)=(\bar{x}_1,\dots,\bar{x}_p)$. In fact, from the previous inequality, we obtain that $x'_1=x_1,\dots,x'_{j-1}=x_{j-1}$. Next we will see that $x'_j>x_j$. Otherwise there exists $h\in\mathbb{N}$ such that $x'_j=x_j,\dots,x'_{j+h}=x_{j+h}$ and $x'_{j+h+1}>x_{j+h+1}$. Then $d_{j+h+1}x_{j+h+1}+\cdots+d_px_p=d_{j+h+1}x'_{j+h+1}+\cdots+d_px'_p$ and thus $d_{j+h+2}x_{j+h+2}+\cdots+d_px_p-d_{j+h+1}\geq 0$, contradicting the maximality of j. Now, by applying that $(x'_1,\dots,x'_p)\leq_{lex}(\bar{x}_1,\dots,\bar{x}_p)$ we deduce that $x'_j=\bar{x}_j=x_j+1$ and $x'_{j+1}=\cdots=x'_{p-1}=0$. In this way $x'_1=\bar{x}_1,\dots,x'_{p-1}=\bar{x}_{p-1}$, by applying $(x'_1,\dots,x'_p)$ and $(\bar{x}_1,\dots,\bar{x}_p)$ are solutions of (1), we obtain that $x'_p=\bar{x}_p$. Therefore $(x'_1,\dots,x'_p)=(\bar{x}_1,\dots,\bar{x}_p)$.

$\square$

Now we can give the announced algorithm for computing all nonnegative integers solutions of (1).

ALGORITHM 60. Input: $(d_1,\ldots,d_p)$-saturated sequence and c nonnegative integer.

Output: The set of nonnegative integers solutions of the equation $d_1x_1+\cdots+d_px_p=c$.

1) $A=\{(0,\ldots,0,c)\}$.
2) $(x_1,\ldots,x_p)=(0,\ldots,0,c)$.
3) while there exists $j=\max\left\{i\in\{1,\ldots,p-1\}\mid d_{i+1}x_{i+1}+\cdots+d_px_p\geq d_i\right\}$ do $(x_1,\ldots,x_p)=(x_1,\ldots,x_{j-1},x_j+1,0,\ldots,0,d_{j+1}x_{j+1}+\cdots+d_px_p-d_j)$ and $A=A\cup\left\{(x_1,\ldots,x_p)\right\}$.
4) Return A.

We illustrate the preceding algorithms with an example.

EXAMPLE 61. Let $(d_1,d_2,d_3)=(6,2,1)$ and $c=10$. We compute all nonnegative integer solutions of the equation $6x_1+2x_2+x_3=10$. We begin with $A=\{(0,0,10)\}$ and $(x_1,x_2,x_3)=(0,0,10)$. Performing the step 3) of the above algorithm we get:

. $(x_1,x_2,x_3)=(0,1,8)$, $A=A\cup\{(0,1,8)\}$;
. $(x_1,x_2,x_3)=(0,2,6)$, $A=A\cup\{(0,2,6)\}$;
. $(x_1,x_2,x_3)=(0,3,4)$, $A=A\cup\{(0,3,4)\}$;
. $(x_1,x_2,x_3)=(0,4,2)$, $A=A\cup\{(0,4,2)\}$;
. $(x_1,x_2,x_3)=(0,5,0)$, $A=A\cup\{(0,5,0)\}$;
. $(x_1,x_2,x_3)=(1,0,4)$, $A=A\cup\{(1,0,4)\}$;
. $(x_1,x_2,x_3)=(1,1,2)$, $A=A\cup\{(1,1,2)\}$;
. $(x_1,x_2,x_3)=(1,2,0)$, $A=A\cup\{(1,2,0)\}$.

Finally,

$$A=\{(0,0,10),(0,1,8),(0,2,6),(0,3,4),(0,4,2),(0,5,0),(1,0,4),\\ (1,1,2),(1,2,0)\}.$$

Now we can give the algorithm announced at the beginning of this section.

ALGORITHM 62. Input: $(d_1,\dots,d_p)$ a saturated sequence and F a positive integer.
Output:

$$\mathcal{L}=\Big\{(t_1,\dots,t_p)\in\big(\mathbb{N}\backslash\{0\}\big)^p \mid \big\{d_1,t_1d_1+t_2d_2,\dots,t_1d_1+\cdots+t_pd_p\big\} \text{ is the minimal SAT-system of generators of a } (d_1,\dots,d_p)- \text{ semigroup with Frobenius number } F\Big\}.$$

1) If F does not verify neither Condition 1) nor Condition 2) of Theorem 55, then return $\mathcal{L}=\emptyset$ and the algorithm stops.
2) If F verifies the Condition 1) of Theorem 55, then $c=F+1-(d_1+\cdots+d_p)$ and go to 4).
3) $c=F+2-(d_1+\cdots+d_p)$.
4) Calculate by applying Algorithm 60 the set A of all nonnegative integer solutions of the equation $d_1x_1+\cdots+d_px_p=c$.
5) $B=A+(1,\dots,1)$.
6) $\mathcal{L}=\Big\{(t_1,\dots,t_p)\in B \mid \gcd\Big\{\frac{d_i}{d_{i+1}},t_{i+1}\Big\}=1 \text{ for all } i\in\{1,\dots,p-1\}\Big\}$.
7) Return $\mathcal{L}$.

We illustrate the preceding algorithm with an example.

EXAMPLE 63. Let us compute all $(6,2,1)$-semigroups with Frobenius number 17. As it checks the condition 2) of Theorem 55, then $c=17+2-(6+2+1)=10$. From Example 61, we compute the set $A=\{(0,0,10),(0,1,8),(0,2,6),(0,3,4),(0,4,2),(0,5,0),(1,0,4),(1,1,2),(1,2,0)\}$ of all nonnegative integer solutions of the equation $6x_1+2x_2+x_3=10$.

Hence the set $B=A+(1,1,1)=$
$\{(1,1,11),(1,2,9),(1,3,7),(1,4,5),(1,5,3),(1,6,1),(2,1,5),(2,2,3),(2,3,1)\}$.

Finally,
$\mathcal{L}=\{(1,1,11),(1,2,9),(1,4,5),(1,5,3),(2,1,5),(2,2,3)\}$.

And thus, the $(6,2,1)$-semigroups with Frobenius number 17 are:

. $\mathrm{Sat}\big(\{6,8,19\}\big)=\{0,6,8,10,12,14,16,18,19,\rightarrow\}$;

. $\mathrm{Sat}\big(\{6,10,19\}\big)=\{0,6,10,12,14,16,18,19\rightarrow\}$;

. $\mathrm{Sat}\big(\{6,14,19\}\big)=\{0,6,12,14,16,18,19,\rightarrow\}$;

. $\mathrm{Sat}\big(\{6,16,19\}\big)=\{0,6,12,16,18,19,\rightarrow\}$;

. $\mathrm{Sat}\big(\{6,14,19\}\big)$ (already appears);

. $\mathrm{Sat}\big(\{6,16,19\}\big)$ (already appears).

REMARK 64. The above example highlights that two distinct elements in $\mathcal{L}$ can produce us the same saturated numerical semigroup. Therefore the representation described in Theorem 51 is not unique.

3.4. An algorithm for computing all saturated numerical semigroups with a given Frobenius number. Recall that an F-saturated sequence is a saturated sequence $(d_1,\ldots,d_{k-1},d_k)$ such that there exist at least one $(d_1,\ldots,d_{k-1},d_k)$-semigroup with Frobenius number F.

Our first aim is to give an algorithmic procedure that allows to calculate all F-saturated sequences with a given positive integer F.

It is clear that the unique saturated sequence of length 1 is (1), $\mathbb{N}$ is the unique (1)-semigroup and $F(\mathbb{N})=-1$. Hence, if F is a positive integer any F-saturated sequence has a length greater than or equal to 2. We say that an F-saturated sequence $(d_1,\ldots,d_{k-1},d_k)$ is of type 1 (respectively type 2) if $\gcd\{F+1,d_{k-1}\}=1$ and $F\not\equiv 0 \bmod d_{k-1}$ (respectively $F+1\equiv 0 \bmod d_{k-1}$). Note that being of type 1 or type 2 is equivalent to fulfill Conditions 1) or 2) of Theorem 55.

The following two results are immediate consequences of Theorem 55.

LEMMA 65. *Let F be a positive integer.*

1) *The set of F-saturated sequences with length 2 and type 1 is equal to* $\{(x,1)\mid x\in\mathbb{Z},\ F+1>x\geq 2,\ \gcd\{F+1,x\}=1$ *and* $F\not\equiv 0 \bmod x\}$.
2) *The set of F-saturated sequences with length 2 and type 2 is equal to* $\{(x,1)\mid x\in\mathbb{Z},\ x\geq 2$ *and* $F+1\equiv 0 \bmod x\}$.

LEMMA 66. *If $k \geq 3$ and $(d_1, \ldots, d_{k-1}, d_k)$ is an F-saturated sequence, then $(d_2, \ldots, d_{k-1}, d_k)$ is also an F-saturated sequence.*

From the previous result we deduce that any F-saturated sequence with length k greater than or equal to 3 can be obtained from an F-saturated sequence with length $k-1$ by joining a first coordinate. As a consequence of Theorem 55 and Lemma 66, we obtain the following result.

LEMMA 67. *Let F and x be positive integers with x greater than or equal to 2.*

1) *Suppose that $(d_1, \ldots, d_{k-1}, d_k)$ is an F-saturated sequence with length k and type 1 and $F+1 \geq xd_1 + d_1 + \cdots + d_{k-1} + d_k$. Then $(xd_1, d_1, \ldots, d_{k-1}, d_k)$ is an F-saturated sequence with length $k+1$ and type 1. Furthermore, all F-saturated sequence with length $k+1$ and type 1 can be obtained of this form.*
2) *Suppose that $(d_1, \ldots, d_{k-1}, d_k)$ is an F-saturated sequence with length k and type 2 and $F+2 \geq xd_1 + d_1 + \cdots + d_{k-1} + d_k$. Then $(xd_1, d_1 \ldots, d_{k-1}, d_k)$ is an F-saturated sequence with length $k+1$ and type 2. Furthermore, all F-saturated sequence with length $k+1$ and type 2 can be obtained of this form.*

The next goal is to give algorithms that allows to obtain all F-saturated sequences with type 1 or 2. As a consequence of Corollary 54, we have that an F-saturated sequence has length less than or equal to $\log_2(F+3)$.

Given a real number q, we denote by $\lfloor q \rfloor$ the integer $\max\{z \in \mathbb{Z} \mid z \leq q\}$ and thus $\lfloor q \rfloor$ is the integer part of q.

ALGORITHM 68. Input: F a positive integer.

Output: $A_2, \ldots, A_{\lfloor \log_2(F+3) \rfloor}$, A_i denotes the set of all F-saturated sequences with length i and type 1.

1) $A_2 = \{(x,1) \mid x \in \mathbb{Z},\ F+1 > x \geq 2,\ \gcd\{F+1, x\} = 1$ and
$F \not\equiv 0 \bmod x\}$.

2) For $i = 3$ to $\lfloor \log_2(F+3) \rfloor$ do

$A_i = \{(xd_1, d_1, \dots, d_{i-1}) \mid (d_1, \dots, d_{i-1}) \in A_{i-1},\ x \geq 2,$ and

$F + 1 \geq xd_1 + d_1 + \cdots + d_{i-1}\}.$

3) Return $A_2, A_3, \dots, A_{\lfloor \log_2(F+3) \rfloor}$.

The previous algorithm computes the F-saturated sequences with type 1 and the following gives us the F-saturated sequences with type 2.

ALGORITHM 69. Input: F a positive integer.

Output: $B_2, \dots, B_{\lfloor \log_2(F+3) \rfloor}$, B_i denotes the set of all F-saturated sequences with length i and type 2.

1) $B_2 = \{(x, 1) \mid x \in \mathbb{Z},\ x \geq 2,\ F + 1 \equiv 0 \bmod x\}$.

2) For $i = 3$ to $\lfloor \log_2(F+3) \rfloor$ do

$B_i = \{(xd_1, d_1, \dots, d_{i-1}) \mid (d_1, \dots, d_{i-1}) \in B_{i-1},\ x \geq 2,$ and

$F + 2 \geq xd_1 + d_1 + \cdots + d_{i-1}\}.$

3) Return $B_2, B_3, \dots, B_{\lfloor \log_2(F+3) \rfloor}$.

Next we illustrate the preceding algorithms with an example.

EXAMPLE 70. Let us compute all 17-saturated sequences.

1) First, and using Algorithm 68, we compute all sequences with type 1.

. $A_2 = \{(5,1), (7,1), (11,1), (13,1)\}$;

. $A_3 = \{(10,5,1)\}$;

. $A_4 = \emptyset$.

2) By applying Algorithm 69, we obtain all sequences with type 2.

. $B_2 = \{(18,1), (9,1), (6,1), (3,1), (2,1)\}$;

. $B_3 = \{(12,6,1), (6,3,1), (9,3,1), (12,3,1), (15,3,1), (4,2,1),$
$(6,2,1), (8,2,1), (10,2,1), (12,2,1), (14,2,1), (16,2,1)\}$.

. $B_4 = \{(8,4,2,1), (12,4,2,1)\}$.

We finish this section by introducing an algorithm that allows us to compute all saturated numerical semigroups with a given Frobenius number.

ALGORITHM 71. Input: F a positive integer.

Output: The set of all saturated numerical semigroups with Frobenius number F.

1) Calculate by applying Algorithm 68 $A_2, A_3, \ldots, A_{\lfloor \log_2(F+3) \rfloor}$.
2) Calculate by applying Algorithm 69 $B_2, B_3, \ldots, B_{\lfloor \log_2(F+3) \rfloor}$.
3) $L = A_2 \cup \cdots \cup A_{\lfloor \log_2(F+3) \rfloor} \cup B_2 \cup \cdots \cup B_{\lfloor \log_2(F+3) \rfloor}$.
4) For each $l \in L$, let $\mathcal{L}_l$ be the output of Algorithm 62 and let
   $C_l = \{\mathrm{Sat}(\{d_1, t_1 d_1 + t_2 d_2, \ldots, t_1 d_1 + \cdots + t_k d_k\}) \mid l = (d_1, \ldots, d_k)$
   and $(t_1, \ldots, t_k) \in \mathcal{L}_l\}$.
5) Return $\cup_{l \in L} C_l$.

Therefore for each positive integer F the previous algorithm computes all F-saturated sequences with type 1 and 2 and for each F-saturated sequence computes all saturated numerical semigroups with Frobenius number F associated with it.

The algorithm has been implemented in `GAP`, and it is available since version 0.98 of the `numericalsgps` GAP [**17**] package [**12**]. Next we give some timings.

For Frobenius number 30,

```
gap> Length(SaturatedNumericalSemigroupsWithFrobeniusNumber(30));
39
```

takes 0 milliseconds, while computing the set of all numerical semigroups with Frobenius number and then filtering those that are saturated takes 30454 milliseconds.

```
gap> Length(Filtered(NumericalSemigroupsWithFrobeniusNumber(30),
IsSaturatedNumericalSemigroup));
39
```

As for 33 we get 31 milliseconds for

```
gap> Length(SaturatedNumericalSemigroupsWithFrobeniusNumber(33));
166
```

while it takes 284172 for

```
gap> Length(Filtered(NumericalSemigroupsWithFrobeniusNumber(33),
```

```
IsSaturatedNumericalSemigroup));time;
```

For 100 we get

```
gap> Length(SaturatedNumericalSemigroupsWithFrobeniusNumber(100));time;
1605
```

with computational time equal to 875 milliseconds.

In the following table there are the results obtained for Frobenius number up to 100. For each positive integer F we wrote the number of saturated numerical semigroups (n_F) of the given Frobenius number (F).

F	n_F	F	n_F	F	n_F	F	n_F	F	n_F	F	n_F	F	n_F	F	n_F	F	n_F	F	n_F
1	1	11	16	21	52	31	175	41	378	51	628	61	1267	71	2228	81	2775	91	5039
2	1	12	7	22	40	32	68	42	88	52	266	62	490	72	197	82	1200	92	1336
3	2	13	21	23	84	33	166	43	439	53	828	63	1208	73	2291	83	3765	93	4574
4	2	14	14	24	20	34	105	44	155	54	170	64	443	74	816	84	282	94	1878
5	4	15	25	25	92	35	240	45	389	55	909	65	1522	75	2124	85	3789	95	5973
6	3	16	18	26	53	36	49	46	233	56	284	66	303	76	779	86	1347	96	463
7	7	17	39	27	103	37	280	47	597	57	865	67	1785	77	2783	87	3752	97	6307
8	5	18	16	28	54	38	131	48	79	58	440	68	528	78	491	88	1196	98	1944
9	9	19	50	29	144	39	285	49	624	59	1210	69	1612	79	3157	89	4681	99	5894
10	8	20	22	30	39	40	113	50	239	60	95	70	662	80	728	90	506	100	1605

CHAPTER 3

Frobenius Problem

The Frobenius problem for numerical semigroups consists in finding formulas, in terms of the elements in a minimal system of generators of a numerical semigroup S, for computing $F(S)$ and $g(S)$. As we mentioned in introduction, this problem remains open for numerical semigroups with embedding dimension greater or equal to three.

This chapter is dedicated to the study of three classes of numerical semigroups, denominated by Mersenne, Thabit and Repunit numerical semigroups. This study is done in sections 1, 2 and 3 and were published in [**34**], [**36**] and [**35**], respectively. In the three cases we give formulas for important invariants of the numerical semigroups, such as: embedding dimension, Frobenius number, type and genus.

1. The Frobenius problem for Mersenne numerical semigroups

A positive integer x is a **Mersenne number** if $x = 2^n - 1$ for some $n \in \mathbb{N}\backslash\{0\}$. We say that a numerical semigroup S is a **Mersenne numerical semigroup** if there exist $n \in \mathbb{N}\backslash\{0\}$ such that $S = \left\langle \left\{ 2^{n+i} - 1 \mid i \in \mathbb{N} \right\} \right\rangle$. The main purpose of this section is to study this class of numerical semigroups and will be denoted by $S(n) = \left\langle \left\{ 2^{n+i} - 1 \mid i \in \mathbb{N} \right\} \right\rangle$. The results presented in this section can be found in [**34**].

1.1. The embedding dimension. Let n be a positive integer. We begin this section by proving that $S(n)$ is a numerical semigroup in which is verified that $2s+1 \in S(n)$ for all $s \in S(n)\backslash\{0\}$.

LEMMA 72. *Let A be a nonempty set of positive integers such that $M = \langle A \rangle$. The following conditions are equivalent:*

(1) $2a+1\in M$ *for all* $a\in A$*;*

(2) $2m+1\in M$ *for all* $m\in M\backslash\{0\}$.

PROOF. 1) *implies* 2). If $m\in M\backslash\{0\}$, then there exist $a_1,\dots,a_k\in A$ such that $m=a_1+\cdots+a_k$. If $k=1$ then $m=a_1$ and therefore $2m+1=2a_1+1\in M$. If $k\geq 2$ then $2m+1=2(a_1+\cdots+a_{k-1})+2a_k+1\in M$, because M is closed under addition.

2) *implies* 1). Trivial. $\square$

PROPOSITION 73. *If* n *is a positive integer, then* $S(n)$ *is a numerical semigroup. Furthermore,* $2s+1\in S(n)$ *for all* $s\in S(n)\backslash\{0\}$.

PROOF. Clearly, $S(1)=\mathbb{N}$ is a numerical semigroup and for every $s\in\mathbb{N}\backslash\{0\}$ we have that $2s+1\in\mathbb{N}$. Suppose now that $n\geq 2$ and let us show that $S(n)$ is a numerical semigroup. Since $S(n)$ is a submonoid of $(\mathbb{N},+)$ it suffices to prove $\mathbb{N}\backslash S$ is finite, this is equivalent to $\gcd(S(n))=1$. Since 2^n-1 and $2^{n+1}-1\in S(n)$, we obtain that $\gcd\left\{2^n-1,2^{n+1}-1\right\}=\gcd\{2^n-1,2\left(2^n-1\right)+1\}=1$.

We have that $S(n)=\langle\left\{2^{n+i}-1\mid i\in\mathbb{N}\right\}\rangle$. If $i\in\mathbb{N}$ then $2(2^{n+i}-1)+1=2^{n+i+1}-1\in S(n)$. By Lemma 72, we conclude that $2s+1\in S(n)$ for all $s\in S(n)\backslash\{0\}$. $\square$

Our next goal is to give the minimal systems of generators of a Mersenne numerical semigroup.

LEMMA 74. *Let* n *be a positive integer and let* $S=\langle\left\{2^{n+i}-1\mid i\in\{0,1,\dots,n-1\}\right\}\rangle$. *Then* $2s+1\in S$ *for all* $s\in S\backslash\{0\}$.

PROOF. If $n=1$ then $S=\mathbb{N}$ and the result is true. Suppose now that $n\geq 2$. If $i\in\{0,1,\dots,n-2\}$, then $2(2^{n+i}-1)+1=2^{n+i+1}-1\in S$. Besides, $2(2^{2n-1}-1)+1=2^{2n}-1=(2^n-1)(2^n+1)\in S$. By applying Lemma 72, we obtain that $2s+1\in S$ for all $s\in S\backslash\{0\}$. $\square$

Before we state the next result, note that if X and Y are non empty sets of positive integer numbers such that $Y\subseteq X$ and $X\subseteq\langle Y\rangle$, then we get that $\langle X\rangle=\langle Y\rangle$

PROPOSITION 75. *If n is a positive integer, then* $S(n) = \langle \{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-1\}\} \rangle$.

PROOF. Let $S = \langle \{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-1\}\} \rangle$. In view of the preceding note, it suffices to prove that $2^{n+i}-1 \in S$ for all $i \in \mathbb{N}$. We use induction on i. For $i = 0$ the result is trivial. Assume that the statement is true for i and let us show it for $i+1$. As $2^{n+i+1}-1 = 2\left(2^{n+i}-1\right)+1$ then by induction hypothesis and Lemma 74 we have that $2^{n+i+1}-1 \in S$. $\square$

The above proposition tells us that $\{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-1\}\}$ is a system of generators of $S(n)$. The next result is fundamental to show that this set is the minimal system of generators of $S(n)$.

LEMMA 76. *Let n be an integer greater than or equal to two. Then* $2^{2n-1}-1 \notin \langle \{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-2\}\} \rangle$.

PROOF. Assume to the contrary that there exist $a_0,\ldots,a_{n-2} \in \mathbb{N}$ such that $2^{2n-1} - 1 = a_0(2^n-1)+\cdots+a_{n-2}(2^{2n-2}-1)$. Then $2^{2n-1}-1 = a_0 2^n+\cdots+a_{n-2}2^{2n-2} - (a_0+\cdots+a_{n-2})$ and consequently $(a_0+\cdots+a_{n-2}) \equiv 1 (\mathrm{mod}\ 2^n)$. Hence $(a_0+\cdots+a_{n-2}) = 1+2^n k$ for some $k \in \mathbb{N}$. If $k = 0$ then $a_0+\cdots+a_{n-2} = 1$ and thus there exist $i \in \{0,1,\ldots n-2\}$ such that $a_i = 1$ and $a_j = 0$ for all $j \in \{0,1,\ldots,n-2\} \setminus \{i\}$. So we deduce that $2^{2n-1}-1 = 2^{n+i}-1$ for some $i \in \{0,1,\ldots n-2\}$, which is absurd. If $k \neq 0$ then $a_0+\cdots+a_{n-2} \geq 1+2^n$. This implies that $a_0(2^n-1)+\cdots+a_{n-2}(2^{2n-2}-1) \geq (a_0+\cdots+a_{n-2})(2^n-1) \geq (1+2^n)(2^n-1) = 2^{2n}-1 > 2^{2n-1}-1$, which is absurd. $\square$

Now we are ready to show the result announced concerning the minimal system of generators of $S(n)$.

THEOREM 77. *Let n be a positive integer and let $S(n)$ be the Mersenne numerical semigroup associated to n, then $e(S(n)) = n$. Furthermore $\{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-1\}\}$ is the minimal system of generators of $S(n)$.*

PROOF. For $n=1$, the result follows trivially. Thus, suppose that $n \geq 2$. By using Proposition 75 we have that $\left\{2^{n+i}-1 \mid i \in \{0,1,\ldots,n-1\}\right\}$ is a system of generators of $S(n)$. If it is not minimal then there exists $h \in \{1,\ldots,n-1\}$ such that $2^{n+h}-1 \in \langle\left\{2^{n+i}-1 \mid i \in \{0,1,\ldots,h-1\}\right\}\rangle$. Let $S = \langle\left\{2^{n+i}-1 \mid i \in \{0,1,\ldots,h-1\}\right\}\rangle$. If $i \in \{0,1,\ldots,h-2\}$ then $2\left(2^{n+i}-1\right)+1 = 2^{n+i+1}-1 \in S$. Moreover $2\left(2^{n+h-1}-1\right)+1 = 2^{n+h}-1 \in S$. By applying Lemma 72 we obtain that $2s+1 \in S$ for all $s \in S\backslash\{0\}$. Now we use induction on i to prove that $2^{n+i}-1 \in S$ for all $i \in \mathbb{N}$. For $i=0$ the result is true. Assume that the result holds for i and and let us prove for $i+1$. As $2^{n+i+1}-1 = 2\left(2^{n+i}-1\right)+1$ by applying the induction hypothesis and $2s+1 \in S$ for all $s \in S\backslash\{0\}$, we obtain that $2^{n+i+1}-1 \in S$. Consequently $2^{2n-1}-1 \in S$, which contradicts Lemma 76. $\square$

Observe that as a consequence of the previous results we obtain that for every positive integer n there exists a unique Mersenne numerical semigroup $S(n)$ with embedding dimension n. In fact, $S(4) = \langle\left\{2^4-1, 2^5-1, 2^6-1, 2^7-1\right\}\rangle = \langle\{15,31,63,127\}\rangle$ is the unique Mersenne numerical semigroup with embedding dimension 4.

1.2. The Apéry set. Let S be a numerical semigroup and let x be one of its nonzero elements. As we have seen before, we define the Apéry set of x in S as $\mathrm{Ap}(S,x) = \{s \in S \mid s-x \notin S\}$.

Our next goal is to describe the set $\mathrm{Ap}(S(n), 2^n-1)$. From now on we will denote by s_i the elements $2^{n+i}-1$ for each $i \in \{0,1,\ldots,n-1\}$. So with this notation we have that $\{s_0, s_1, \ldots, s_{n-1}\}$ is the minimal system of generators of $S(n)$. It is easy to deduce the following equalities.

LEMMA 78. *Let n be an integer greater than or equal to two. Then:*

(1) if $0 < i \leq j < n-1$ then $s_i + 2s_j = 2s_{i-1} + s_{j+1}$;
(2) if $0 < i \leq n-1$ then $s_i + 2s_{n-1} = 2s_{i-1} + (2^n+1)s_0$.

We say that a sequence $(a_1,\dots,a_k)$ is a residual k-tuple if satisfies the following conditions:

(1) for every $i\in\{1,\dots,k\}$ we have that $a_i\in\{0,1,2\}$;
(2) if $i\in\{2,\dots,k\}$ and $a_i=2$ then $a_1=\cdots=a_{i-1}=0$.

LEMMA 79. *Let n be an integer greater than or equal to two. If $x\in \mathrm{Ap}(S(n),s_0)$ then there exist a residual $(n-1)$-tuple $(a_1,\dots,a_{n-1})$ such that $x=a_1s_1+\cdots+a_{n-1}s_{n-1}$.*

PROOF. We proceed by induction on x. The result is clear for $x=0$. Suppose that $x>0$ and let $j=\min\{i\in\{0,\dots,n-1\}\mid x-s_i\in S(n)\}$. Observe that $j\neq 0$ because $x\in \mathrm{Ap}(S(n),s_0)$. By induction hypothesis there exist a residual $(n-1)$-tuple $(a_1,\dots,a_{n-1})$ such that $x-s_j=a_1s_1+\cdots+a_{n-1}s_{n-1}$. Hence $x=a_1s_1+\cdots+(a_j+1)s_j+\cdots+a_{n-1}s_{n-1}$. To conclude the proof we only need to show that $\left(a_1,\dots,a_j+1,\dots,a_{n-1}\right)$ is a residual $(n-1)$-tuple. If $a_j+1=3$ then, by applying Lemma 78, we get that either $(a_j+1)s_j=3s_j=2s_{j-1}+s_{j+1}$ in the case $j<n-1$ or $(a_j+1)s_j=3s_j=2s_{j-1}+(2^n+1)s_0$ in the case $j=n-1$. In both cases this leads to $x-s_{j-1}\in S(n)$, contradicting the minimality of j. If there exist $k>j$ such that $a_k=2$ then, by using again Lemma 78, we obtain that either $s_j+2s_k=2s_{j-1}+s_{k+1}$ in the case $k<n-1$ or $s_j+2s_k=2s_{j-1}+(\ 2^n+1)s_0$ in the case $k=n-1$. In both cases we get once again that $x-s_{j-1}\in S(n)$ contradicting the minimality of j. Now by the minimality of j we have that $a_1=\cdots=a_{j-1}=0$ and consequently $\left(a_1,\dots,a_j+1,\dots a_{n-1}\right)$ is a residual $(n-1)$-tuple. $\square$

Note that if h is a positive integer, then the sequence of numbers $2^n,2^{n+1},\dots,2^{n+h}$ is a geometric progression with common ratio 2 and so the sum of its terms $2^n+2^{n+1}+\cdots+2^{n+h}$ is equal to $2^{n+h+1}-2^n$.

THEOREM 80. *Let n be an integer greater than or equal to two and let $S(n)$ be the Mersenne numerical semigroup minimally generated by $\{s_0,s_1,\dots,s_{n-1}\}$. Then* $\mathrm{Ap}(S(n),s_0)=\{a_1s_1+\cdots+a_{n-1}s_{n-1}\mid(a_1,\dots,a_{n-1})$ is a residual $(n-1)-$tuple$\}$.

PROOF. Let us denote by R the set of all residual $(n-1)$-tuples. It is clear that $R=\{0,1\}^{n-1}\cup\{(a_1,\ldots,a_{n-1})\in R \mid a_1=2\}\cup\cdots\cup\{(a_1,\ldots,a_{n-1})\in R \mid a_{n-1}=2\}$. Observe that R is the disjoint union of these sets, consequently the cardinality of R is equal to $2^{n-1}+2^{n-2}+\cdots+2^0=2^n-1=s_0$.

By Lemma 79, we know that $\mathrm{Ap}(S(n),s_0)\subseteq\{a_1s_1+\cdots+a_{n-1}s_{n-1} \mid (a_1,\ldots,a_{n-1})\in R\}$. Besides, from previous paragraph, we get that the cardinality of the set $\{a_1s_1+\cdots+a_{n-1}s_{n-1} \mid (a_1,\ldots,a_{n-1})\in R\}$ is less than or equal to s_0. In view of Lemma 5 the cardinality of $\mathrm{Ap}(S(n),s_0)$ is exactly s_0, and consequently $\mathrm{Ap}(S(n),s_0)=\{a_1s_1+\cdots+a_{n-1}s_{n-1} \mid (a_1,\ldots,a_{n-1})\in R\}$. $\square$

As an immediate consequence of the proof of previous theorem we have the following result.

COROLLARY 81. *Let n be an integer greater than or equal to two and let $(a_1,\ldots,a_{n-1})$ and $(b_1,\ldots,b_{n-1})$ be two distinct residual $(n-1)$-tuples. Then we have that $a_1s_1+\cdots+a_{n-1}s_{n-1}\neq b_1s_1+\cdots+b_{n-1}s_{n-1}$*

We illustrate the preceding theorem with an example.

EXAMPLE 82. Let us compute $\mathrm{Ap}(S(4),s_0)$. We have that $s_0=15$ and $S(4)=\langle\{15,31,63,127\}\rangle$. The residual 3-tuples are $(0,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,1,1)$, $(1,0,0)$, $(1,1,0)$, $(1,0,1)$, $(1,1,1)$, $(2,0,0)$, $(2,1,0)$, $(2,0,1)$, $(2,1,1)$, $(0,2,0)$, $(0,2,1)$ and $(0,0,2)$. Since $s_1=31$ $s_2=63$ and $s_3=127$, by Theorem 80, we obtain that $\mathrm{Ap}(S(4),s_0)=\{0,63,127,190,31,94,158,221,62,125,189,252,126,253,254\}$.

Next we give a procedure to see if a positive integer belongs or not to $S(4)$.

Recall that if S is a numerical semigroup, $x\in S\backslash\{0\}$ then $Ap(S,x)=\{w(0)=0,w(1),\cdots,w(x-1)\}$ where $w(i)$ is the smallest element of S that is congruent with i modulo x. Then using Lemma 5 we can conclude that an integer z belongs to S if and only if $z\geq w(z \bmod x)$ (where $z \bmod x$ denotes the remainder of the division of z by x).

As we see in previous example

$\mathrm{Ap}(S(4), s_0) = \{w(0) = 0, w(1) = 31, w(2) = 62, w(3) = 63, w(4) = 94,$
$w(5) = 125, w(6) = 126, w(7) = 127, w(8) = 158, w(9) = 189, w(10) = 190,$
$w(11) = 221, w(12) = 252, w(13) = 253, w(14) = 254\}.$

From this and Remark 7 it easily follows that $172 \in S(4)$ and $222 \notin S(4)$, because $172 \geq w(172 \bmod 15) = w(7) = 127$ and $222 < w(222 \bmod 15) = w(12) = 252$.

1.3. The Frobenius problem. The next aim is to give a formula for the greatest integer that does not belong to $S(n)$ (i.e. Frobenius number). It is easy to prove our next result.

LEMMA 83. *Let n be an integer greater than or equal to two and let R be the set of all residual $(n-1)$-tuples. Then the maximal elements (with respect to the product order) in R are* $(2,1,\ldots,1)$, $(0,2,1,\ldots,1)$ *and* $(0,0,\ldots,2)$.

In the following result we will see that $2s_1 + s_2 + \cdots + s_{n-1}, 2s_2 + s_3 + \cdots + s_{n-1}, \ldots, 2s_{n-1}$ is a sequence of integers wherein each term is obtained from the previous by adding a unit.

LEMMA 84. *Let n be an integer greater than or equal to three and let $i \in \{1,\ldots,n-2\}$. Then $2s_i + s_{i+1} + \cdots + s_{n-1} + 1 = 2s_{i+1} + s_{i+2} \cdots + s_{n-1}$.*

PROOF. This is equivalent to prove that $2s_i + 1 = s_{i+1}$. But this is clear, because $2(2^{n+i} - 1) + 1 = 2^{n+i+1} - 1$. □

Now we can give a formula for the Frobenius number of a Mersenne numerical semigroup.

THEOREM 85. *Let n be an integer greater than or equal to two and let $S(n)$ be the Mersenne numerical semigroup associated to n. Then* $\mathrm{F}(S(n)) = 2^{2n} - 2^n - 1$.

PROOF. By applying Theorem 80 and Lemmas 83 and 84, we deduce that $\max(\mathrm{Ap}(S(n), s_0)) = 2s_{n-1}$. Using now Proposition 10, we get that $\mathrm{F}(S(n)) = 2s_{n-1} - s_0 = 2\left(2^{2n-1} - 1\right) - (2^n - 1) = 2^{2n} - 2^n - 1$. □

Our next goal is to determine the set of all pseudo-Frobenius number and the type of $S(n)$.

THEOREM 86. *Let n be an integer greater than or equal to two and let $S(n)$ be the Mersenne numerical semigroup associated to n. Then* $\mathrm{t}(S(n)) = n-1$. *Furthermore*

$$\mathrm{PF}(S(n)) = \{\mathrm{F}(S(n)), \mathrm{F}(S(n))-1, \ldots, \mathrm{F}(S(n))-(n-2)\}.$$

PROOF. From Theorem 80 and Lemma 83, we deduce that $\max_{\leq_{S(n)}} \mathrm{Ap}(S(n), s_0) \subseteq \{2s_1+s_2+\cdots+s_{n-1}, 2s_2+s_3+\cdots+s_{n-1}, \ldots, 2s_{n-1}\}$. By using Lemma 84, we have that the elements in this set are consecutive positive integers and thus the difference between any two of its elements is smallest than or equal to $n-2$. Since 2^n-1 is the smaller positive integer in $S(n)$ and $2^n-1 > n-2$ then we conclude that the difference between two distinct elements of $\{2s_1+s_2+\cdots+s_{n-1}, 2s_2+s_3+\cdots+s_{n-1}, \ldots, 2s_{n-1}\}$ is not in $S(n)$. Hence $\max_{\leq_{S(n)}} \mathrm{Ap}(S(n), s_0) = \{2s_1+s_2+\cdots+s_{n-1}, 2s_2+s_3+\cdots+s_{n-1}, \ldots, 2s_{n-1}\}$. By the proof of Theorem 85, we have that $\mathrm{F}(S(n)) = 2s_{n-1} - s_0$. From Lemma 84, we get that $\mathrm{Maximales}_{\leq_{S(n)}} \mathrm{Ap}(S(n), s_0) = \{\mathrm{F}(S(n))+s_0, \mathrm{F}(S(n))+s_0-1, \ldots, \mathrm{F}(S(n))+s_0-(n-2)\}$. Finally, by Lemma 11, we obtain $\mathrm{PF}(S(n)) = \{\mathrm{F}(S(n)), \mathrm{F}(S(n))-1, \ldots, \mathrm{F}(S(n))-(n-2)\}$. $\square$

Observe that the previous theorem is not true for $n=1$, since $S(1) = \mathbb{N}$, $\mathrm{PF}(S(1)) = \{-1\}$ and consequently $\mathrm{t}(S(1)) = 1$.

The next result gives the formula for the genus of the Mersenne numerical semigroup $S(n)$.

THEOREM 87. *Let n be a positive integer and let $S(n)$ be the Mersenne numerical semigroup associated to n. Then* $\mathrm{g}(S(n)) = 2^{n-1}(2^n+n-3)$.

PROOF. For $n=1$ the result is trivial. Suppose that $n \geq 2$ and let R be the set of all residual $(n-1)$-tuples. By applying Proposition 10, Theorem 80 and Corollary 81 we have that

$$\mathrm{g}\left(S(n)\right)=\frac{1}{s_0}\left(\sum_{(a_1,\ldots,a_{n-1})\in R} a_1s_1+\cdots+a_{n-1}s_{n-1}\right)-\frac{s_0-1}{2}.$$

It is clear that

$$\sum_{(a_1,\ldots,a_{n-1})\in R} a_1s_1+\cdots+a_{n-1}s_{n-1}=\sum_{(a_1,\ldots,a_{n-1})\in R,\ a_1=1} s_1+$$
$$+\sum_{(a_1,\ldots,a_{n-1})\in R,\ a_1=2} 2s_1+\cdots+\sum_{(a_1,\ldots,a_{n-1})\in R,\ a_{n-1}=1} s_{n-1}+\sum_{(a_1,\ldots,a_{n-1})\in R,\ a_{n-1}=2} 2s_{n-1}.$$

Let $i\in\{1,\cdots,n-1\}$. The reader can prove the following:

- the cardinality of $\{(a_1,\ldots,a_{n-1})\in R \mid a_i=2\}$ is 2^{n-1-i};
- the cardinality of $\{(a_1,\ldots,a_{n-1})\in R \mid a_i=1 \text{ and } 2\notin\{a_1,\ldots,a_{i-1}\}\}$ is 2^{n-2};
- if $1\leq j<i$ then the cardinality of $\left\{(a_1,\ldots,a_{n-1})\in R \mid a_i=1 \text{ and } a_j=2\right\}$ is 2^{n-j-2}.

Whence we deduce that

$$\sum_{(a_1,\ldots,a_{n-1})\in R} a_1s_1+\cdots+a_{n-1}s_{n-1}=\sum_{i=1}^{n-1}\left(2^{n-2}+2^{n-3}+\cdots+2^{n-1-i}\right)s_i+$$
$$+\sum_{i=1}^{n-1}2^{n-1-i}2s_i=\sum_{i=1}^{n-1}\left(2^{n-1}-2^{n-1-i}\right)s_i+\sum_{i=1}^{n-1}2^{n-i}s_i=$$
$$=\sum_{i=1}^{n-1}\left(2^{n-1}-2^{n-1-i}+2^{n-i}\right)s_i=\sum_{i=1}^{n-1}\left(2^{n-1}+2^{n-1-i}\right)s_i=$$
$$=\sum_{i=1}^{n-1}\left(2^{n-1}+2^{n-1-i}\right)\left(2^{n+i}-1\right)=\sum_{i=1}^{n-1}\left(2^{2n+i-1}+2^{2n-1}-2^{n-1}-2^{n-1-i}\right)=$$
$$=2^{3n-1}-2^{2n}+(n-1)\,2^{2n-1}-(n-1)\,2^{n-1}-\left(2^{n-1}-1\right)=$$
$$=2^{3n-1}-2^{2n}+(n-1)2^{2n-1}-n2^{n-1}+1=$$
$$=\left(2^n-1\right)\left(2^{2n-1}-2^n+n2^{n-1}-1\right).$$

Consequently, we get that

$$\mathrm{g}(S(n)) = 2^{2n-1} - 2^n + n2^{n-1} - 1 - \frac{2^n - 2}{2} = 2^{2n-1} - 2^n + (n-1)\,2^{n-1} =$$
$$= 2^{n-1}\,(2^n + n - 3)\,.$$

□

We conclude the study of the Mersenne numerical semigroups by giving an example that illustrates the previous results.

EXAMPLE 88. Let us compute the Frobenius number, the type and genus of the Mersenne numerical semigroup $S(4)$. From Theorem 85 we obtain that $\mathrm{F}(S(4)) = 2^8 - 2^4 - 1 = 239$. By using Theorem 86 we get that $\mathrm{t}(S(4)) = 3$ and $\mathrm{PF}(S(4)) = \{239, 238, 237\}$. Finally, by applying now Theorem 87 we have that $\mathrm{g}(S(4)) = 2^3\left(2^4 + 4 - 3\right) = 136$.

2. The Frobenius problem for Thabit numerical semigroups

A positive integer x is a **Thabit number** if $x = 3.2^n - 1$ for some $n \in \mathbb{N}$. We say that a numerical semigroup S is a **Thabit numerical semigroup** if there exist $n \in \mathbb{N}$ such that $S = \left\langle \left\{3.2^{n+i} - 1 \mid i \in \mathbb{N}\right\}\right\rangle$. The main purpose of this section is to study this class of numerical semigroups. We will denote by $T(n)$ the numerical semigroup $\left\langle \left\{3.2^{n+i} - 1 \mid i \in \mathbb{N}\right\}\right\rangle$. The results presented in this section can be found in [**36**].

2.1. The embedding dimension. If n is a nonnegative integer, then $T(n)$ is a submonoid of $(\mathbb{N}, +)$. Moreover $\left\{3.2^n - 1, 3.2^{n+1} - 1\right\} \subseteq T(n)$ and $\gcd\left\{3.2^n - 1, 3.2^{n+1} - 1\right\} = \gcd\left\{3.2^n - 1, 2\,(3.2^n - 1) + 1\right\} = 1$. Hence $\gcd(T(n)) = 1$ and so $T(n)$ is a numerical semigroup.

The next result is fundamental to the development of this work.

PROPOSITION 89. *If n is a nonnegative integer, then $2t + 1 \in T(n)$ for all $t \in T(n)\backslash\{0\}$.*

PROOF. Let $n \in \mathbb{N}$ and let $T(n) = \langle \{3.2^{n+i} - 1 \mid i \in \mathbb{N}\} \rangle$. Clearly $2(3.2^{n+i} - 1) + 1 = 3.2^{n+i+1} - 1 \in T(n)$. From Lemma 72, we obtain that $2t + 1 \in T(n)$ for all $t \in T(n) \backslash \{0\}$. $\square$

Our aim is to prove Theorem 93, which ensures $\{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n+1\}\}$ is the minimal system of generators of $T(n)$. To this purpose, we need some preliminary results.

LEMMA 90. *Let n be a nonnegative integer and let $S = \langle \{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n+1\}\} \rangle$. Then $2s + 1 \in S$ for all $s \in S \backslash \{0\}$.*

PROOF. If $i \in \{0, 1, \ldots, n\}$, then $2(3.2^{n+i} - 1) + 1 = 3.2^{n+i+1} - 1 \in S$. Furthermore, $2\left(3.2^{2n+1} - 1\right) + 1 = 3.2^{2n+2} - 1 = (3.2^n - 1)^2 + \left(3.2^{n+1} - 1\right) + \left(3.2^{2n} - 1\right) \in S$. By using now the Lemma 72, we obtain the desired result. $\square$

The next result gives a system of generators of $T(n)$.

LEMMA 91. *If n is a nonnegative integer, then $T(n) = \langle \{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n+1\}\} \rangle$.*

PROOF. Let $S = \langle \{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n+1\}\} \rangle$. It is clear that $S \subseteq T(n)$. To prove the other inclusion, we need to show that $3.2^{n+i} - 1 \in S$ for all $i \in \mathbb{N}$. We use induction on i. For $i = 0$ the result is trivial. Assume that the statement holds for i and let us show it for $i + 1$. Since $3.2^{n+i+1} - 1 = 2\left(3.2^{n+i} - 1\right) + 1$ then, by induction hypothesis and Lemma 90, we get that $3.2^{n+i+1} - 1 \in S$. $\square$

The next result show that $3.2^{2n+1} - 1$ belongs to the minimal system of generators of $T(n)$.

LEMMA 92. *If n is a nonnegative integer, then $3.2^{2n+1} - 1 \notin \langle \{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n\}\} \rangle$.*

PROOF. Let us suppose $3.2^{2n+1} - 1 \in \langle \{3.2^{n+i} - 1 \mid i \in \{0, 1, \ldots, n\}\} \rangle$. Then there exists $a_0, \ldots, a_n \in \mathbb{N}$ such that $3.2^{2n+1} - 1 = a_0 (3.2^n - 1) + \cdots + a_n \left(3.2^{2n} - 1\right) =$

$3\left(a_0 2^n+\cdots+a_n 2^{2n}\right)-\left(a_0+\cdots+a_n\right)$ and consequently $a_0+\cdots+a_n\equiv 1(\bmod 3.2^n)$. Hence $a_0+\cdots+a_n=1+k3.2^n$ for some $k\in\mathbb{N}$. Besides, it is clear that $k\neq 0$ and so $a_0+\cdots+a_n\geq 1+3.2^n$. Therefore $a_0\left(3.2^n-1\right)+\cdots+a_n\left(3.2^{2n}-1\right)\geq \left(a_0+\cdots+a_n\right)\left(3.2^n-1\right)\geq\left(1+3.2n\right)\left(3.2n-1\right)=9.2^{2n}-1>3.2^{2n+1}-1$, which is absurd. $\square$

We are already in conditions to prove the result mentioned above.

THEOREM 93. *Let n be a nonnegative integer and let $T(n)$ be the Thabit numerical semigroup associated to n, then $e(T(n))=n+2$. Furthermore $\left\{3.2^{n+i}-1\mid i\in\{0,1,\ldots,n+1\}\right\}$ is the minimal system of generators of $T(n)$.*

PROOF. From Lemma 91, we know that $\left\{3.2^{n+i}-1\mid i\in\{0,1,\ldots,n+1\}\right\}$ is a system of generators of $T(n)$. If it is not a minimal system of generators of $T(n)$, then there exists $h\in\{1,\ldots,n+1\}$ such that $3.2^{n+h}-1\in\langle\left\{3.2^{n+i}-1\mid i\in\{0,1,\ldots,h-1\}\right\}\rangle$.

Assume that $S=\langle\left\{3.2^{n+i}-1\mid i\in\{0,1,\ldots,h-1\}\right\}\rangle$. If $i\in\{0,\ldots,h-2\}$ then $2\left(3.2^{n+i}-1\right)+1=3.2^{n+i+1}-1\in S$. Moreover, in view of the previous paragraph $2\left(3.2^{n+h-1}-1\right)+1=3.2^{n+h}-1\in S$. Hence by applying Lemma 72 we obtain that $2s+1\in S$ for all $s\in S\backslash\{0\}$.

Now we use induction on i to prove that $3.2^{n+i}-1\in S$ for all $i\in\mathbb{N}$. For $i=0$, the result follows trivially. Assume that the result is true for i and let us prove it for $i+1$. As $3.2^{n+i+1}-1=2\left(3.2^{n+i}-1\right)+1$, from the induction hypothesis and the end of the last paragraph, we get that $3.2^{n+i+1}-1\in S$. In particular we obtain $3.2^{2n+1}-1\in S\subseteq\langle\left\{3.2^{n+i}-1\mid i\in\{0,1,\ldots,n\}\right\}\rangle$, which contradicts Lemma 92. $\square$

Gathering all this information we obtain that for each integer k greater than or equal to 2 there exists an unique Thabit numerical semigroup $T(n)$ with embedding dimension k. For example $T(2)=\langle\left\{3.2^2-1,3.2^3-1,3.2^4-1,3.2^5-1\right\}\rangle=\langle\{11,23,47,95\}\rangle$ is the unique Thabit numerical semigroup with embedding dimension 4.

2.2. The Apéry set. Our first purpose is to get an explicit description of the elements in the $\mathrm{Ap}(T(n), 3.2^n - 1)$. From now on we will denote by s_i the elements $3.2^{n+i} - 1$ for each $i \in \{0, 1, \ldots, n+1\}$. Thus with this notation we have that $\{s_0, s_1, \ldots, s_{n+1}\}$ is the minimal system of generators of $T(n)$.

LEMMA 94. *Let n be a nonnegative integer. Then:*

(1) if $0 < i \leq j < n+1$ *then* $s_i + 2s_j = 2s_{i-1} + s_{j+1}$*;*
(2) if $0 < i \leq n+1$ *then* $s_i + 2s_{n+1} = 2s_{i-1} + s_0^2 + s_1 + s_n$.

PROOF.

(1) If $0 < i \leq j < n+1$, then we have that $s_i + 2s_j = 3.2^{n+i} - 1 + 2\left(3.2^{n+j} - 1\right) = 2\left(3.2^{n+i-1} - 1\right) + 3.2^{n+j+1} - 1 = 2s_{i-1} + s_{j+1}$.
(2) If $0 < i \leq n+1$, then we get that $s_i + 2s_{n+1} = 3.2^{n+i} - 1 + 2\left(3.2^{2n+1} - 1\right) = 3.2^{n+i} - 2 + 3.2^{2n+2} - 1 = 2\left(3.2^{2n+i-1} - 1\right) + (3.2^n - 1)^2 + \left(3.2^{n+1} - 1\right) + \left(3.2^{2n} - 1\right) = 2s_{i-1} + s_0^2 + s_1 + s_n$.

$\square$

Denote by $A(n)$ the set of all elements $(a_1, \ldots, a_{n+1}) \in \{0, 1, 2\}^{n+1}$ fulfilling the following condition: if $1 \leq i < j \leq n+1$ and $a_j = 2$ then $a_i = 0$.

LEMMA 95. *Let* $n \in \mathbb{N}$ *and let* $T(n)$ *be a Thabit numerical semigroup minimally generated by* $\{s_0, s_1, \ldots, s_{n+1}\}$*. Then* $\mathrm{Ap}(T(n), s_0) \subseteq \{a_1 s_1 + \cdots + a_{n+1} s_{n+1} \mid (a_1, \ldots, a_{n+1}) \in A(n)\}$.

PROOF. Let $x \in \mathrm{Ap}(T(n), s_0)$. We use induction on x to prove that $x = a_1 s_1 + \cdots + a_{n+1} s_{n+1}$ with $(a_1, \ldots, a_{n+1}) \in A(n)$. For $x = 0$ then $x = 0s_1 + \cdots + 0s_{n+1}$ and the result is clear. Assume that $x > 0$ and $j = \min\{i \in \{0, \ldots, n+1\} \mid x - s_i \in T(n)\}$. Since $x \in \mathrm{Ap}(T(n), s_0)$ observe that $j \neq 0$ and $x - s_j \in \mathrm{Ap}(T(n), s_0)$. By induction hypothesis, there exists $(a_1, \ldots, a_{n+1}) \in A(n)$ such that $x - s_j = a_1 s_1 + \cdots + a_{n+1} s_{n+1}$. Hence $x = a_1 s_1 + \cdots + (a_j + 1) s_j + \cdots + a_{n+1} s_{n+1}$. To conclude the proof, it suffices to check that $\left(a_1, \ldots, a_j + 1, \ldots, a_{n+1}\right) \in A(n)$. In order to prove $\left(a_1, \ldots, a_j + 1, \ldots, a_{n+1}\right) \in$

$\{0,1,2\}^{n+1}$ it is enough to see that $a_j+1 \neq 3$. Suppose that $a_j+1=3$. By using Lemma 94, we distinguish two cases depending on the value of j:

. if $j<n+1$ then $(a_j+1)s_j = 3s_j = 2s_{j-1}+s_{j+1}$;
. if $j=n+1$ then $(a_j+1)s_j = 3s_j = 2s_{j-1}+s_0^2+s_1+s_n$.

In both cases, we deduce that $x-s_{j-1} \in T(n)$ contradicting the minimality of j. Whence from the minimality of j we have that $a_i=0$ for $1 \leq i < j$. Let us see that does not exists $k>j$ such that $a_k=2$. Assume to the contrary that $a_k=2$, by Lemma 94 we have that:

. if $k<n+1$ then $s_j+2s_k = 2s_{j-1}+s_{k+1}$;
. if $k=n+1$ then $s_j+2s_k = 2s_{j-1}+s_0^2+s_1+s_n$.

In both cases, we get that $x-s_{j-1} \in T(n)$ contradicting again the minimality of j. Therefore $\left(a_1,\ldots,a_j+1,\ldots,a_{n+1}\right) \in A(n)$. □

We will see in the next example that the equality in Lemma 95 does not hold in general.

EXAMPLE 96. We have that $T(1) = \langle\{5,11,23\}\rangle$ and $\mathrm{Ap}(T(1),5) = \{0,11,22,23,34\}$.

It is clear that $A(1) = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,1)\}$ and thus $\{a_1 11 + a_2 23 \mid (a_1,a_2) \in A(1)\} = \{0,23,46,11,34,22,45\}$.

Now our purpose is to find a subset $R(n)$ of $A(n)$ such that the equality holds in Lemma 95 if we substitute $A(n)$ by $R(n)$.

LEMMA 97. *Under the standing notation and $n \in \mathbb{N}$. If $x \in T(n)$ and $x \not\equiv 0 \bmod s_0$ then $x-1 \in T(n)$.*

PROOF. If $x \in T(n)$ then there exists there exist $a_0,\ldots,a_{n+1} \in \mathbb{N}$ such that $x = a_0s_0+\cdots+a_{n+1}s_{n+1}$. On the other hand, if $x \not\equiv 0 \bmod s_0$ then there exists

$i \in \{1,\ldots,n+1\}$ such that $a_i \neq 0$. Hence

$$\begin{aligned} x-1 &= a_0s_0+\cdots+(a_i-1)s_i+\cdots+a_{n+1}s_{n+1}+3.2^{n+i}-2= \\ &= a_0s_0+\cdots+(a_i-1)s_i+\cdots+a_{n+1}s_{n+1}+2\left(3.2^{n+i-1}-1\right)= \\ &= a_0s_0+\cdots+(a_{i-1}+2)s_{i-1}+(a_i-1)s_i+\cdots+a_{n+1}s_{n+1}\in T(n). \end{aligned}$$

□

The next result shows that if $\mathrm{Ap}(T(n),s_0)=\{w(0),w(1),\ldots,w(s_0-1)\}$ then $w(0)<w(1)<\cdots<w(s_0-1)$.

LEMMA 98. *Let $n\in\mathbb{N}$ and let $w(i)$ be the least element of $T(n)$ congruent with i modulo s_0 for all $i\in\{0,\ldots,s_0-1\}$. Then $w(0)<w(1)<\cdots<w(s_0-1)$.*

PROOF. Let us show that $w(i)<w(i+1)$ for all $i\in\{0,\ldots,s_0-2\}$. Since $w(i+1)\in T(n)$ and $w(i+1)\not\equiv 0 \bmod s_0$, we have that $w(i+1)-1\in T(n)$ by Lemma 97. As $w(i+1)-1\equiv i \bmod s_0$, we can deduce that $w(i)\leq w(i+1)-1$. □

As a consequence of previous lemma we get that $w(s_0-1)=\max(\mathrm{Ap}(T(n),s_0))$

LEMMA 99. *Under the standing notation. If $n\in\mathbb{N}$ then* $\max(\mathrm{Ap}(T(n),s_0))\leq s_n+s_{n+1}$.

PROOF. Since $s_n+s_{n+1}=3.2^{2n}-1+3.2^{2n+1}-1=2^n(3.2^n-1)+(2^n-1)+2^{n+1}(3.2^n-1)+\left(2^{n+1}-1\right)=\left(2^n+2^{n+1}\right)s_0+2^n-1+2^{n+1}-1=\left(2^n+2^{n+1}\right)s_0+s_0-1$, we can conclude that $s_n+s_{n+1}\equiv s_0-1 \bmod s_0$. Therefore $w(s_0-1)\leq s_n+s_{n+1}$ and by Lemma 98 we obtain the desired result. □

As a consequence of Lemma 99, we obtain the following result.

LEMMA 100. *Under the standing notation. If $n\in\mathbb{N}$ then*

(1) $2s_{n+1}\notin\mathrm{Ap}(T(n),s_0)$*;*
(2) $s_n+s_{n+1}+s_i\notin\mathrm{Ap}(T(n),s_0)$ *for all $i\in\{0,\ldots,n+1\}$.*

From now on we will suppose that n is a integer greater than or equal to 1. We will denote by $R(n)$ the set of the sequences $(a_1,\ldots,a_{n+1}) \in A(n)$ satisfying the following conditions:

(1) $a_{n+1} \in \{0,1\}$;
(2) if $a_n = 2$ then $a_{n+1} = 0$;
(3) if $1 \leq i < n$ and $a_n = a_{n+1} = 1$ then $a_i = 0$.

Our goal now is to prove that $\mathrm{Ap}(T(n), s_0) = \{a_1 s_1 + \cdots + a_{n+1}s_{n+1} \mid (a_1,\ldots,a_{n+1}) \in R(n)\}$.

REMARK 101. Observe that if $n \geq 2$ then $R(n)$ is the set of the sequences $(a_1,\ldots,a_{n+1}) \in A(n)$ satisfying the following conditions:

(1) $(a_1,\ldots,a_{n+1}) \neq (0,\ldots,0,2)$;
(2) $(a_1,\ldots,a_{n+1}) \neq (0,\ldots,2,1)$;
(3) if $a_n = a_{n+1} = 1$ then $a_1 = \cdots = a_{n-1} = 0$.

LEMMA 102. *Under the standing notation. If n is a positive integer, then* $\#R(n) = 3.2^n - 1$ *(where $\#X$ stands for cardinality of X).*

PROOF. We distinguish two cases.

1) If $(a_1,\ldots,a_{n+1}) \in R(n)$ and $2 \notin \{a_1,\ldots,a_{n+1}\}$, then $a_i \in \{0,1\}$ for all $i \in \{1,\ldots,n-1\}$ and furthermore either $a_n = a_{n+1} = 0$ or $a_n = 0$ and $a_{n+1} = 1$ or $a_n = 1$ and $a_{n+1} = 0$ or $(a_1,\ldots,a_{n+1}) = (0,\ldots,0,1,1)$. Whence $\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid 2 \notin \{a_1,\ldots,a_{n+1}\}\} = 3.2^{n-1} + 1$.
2) If $(a_1,\ldots,a_{n+1}) \in R(n)$ and $2 \in \{a_1,\ldots,a_{n+1}\}$, then there exists an unique $i \in \{1,\ldots,n\}$ such that $a_i = 2$. Besides, if $i = n$ then $(a_1,\ldots,a_{n+1}) = (0,\ldots 0,2,0)$. On the other hand, if $i \in \{1,\ldots,n-1\}$ then $a_1 = \cdots = a_{i-1} = 0$, $a_{i+1},\ldots,a_{n-1} \in \{0,1\}$ and either $a_n = a_{n+1} = 0$ or $a_n = 0$ and $a_{n+1} = 1$ or $a_n = 1$ and $a_{n+1} = 0$. Hence $\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid 2 \in \{a_1,\ldots,a_{n+1}\}\} = 3\sum_{i=1}^{n-1} 2^{n-i-1} + 1$.

Consequently $\#R(n) = 3.2^{n-1} + 1 + 3\sum_{i=1}^{n-1} 2^{n-i-1} + 1 = 3.2^{n-1} + 1 + 3(2^{n-1} - 1) + 1 = 6.2^{n-1} - 1 = 3.2^n - 1$. □

Now we can state the result announced above.

THEOREM 103. *Let $n \in \mathbb{N}\backslash\{0\}$ and let $T(n)$ be a Thabit numerical semigroup minimally generated by $\{s_0, s_1, \ldots, s_{n+1}\}$. Then* $\mathrm{Ap}(T(n), s_0) = \{a_1 s_1 + \cdots + a_{n+1} s_{n+1} \mid (a_1, \ldots, a_{n+1}) \in R(n)\}$.

PROOF. As a consequence of Lemmas 95 and 100 we obtain that $\mathrm{Ap}(T(n), s_0) \subseteq \{a_1 s_1 + \cdots + a_{n+1} s_{n+1} \mid (a_1, \ldots, a_{n+1}) \in R(n)\}$. By using Lemmas 5 and 102, we get that $\#\{a_1 s_1 + \cdots + a_{n+1} s_{n+1} \mid (a_1, \ldots, a_{n+1}) \in R(n)\} \leq \#R(n) = 3.2^n - 1 = s_0 = \#\mathrm{Ap}(T(n), s_0)$. Hence $\mathrm{Ap}(T(n), s_0) = \{a_1 s_1 + \cdots + a_{n+1} s_{n+1} \mid (a_1, \ldots, a_{n+1}) \in R(n)\}$. □

As a consequence of the proof of previous theorem we have the following result.

COROLLARY 104. *Under the standing hypothesis and notation. If $(a_1, \ldots, a_{n+1}), (b_1, \ldots, b_{n+1}) \in R(n)$ and $(a_1, \ldots, a_{n+1}) \neq (b_1, \ldots, b_{n+1})$, then we have that $a_1 s_1 + \cdots + a_{n+1} s_{n+1} \neq b_1 s_1 + \cdots + b_{n+1} s_{n+1}$.*

Observe, by Remark 101 that, since $(0, 0, \ldots, 1, 1)$ belongs to $R(n)$ if $n \in \mathbb{N}\backslash\{0\}$ then $s_n + s_{n+1} \in \mathrm{Ap}(T(n), s_0)$. Using Lemma 99 we have that $\max(\mathrm{Ap}(T(n), s_0)) = s_n + s_{n+1}$. Now by Proposition 10 we obtain the following result, which gives a formula for the Frobenius number.

COROLLARY 105. *Under the standing notation. If $n \in \mathbb{N}\backslash\{0\}$ then* $\mathrm{F}(T(n)) = s_n + s_{n+1} - s_0 = 9.2^{2n} - 3.2^n - 1$.

We illustrate some of these results with an example.

EXAMPLE 106.

. Let $T(1) = \langle\{5, 11, 23\}\rangle$. From Corollary 105, we obtain that $\mathrm{F}(T(1)) = 11 + 23 - 5 = 29$. We have that $R(1) = \{(0,0), (0,1), (1,0), (1,1), (2,0)\}$ and thus, by Theorem 103, $\mathrm{Ap}(T(1), 5) = \{0, 23, 11, 34, 22\}$.

. Let $T(2) = \langle\{11,23,47,95\}\rangle$. By using again Corollary 105, we get that $\mathrm{F}(T(2)) = 95+47-11 = 131$. It is easy to check that

$$R(2) = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(0,2,0),(1,0,0),(1,0,1),\\ (1,1,0),(2,0,0),(2,0,1),(2,1,0)\}.$$

Hence $\mathrm{Ap}(T(2),11) = \{0,95,47,142,94,23,118,70,46,141,93\}$.

Observe that for $T(2)$ we have that $\mathrm{Ap}(T(2),11) = \{w(0)=0, w(1)=23, w(2)=46, w(3)=47, w(4)=70, w(5)=93, w(6)=94, w(7)=95, w(8)=118, w(9)=141, w(10)=142\}$.

Thus using Remark 7, for example $129 \in T(2)$ and $119 \notin T(2)$, since $129 \geq w(129 \bmod 11) = w(8) = 118$ and $119 < w(119 \bmod 11) = w(9) = 141$.

2.3. Pseudo-Frobenius numbers and type. Note that if $w, w' \in \mathrm{Ap}(S,x)$, then $w'-w \in S$ if and only if $w'-w \in \mathrm{Ap}(S,x)$. Therefore $\mathrm{maximals}_{\leq_S}(\mathrm{Ap}(S,x)) = \{w \in \mathrm{Ap}(S,x) \mid w'-w \notin \mathrm{Ap}(S,x)\setminus\{0\} \text{ for all } w' \in \mathrm{Ap}(S,x)\}$. Consequently, we have that $\mathrm{maximals}_{\leq_{T(1)}}(\mathrm{Ap}(T(1),5)) = \{22,34\}$ (see Example 106). From Lemma 11, we get that $\mathrm{PF}(T(1)) = \{17,29\}$ and so $\mathrm{t}(T(1)) = 2$.

Let n be an integer greater than or equal to 2. It is clear that maximal elements in $R(n)$ (with respect to the product order) are

$$(2,1,\ldots,1,0),(0,2,1,\ldots,1,0),\ldots,(0,\ldots,0,2,1,0),(0,\ldots,0,2,0)\\ (2,1,\ldots,1,0,1),(0,2,1,\ldots,1,0,1),\ldots,(0,\ldots,0,2,0,1),(0,\ldots,0,1,1).$$

Moreover, since $2s_i+1 = s_{i+1}$ for all $i \in \{1,\ldots,n\}$, we have that

$$\{a_1s_1+\cdots+a_{n+1}s_{n+1} \mid (a_1,\ldots,a_{n+1}) \in \{(2,1,\ldots,1,0),(0,2,1,\ldots,1,0),\\ \ldots,(0,\ldots,0,2,1,0),(0,\ldots,0,2,0)\}\} = \{2s_n-(n-1),\ldots,2s_n-1,2s_n\},$$

and

$$\{a_1s_1+\cdots+a_{n+1}s_{n+1} \mid (a_1,\ldots,a_{n+1})\in\{(2,1,\ldots,1,0,1),(0,2,1,\ldots,1,0,1),$$
$$\ldots,(0,\ldots,0,2,0,1),(0,\ldots,0,1,1)\}\}=\{s_n+s_{n+1}-(n-1),\ldots,s_n+s_{n+1}-1,s_n+s_{n+1}\}.$$

As a consequence of Theorem 103, we obtain the following.

LEMMA 107. *Under the standing notation. If n is an integer greater than or equal to two, then* $\mathit{maximals}_{\leq_{T(n)}}(\mathrm{Ap}(T(n),s_0)) =$ $\mathit{maximals}_{\leq_{T(n)}}\{2s_n,2s_n-1,\ldots,2s_n-(n-1),s_n+s_{n+1},s_n+s_{n+1}-1,\ldots$
$\ldots,s_n+s_{n+1}-(n-1)\}$

We are now able to give the next result that is central in this study.

THEOREM 108. *Let n be an integer greater than or equal to two and let $T(n)$ be the Thabit numerical semigroup associated to n. Then* $\mathit{maximals}_{\leq_{T(n)}}(\mathrm{Ap}(T(n),s_0)) =$ $\{2s_n-(n-1),s_n+s_{n+1},s_n+s_{n+1}-1,\ldots,s_n+s_{n+1}-(n-1)\}$.

PROOF. Let $i\in\{0,\ldots,n-2\}$. Then $s_n+s_{n+1}-(i+1)-(2s_n-i)=s_n+s_{n+1}-(2s_n+1)=s_n+s_{n+1}-s_{n+1}=s_n$ and consequently have that $(2s_n-i)\leq_{T(n)} s_n+s_{n+1}-(i+1)$. From Lemma 107 we obtain that $\mathrm{maximals}_{\leq_{T(n)}}(\mathrm{Ap}(T(n),s_0)) = \mathrm{maximals}_{\leq_{T(n)}}\{2s_n-(n-1),s_n+s_{n+1},s_n+s_{n+1}-1,\ldots,s_n+s_{n+1}-(n-1)\}$.

As $2s_n-(n-1)<s_n+s_{n+1}-(n-1)$, the elements $s_n+s_{n+1}-(n-1),\ldots,s_n+s_{n+1}-1,s_n+s_{n+1}$ are n consecutive positive integers and $n<3.2^n-1$, then we deduce that $\{s_n+s_{n+1}-(n-1),\ldots,s_n+s_{n+1}-1,s_n+s_{n+1}\}\subseteq \mathrm{maximals}_{\leq_{T(n)}}(\mathrm{Ap}(T(n),s_0))$.

Finally, we show that $s_n+s_{n+1}-i-(2s_n-(n-1))\notin T(n)$ for all $i\in\{0,\ldots,n-1\}$, or equivalently, $s_n+n-i\notin T(n)$ for all $i\in\{0,\ldots,n-1\}$. Assume that there exists $i\in\{0,\ldots,n-1\}$ such that $s_n+n-i\in T(n)$. Since $s_n+n-i=3.2^{2n}-1+n-i=2^n(3.2^n-1)+2^n-1+n-i$ and $1\leq 2^n-1+n-i<3.2^n-1$, by Lemma 97, we conclude that $s_n+1\in T(n)$. Then there exists $a_0,\ldots,a_{n-1}\in\mathbb{N}$ such that $s_n+1=a_0s_0+\cdots+a_{n-1}s_{n-1}$. As $s_n+1\not\equiv 0 \bmod s_0$, then there exists

$j \in \{1,\ldots,n-1\}$ such that $a_j \neq 0$. Therefore $s_n = a_0s_0 + \cdots + (a_j-1)s_j + \cdots + a_{n-1}s_{n-1} + 3.2^{n+j} - 2 = a_0s_0 + \cdots + (a_j-1)s_j + \cdots + a_{n-1}s_{n-1} + 2\left(3.2^{n+j-1} - 1\right) = a_0s_0 + \cdots + (a_j-1)s_j + \cdots + a_{n-1}s_{n-1} + 2s_{j-1}$. Hence $s_n \in \langle\{s_0,\ldots,s_{n-1}\}\rangle$ which contradicts Theorem 93. $\square$

By applying now Lemma 11 and Corollary 105 we obtain the following result.

COROLLARY 109. *Let n be a positive integer and let $T(n)$ be the Thabit numerical semigroup associated to n. Then*

$$\mathrm{PF}(T(n)) = \{\mathrm{F}(T(n)) - i \mid i \in \{0,\ldots,n-1\}\} \cup \{2s_n - s_0 - (n-1)\}$$

and $\mathrm{t}(T(n)) = n+1$.

Next we give an example.

EXAMPLE 110.

Let $T(2) = \langle\{11,23,47,95\}\rangle$. From Corollary 105, we know that $\mathrm{F}(T(2)) = 95 + 47 - 11 = 131$. Moreover, we have that $2s_n - s_0 - (n-1) = 2.47 - 11 - 1 = 82$. By applying Corollary 109, we get that $\mathrm{PF}(T(2)) = \{131,130,82\}$.

2.4. The genus. Our next purpose is to prove Theorem 113, which gives the a formula for the genus of $T(n)$. To this purpose, we need some preliminary results.

LEMMA 111. *Let n be an integer greater than or equal to* 2 *and let* $i \in \{1,\ldots,n+1\}$. *Then*

$$\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid a_i = 2\} = \begin{cases} 3.2^{n-i-1} & \text{if } i \in \{1,\ldots,n-1\}, \\ 1 & \text{if } i = n, \\ 0 & \text{if } i = n+1. \end{cases}$$

PROOF. If $i \in \{1,\ldots,n-1\}$, $(a_1,\ldots,a_{n+1}) \in R(n)$ and $a_i = 2$, then we have that $a_1 = a_2 = \cdots = a_{i-1} = 0$, $a_{i+1},\ldots,a_{n+1} \in \{0,1\}$ and furthermore either $a_n = 0$ or $a_{n+1} = 0$. Hence $\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid a_i = 2\} = 3.2^{n-i-1}$. On the other hand, it is also clear that $\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_n = 2\} = \{(0,\dots,0,2,0)\}$ and $\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_{n+1} = 2\} = \emptyset$. $\square$

LEMMA 112. *Let n be an integer greater than or equal to 2 and let $i \in \{1,\dots,n+1\}$. Then*

$$\#\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_i = 1\} = \begin{cases} 3\left(2^{n-1}-2^{n-i-1}\right) & \textit{if } \ i \in \{1,\dots,n-1\}, \\ 2^n & \textit{if } \ i \in \{n,n+1\}. \end{cases}$$

PROOF.

1) Let $i \in \{1,\dots,n-1\}$. We distinguish two cases.
 1.1) If $2 \notin \{a_1,\dots,a_{i-1}\}$, then $a_1,\dots,a_{i-1},a_{i+1},\dots,a_{n+1} \in \{0,1\}$ and either $a_n = 0$ or $a_{n+1} = 0$. Therefore $\#\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_i = 1 \text{ and } 2 \notin \{a_1,\dots,a_{i-1}\}\} = 3.2^{n-2}$.
 1.2) If $2 \in \{a_1,\dots,a_{i-1}\}$, then $a_j = 2$ for some $j \in \{1,\dots,i-1\}$. Thus $a_1 = \cdots = a_{j-1} = 0$, $a_{j+1},\dots,a_{n+1} \in \{0,1\}$, $a_i = 1$ and either $a_n = 0$ or $a_{n+1} = 0$. Hence $\#\left\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_i = 1 \text{ and } a_j = 2\right\} = 3.2^{n-j-2}$.

 Consequently $\#\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_i = 1\} = 3.2^{n-2} + \sum_{j=1}^{i-1} 3.2^{n-j-2} = 3.2^{n-2} + 3.2^{n-3} + \cdots + 3.2^{n-i-1} = 3\left(2^{n-1}-2^{n-i-1}\right)$.

2) Let $i = n$. We distinguish two cases.
 2.1) If $2 \notin \{a_1,\dots,a_{n-1}\}$, then $a_1,\dots,a_{n-1},a_{n+1} \in \{0,1\}$. Besides if $a_{n+1} = 1$, then $a_1 = \cdots = a_{n-1} = 0$. Hence $\#\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_n = 1 \text{ and } 2 \notin \{a_1,\dots,a_{n-1}\}\} = 2^{n-1}+1$.
 2.2) If $2 \in \{a_1,\dots,a_{n-1}\}$, then $a_j = 2$ for some $j \in \{1,\dots,n-1\}$. In this way $a_1 = \cdots = a_{j-1} = 0$, $a_{j+1},\dots,a_{n+1} \in \{0,1\}$ and $a_{n+1} = 0$. Whence $\#\left\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_n = 1 \text{ and } a_j = 2\right\} = 2^{n-j-1}$.

 Accordingly, $\#\{(a_1,\dots,a_{n+1}) \in R(n) \mid a_n = 1\} = 2^{n-1} + 1 + \sum_{j=1}^{n-1} 2^{n-j-1} = 2^{n-1} + 2^{n-2} + \cdots + 2^0 + 1 = 2^n$.

3) Let $i = n+1$. We distinguish two cases.

3.1) If $2 \notin \{a_1,\ldots,a_n\}$, then $a_1,\ldots,a_n \in \{0,1\}$. Furthermore, if $a_n = 1$ we have that $a_1 = \cdots = a_{n-1} = 0$. Therefore, $\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid a_{n+1} = 1 \text{ and } 2 \notin \{a_1,\ldots,a_n\}\} = 2^{n-1}+1$.

3.2) Now assume that $2 \in \{a_1,\ldots,a_n\}$. We deduce that there exists $j \in \{1,\ldots,n-1\}$ such that $a_j = 2$ and thus $a_1 = \cdots = a_{j-1} = 0$, $a_{j+1},\ldots,a_{n-1} \in \{0,1\}$ and $a_n = 0$ (observe that in this case does not exist elements such that $a_n = 2$ and $a_{n+1} = 1$). Hence $\#\left\{(a_1,\ldots,a_{n+1}) \in R(n) \mid a_{n+1} = 1 \text{ and } a_j = 2\right\} = 2^{n-j-1}$.

Consequently $\#\{(a_1,\ldots,a_{n+1}) \in R(n) \mid a_{n+1} = 1\} = 2^{n-1} + 1 + \sum_{j=1}^{n-1} 2^{n-j-1} = 2^n$.

□

We are ready to prove the next result.

THEOREM 113. *Let n be a nonnegative integer and let $T(n)$ be the Thabit numerical semigroup associated to n. Then* $\mathrm{g}(T(n)) = 9.2^{2n-1} + (3n-5)\,2^{n-1}$.

PROOF. The reader can check that the result is also true for $n \in \{0,1\}$.

Now, we can suppose that $n \geq 2$. Applying Proposition 10, Theorem 103 and Corollary 104 we have that

$$\mathrm{g}(T(n)) = \frac{1}{s_0}\left(\sum_{(a_1,\ldots,a_{n+1})\in R(n)} a_1 s_1 + \cdots + a_{n+1}s_{n+1}\right) - \frac{s_0-1}{2}.$$

Clearly,

$$\begin{aligned}
&\sum_{(a_1,\ldots,a_{n+1})\in R(n)} a_1 s_1 + \cdots + a_{n+1}s_{n+1} = \\
&\qquad = \sum_{(a_1,\ldots,a_{n+1})\in R(n),\ a_1=1} s_1 + \sum_{(a_1,\ldots,a_{n+1})\in R(n),\ a_1=2} 2s_1 + \cdots \\
&\qquad\qquad \cdots + \sum_{(a_1,\ldots,a_{n+1})\in R(n),\ a_{n+1}=1} s_{n+1} + \sum_{(a_1,\ldots,a_{n+1})\in R(n),\ a_{n+1}=2} 2s_{n+1}.
\end{aligned}$$

By using Lemmas 111 and 112, we obtain that

$$\sum_{(a_1,\ldots,a_{n-1})\in R(n)} a_1 s_1+\cdots+a_{n+1}s_{n+1} =$$

$$=\sum_{i=1}^{n-1} 3.2^{n-i-1}2s_i+2s_n+\sum_{i=1}^{n-1}3\left(2^{n-1}-2^{n-i-1}\right)s_i+2^n s_n+$$

$$2^n s_{n+1}=3\sum_{i=1}^{n-1}2^{n-i}\left(3.2^{n+i}-1\right)+3.2^{n-1}\sum_{i=1}^{n-1}\left(3.2^{n+i}-1\right)-$$

$$3\sum_{i=1}^{n-1}2^{n-i-1}\left(3.2^{n+i}-1\right)+\left(2^n+2\right)s_n+2^n s_{n+1}=$$

$$=3\sum_{i=1}^{n-1}3.2^{2n}-3\sum_{i=1}^{n-1}2^{n-i}+9.2^{n-1}\sum_{i=1}^{n-1}2^{n+i}-(n-1)\,3.2^{n-1}-$$

$$9\sum_{i=1}^{n-1}2^{2n-1}+3\sum_{i=1}^{n-1}2^{n-i-1}+\left(2^n+2\right)s_n+s_{n+1}=$$

$$=9\,(n-1)\,2^{2n}-3\left(2^n-2\right)+9.2^{n-1}\left(2^{2n}-2^{n+1}\right)-3\,(n-1)\,2^{n-1}-$$

$$9\,(n-1)\,2^{2n-1}+3\left(2^{n-1}-1\right)+\left(2^n+2\right)\left(3.2^{2n}-1\right)+2^n\left(3.2^{2n+1}-1\right)=$$

$$=27.2^{3n-1}+(9n-15)\,2^{2n-1}-(3n+4)\,2^{n-1}+1=$$

$$=\left(3.2^n-1\right)\left(9.2^{2n-1}+(3n-2)\,2^{n-1}-1\right).$$

Therefore,

$$\mathrm{g}\left(T(n)\right)=9.2^{2n-1}+(3n-2)\,2^{n-1}-1-\frac{3.2^n-2}{2}=$$

$$=9.2^{2n-1}+(3n-5)\,2^{n-1}.$$

$\square$

We conclude this section by illustrating the previous results with an example.

EXAMPLE 114. Consider the Thabit numerical semigroup $T(3)$. By applying Theorem 93, we obtain that $e(T(3))=5$ and $\{23,47,95,191,383\}$ is its minimal set of generators. From Corollary 105, we know that $\mathrm{F}(T(3))=551$ and, by Theorem 113, we have that $\mathrm{g}(T(3)) = 304$. Now using Corollary 109, we get that $\mathrm{t}(T(3)) = 4$ and $\mathrm{PF}(T(3)) = \{551, 550, 549, 357\}$

It also follows easily from the definition of $R(n)$ that

$$\begin{aligned} R(3) = \{&(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,0,2,0), \\ &(0,1,0,0), (0,1,0,1), (0,1,1,0), (0,2,0,0), (0,2,0,1), (0,2,1,0), \\ &(1,0,0,0), (1,0,0,1), (1,0,1,0), (1,1,0,0), (1,1,0,1), (1,1,1,0), \\ &(2,0,0,0), (2,0,0,1), (2,0,1,0), (2,1,0,0), (2,1,0,1), (2,1,1,0)\} \end{aligned}$$

and, by Theorem 103, we get that

$$\begin{aligned} \mathrm{Ap}(T(3), 23) = \{&0, 383, 191, 574, 382, 95, 478, 286, 190, 573, 381, 47, 430, \\ &238, 142, 525, 333, 94, 477, 285, 189, 572, 380\}. \end{aligned}$$

Moreover, from Lemma 98 ordering the previous set in increasing order we have that $w(0) = 0$, $w(1) = 47$, $w(2) = 94$, $w(3) = 95$, $w(4) = 142$, $w(5) = 189$, $w(6) = 190$, $w(7) = 191$, $w(8) = 238$, $w(9) = 285$, $w(10) = 286$, $w(11) = 333$, $w(12) = 380$, $w(13) = 381$, $w(14) = 382$, $w(15) = 383$, $w(16) = 430$, $w(17) = 477$, $w(18) = 478$, $w(19) = 525$, $w(20) = 572$, $w(21) = 573$, $w(22) = 574$, where $w(i)$ is the least element of $T(3)$ congruent with i modulo 23.

3. The Frobenius problem for Repunit numerical semigroups

In number theory, a Repunit is a number consisting of copies of the single digit 1. The numbers 1, 11, 111 or 1111, etc., are examples of Repunits. The term stands for repeated unit and was coined by Albert H. Beiler in [**3**]. In general, the set of Repunits in base b is $\left\{\frac{b^n-1}{b-1} \mid n \in \mathbb{N}\backslash\{0\}\right\}$. In binary, these are known like Mersenne numbers. In the literature there are many problems related to this kind of numbers (see, for example, [**45**] and [**52**]). The results presented in this section can be found in [**35**].

A numerical semigroup S is a **Repunit numerical semigroup** if there exist integers $b\in\mathbb{N}\setminus\{0,1\}$ and $n\in\mathbb{N}\setminus\{0\}$ such that $S=\left\langle\left\{\frac{b^{n+i}-1}{b-1}\mid i\in\mathbb{N}\right\}\right\rangle$ and it will be denoted by $S(b,n)$.

3.1. The embedding dimension. Along this section, b denotes an integer greater than 2 and n denotes a positive integer. It is clear that $b^n-1=(b-1)\left(b^{n-1}+b^{n-2}+\cdots+b+1\right)$ and thus $\frac{b^n-1}{b-1}$ is a positive integer. Besides $\gcd\left\{\frac{b^n-1}{b-1},\frac{b^{n+1}-1}{b-1}\right\}=\frac{1}{b-1}\left(\gcd\left\{b^n-1,b^{n+1}-1\right\}\right)=\frac{1}{b-1}\left(\gcd\left\{b^n-1,b\left(b^n-1\right)+b-1\right\}\right)=\frac{1}{b-1}\left(\gcd\left\{b^n-1,b-1\right\}\right)=\frac{1}{b-1}(b-1)=1$. This proves the following result.

PROPOSITION 115. *$S(b,n)$ is a numerical semigroup.*

Let $M(b,n)$ be a submonoid of $(\mathbb{N},+)$ generated by $\left\{b^{n+i}-1\mid i\in\mathbb{N}\right\}$. It is clear that $S(b,n)=\left\{\frac{m}{b-1}\mid m\in M(b,n)\right\}$. Hence, we get that $\gcd(M(b,n))=b-1$ and the map $\varphi: M(b,n)\longrightarrow S(b,n)$, defined by $\varphi(m)=\frac{m}{b-1}$, is a monoid isomorphism. Consequently, if X is the minimal system of generators of $M(b,n)$, then $\left\{\frac{x}{b-1}\mid x\in X\right\}$ is the minimal system of generators of $S(b,n)$.

The next result gives a property verified in the monoid $M(b,n)$, which is the key to the development of this study.

LEMMA 116. *If $m\in M(b,n)\setminus\{0\}$, then $bm+b-1\in M(b,n)$.*

PROOF. Since $M(b,n)=\langle\left\{b^{n+i}-1\mid i\in\mathbb{N}\right\}\rangle$ and $b\left(b^{n+i}-1\right)+b-1=b^{n+i+1}-1\in M(b,n)$ then, by Lemma 72, we get that $bm+b-1\in M(b,n)$ for all $m\in M(b,n)\setminus\{0\}$. $\square$

Observe that if X and Y are non empty sets of positive integer numbers such that $Y\subseteq X$ and $X\subseteq\langle Y\rangle$, then clearly $\langle X\rangle=\langle Y\rangle$.

LEMMA 117. *The set $\left\{b^{n+i}-1\mid i\in\{0,\ldots,n-1\}\right\}$ is a system of generators of $M(b,n)$.*

PROOF. Assume that $M = \langle \{ b^{n+i} - 1 \mid i \in \{0, \ldots, n-1\} \} \rangle$. First, we show that if $m \in M \backslash \{0\}$ then $bm + b - 1 \in M$. For $n = 1$ the result is true. Thus, suppose that $n \geq 2$. If $i \in \{0, \ldots, n-2\}$, then $b\left(b^{n+i} - 1\right) + b - 1 = b^{n+i+1} - 1 \in M$. Moreover $b\left(b^{2n-1} - 1\right) + b - 1 = b^{2n} - 1 = (b^n - 1)(b^n + 1) \in M$. By using Lemma 72, we obtain that $bm + b - 1 \in M$ for all $m \in M \backslash \{0\}$.

Now we show that $M(b,n) = M$. It is enough to show that $b^{n+i} - 1 \in M$ for all $i \in \mathbb{N}$. We proceed by induction on i. For $i = 0$ the result is true. Assume that the statement is true for i and let us show it for $i+1$. As $b^{n+i+1} - 1 = b\left(b^{n+i} - 1\right) + b - 1$ then, by the hypothesis of induction and from de fact that $bm + b - 1 \in M$ for all $m \in M \backslash \{0\}$, we get that $b^{n+i+1} - 1 \in M$. $\square$

Now we are ready to show that the system of generators of $M(b,n)$ given in previous lemma is minimal.

THEOREM 118. *The set* $\left\{ b^{n+i} - 1 \mid i \in \{0, 1, \ldots, n-1\} \right\}$ *is the minimal system of generators of* $M(b,n)$.

PROOF. If $n = 1$ the result is trivially true. Thus, suppose that $n \geq 2$. First we prove that $b^{2n-1} - 1 \notin \langle \{ b^{n+i} - 1 \mid i \in \{0, 1, \ldots, n-2\} \} \rangle$. Otherwise, there exist $a_0, \ldots, a_{n-2} \in \mathbb{N}$ such that $b^{2n-1} - 1 = a_0(b^n - 1) + \cdots + a_{n-2}\left(b^{2n-2} - 1\right) = a_0 b^n + \cdots + a_{n-2} b^{2n-2} - (a_0 + \cdots + a_{n-2})$. Therefore $a_0 + \cdots + a_{n-2} \equiv 1 (\bmod\ b^n)$ implies that $a_0 + \cdots + a_{n-2} = 1 + kb^n$ for some $k \in \mathbb{N} \backslash \{0\}$ and thus $a_0 + \cdots + a_{n-2} \geq 1 + b^n$. Consequently, we have that $b^{2n-1} - 1 = a_0(b^n - 1) + \cdots + a_{n-2}\left(b^{2n-2} - 1\right) \geq (a_0 + \cdots + a_{n-2})(b^n - 1) \geq (1 + b^n)(b^n - 1) = b^{2n} - 1 > b^{2n-1} - 1$, which is impossible.

Now by Lemma 117, we know that $\left\{ b^{n+i} - 1 \mid i \in \{0, \ldots, n-1\} \right\}$ is a system of generators of $M(b,n)$. If it is not the minimal system of generators, then there exists $h \in \{1, \ldots, n-1\}$ such that $b^{n+h} - 1 \in \langle \{ b^{n+i} - 1 \mid i \in \{0, 1, \ldots, h-1\} \} \rangle$.

Let $M = \langle \{ b^{n+i} - 1 \mid i \in \{0, \ldots, h-1\} \} \rangle$. If $i \in \{0, \cdots, h-2\}$ then $b\left(b^{n+i} - 1\right) + b - 1 = b^{n+i+1} - 1 \in M$. Furthermore, in view of the previous paragraph,

$b\left(b^{n+h-1}-1\right)+b-1=b^{n+h}-1\in M$. Therefore, by using Lemma 72 we obtain that $bm+b-1\in M$ for all $m\in M\backslash\{0\}$.

Using induction on i it is easy to show that $b^{n+i}-1\in M$ for all $i\in\mathbb{N}$. For $i=0$ the result is true. Assume that the result holds for i. By induction hypothesis and setting $b^{n+i+1}-1=b\left(b^{n+i}-1\right)+b-1$ we can deduce that $b^{n+i+1}-1\in M$. As a consequence we have that $b^{2n-1}-1\in M\subseteq\langle\left\{b^{n+i}-1\mid i\in\{0,\ldots,n-2\}\right\}\rangle$, which contradicts the fact that $b^{2n-1}-1\notin\langle\left\{b^{n+i}-1\mid i\in\{0,1,\ldots,n-2\}\right\}\rangle$. $\square$

As a consequence of the previous theorem, we have the following statement.

COROLLARY 119. *The numerical semigroup $S(b,n)$ has embedding dimension n. Moreover, its minimal system of generators is* $\left\{\frac{b^{n+i}-1}{b-1}\mid i\in\{0,\ldots,n-1\}\right\}$.

Note that $\mathbb{N}$ is the unique Repunit numerical semigroup with embedding dimension 1. Clearly, as a consequence of previous results in this section, we obtain that, for n greater than or equal 2, there are infinitely many Repunit numerical semigroups with embedding dimension n. Specifically, this set is equal to $\{S(b,n)\mid b\in\mathbb{N}\backslash\{0,1\}\}$. For example, the set of all Repunit numerical semigroups with embedding dimension 3 is equal to $\{S(b,3)\mid b\in\mathbb{N}\backslash\{0,1\}\}=\left\{\langle\frac{b^3-1}{b-1},\frac{b^4-1}{b-1},\frac{b^5-1}{b-1}\rangle\mid b\in\mathbb{N}\backslash\{0,1\}\right\}=\{\langle\{7,15,31\}\rangle,\langle\{13,40,121\}\rangle,\ldots\}$.

3.2. The Apéry set. The knowledge of $\mathrm{Ap}(S,x)$ for some $x\in S\backslash\{0\}$ gives us enough information about S.

Motivated by the definitions, Lemma 5 and Proposition 10 we will extend the concept of the Apéry set of numerical semigroups to the submonoids of $(\mathbb{N},+)$. If M is a submonoid of $(\mathbb{N},+)$ and $m\in M\backslash\{0\}$ then Apéry set of m in M is $\mathrm{Ap}(M,m)=\{x\in M\mid x-m\notin M\}$. The next result is easy to prove.

LEMMA 120. *Let M be a submonoid of $(\mathbb{N},+)$ such that $M\neq\{0\}$ and let $d=\gcd(M)$. Then:*

(1) $S=\left\{\frac{m}{d}\mid m\in M\right\}$ is a numerical semigroup;

(2) if $m\in M\backslash\{0\}$ then $\mathrm{Ap}(M,m)=\left\{dw\mid w\in\mathrm{Ap}(S,\frac{m}{d})\right\}$;

(3) the cardinality of $\mathrm{Ap}(M,m)$ *is* $\frac{m}{d}$.

From now on we will denote by m_i the elements $b^{n+i}-1$ for each $i \in \{0,1,\ldots,n-1\}$. Observe that with this notation we have that $\{m_0, m_1, \ldots, m_{n-1}\}$ is the minimal system of generators of $M(b,n)$ and $\left\{\frac{m_0}{b-1}, \frac{m_1}{b-1}, \ldots, \frac{m_{n-1}}{b-1}\right\}$ is the minimal system of generators of $S(b,n)$.

Our next aim is to prove Theorem 123, which describes the set $\mathrm{Ap}(M(b,n),m_0)$. It is easy to prove the following result.

LEMMA 121. *Let n be an integer grater than or equal* 2. *Then:*

(1) if $0 < i \leq j < n-1$ *then* $m_i + bm_j = bm_{i-1} + m_{j+1}$*;*

(2) if $0 < i \leq n-1$ *then* $m_i + bm_{n-1} = bm_{i-1} + (b^n+1)\,m_0$.

We denote by $R(b,n)$ the set of all $n-1$-tuple $(a_1, \ldots, a_{n-1})$ that verify the following conditions:

(1) for every $i \in \{1,\ldots,n-1\}$ we have that $a_i \in \{0,1,\ldots,b\}$;

(2) if $i \in \{2,\ldots,n-1\}$ and $a_i = b$ then $a_1 = \cdots = a_{i-1} = 0$.

The following result expresses the interest of the set $R(b,n)$.

LEMMA 122. *Let n be an integer greater than or equal to two. If* $x \in \mathrm{Ap}(M(b,n),m_0)$ *then there exists* $(a_1,\ldots,a_{n-1}) \in R(b,n)$ *such that* $x = a_1m_1 + \cdots + a_{n-1}m_{n-1}$.

PROOF. We can use induction on x for the proof. For $x=0$ the result is clear. Suppose that $x>0$ and let $j = \min\{i \in \{0,\ldots,n-1\} \mid x - m_i \in M(b,n)\}$. Observe that, as $x \in \mathrm{Ap}(M(b,n),m_0)$ we obtain that $j \neq 0$. By induction hypothesis, there exists $(a_1,\ldots,a_{n-1}) \in R(b,n)$ such that $x - m_j = a_1m_1 + \cdots + a_{n-1}m_{n-1}$. Therefore $x = a_1m_1 + \cdots + \left(a_j+1\right)m_j + \cdots + a_{n-1}m_{n-1}$.

To conclude the proof, we will see that $\left(a_1,\ldots,a_j+1,\ldots,a_{n-1}\right) \in R(b,n)$. If $a_j + 1 = b+1$, then by applying Lemma 121, it fulfills one of the conditions.

. if $j < n-1$ then $(a_j+1)m_j = (b+1)\,m_j = bm_{j-1} + m_{j+1}$;

. if $j = n-1$ then $(a_j+1)m_j = (b+1)m_j = bm_{j-1} + (b^n+1)m_0$.

In both cases we deduce that $x - m_{j-1} \in M(b,n)$, which contradicts the minimality of j.

Now suppose there exists $k > j$ such that $a_k = b$ then, by using Lemma 121, we have the following conditions.

. if $k < n-1$ then $m_j + bm_k = bm_{j-1} + m_{k+1}$;

. if $k = n-1$ then $m_j + bm_k = bm_{j-1} + (b^n+1)m_0$.

In both cases we obtain again that $x - m_{j-1} \in M(b,n)$, which contradicts the minimality of j. Moreover, from the minimality of j we know that $a_1 = \cdots = a_{j-1} = 0$.

So we can conclude that $\left(a_1, \ldots, a_j+1, \ldots, a_{n+1}\right) \in R(b,n)$. $\square$

Before we state the following result let us observe that if h is a positive integer, then the sequence of numbers $b^n, b^{n+1}, \ldots, b^{n+h}$ is a geometric progression with common ratio b, it follows that $b^n + b^{n+1} + \cdots + b^{n+h} = \frac{b^{n+h+1}-b^n}{b-1}$.

THEOREM 123. *Let n be an integer greater than or equal to two. Then* $\mathrm{Ap}(M(b,n), m_0) = \{a_1m_1 + \cdots + a_{n-1}m_{n-1} \mid (a_1, \ldots, a_{n-1}) \in R(b,n)\}$.

PROOF. Clearly $R(b,n) = \{0, \ldots, b-1\}^{n-1} \cup \{(a_1, \ldots, a_{n-1}) \in R(b,n) \mid a_1 = b\} \cup \cdots \cup \{(a_1, \ldots, a_{n-1}) \in R(b,n) \mid a_{n-1} = b\}$. Then $R(b,n)$ is the disjoint union of these sets and therefore the cardinality of $R(b,n)$ is equal to $b^{n-1} + b^{n-2} + \cdots + b^0 = \frac{b^n-1}{b-1} = \frac{m_0}{b-1}$. By using Lemma 122, we have that $\mathrm{Ap}(M(b,n), m_0) \subseteq \{a_1m_1 + \cdots + a_{n-1}m_{n-1} \mid (a_1, \ldots, a_{n-1}) \in R(b,n)\}$. In view of Lemma 120 the cardinality of $\mathrm{Ap}(M(b,n), m_0)$ is equal to $\frac{m_0}{b-1}$. Furthermore, from the previous paragraph, we know that the cardinality of the set $\{a_1m_1 + \cdots + a_{n-1}m_{n-1} \mid (a_1, \ldots, a_{n-1}) \in R(b,n)\}$ is less than or equal to $\frac{m_0}{b-1}$. Hence we conclude that $\mathrm{Ap}(M(b,n), m_0) = \{a_1m_1 + \cdots + a_{n-1}m_{n-1} \mid (a_1, \ldots, a_{n-1}) \in R(b,n)\}$. $\square$

As a consequence of the proof of previous theorem we obtain the following result.

COROLLARY 124. *Let n be an integer greater than or equal to two and let $(a_1,\dots,a_{n-1})$ and $(b_1,\dots,b_{n-1})$ be two distinct elements in $R(b,n)$. Then it follows that $a_1m_1+\cdots+a_{n-1}m_{n-1}\neq b_1m_1+\cdots+b_{n-1}m_{n-1}$*

As an immediate consequence of Lemma 120 and Theorem 123, we have the following result.

COROLLARY 125. *Let n be an integer greater than or equal to two. Then* $\mathrm{Ap}\left(S(b,n),\frac{m_0}{b-1}\right)=\left\{a_1\frac{m_1}{b-1}+\cdots+a_{n-1}\frac{m_{n-1}}{b-1}\mid(a_1,\dots,a_{n-1})\in R(b,n)\right\}$.

We alrealdy seen that the least positive integer belonging to a numerical semigroup S is called its multiplicity and is denoted by m(S). Recall that a numerical semigroup S has a monotonic Apéry set if $w(1)<w(2)<\cdots<w(m(S)-1)$ where $w(i)$ is the least element of S congruent with i modulo m(S) for all $i\in\{1,\dots,m(S)-1\}$. Our next goal is to prove that $S(b,n)$ is a numerical semigroups with a monotonic Apéry set. Note that not every numerical semigroup is of this form. In fact, if $S=\langle 5,7,9\rangle$ then we have that $\mathrm{m}(S)=5$ and $\mathrm{Ap}(S,5)=\{w(0)=0,w(1)=16,w(2)=7,w(3)=18,w(4)=9\}$.

LEMMA 126. *Under the standing notation and $n\geq 2$. If $x\in S(b,n)$ and $x\not\equiv 0 \bmod \frac{m_0}{b-1}$ then $x-1\in S(b,n)$.*

PROOF. If $x\in S(b,n)$, then there exists $a_0,\dots,a_{n-1}\in\mathbb{N}$ such that $x=a_0\frac{m_0}{b-1}+\cdots+a_{n-1}\frac{m_{n-1}}{b-1}$. Besides, if $x\not\equiv 0 \bmod \frac{m_0}{b-1}$ then there exists $i\in\{1,\dots,n-1\}$ such that $a_i\neq 0$. Therefore,

$$x-1=a_0\frac{m_0}{b-1}+\cdots+(a_i-1)\frac{m_i}{b-1}+\cdots+a_{n-1}\frac{m_{n-1}}{b-1}+\frac{m_i}{b-1}-1.$$

Since $\frac{m_i}{b-1}-1=\frac{b^{n+i}-1}{b-1}-1=\frac{b^{n+i}-b}{b-1}=b\frac{b^{n+i-1}-1}{b-1}=b\frac{m_i}{b-1}$, we have that

$$x-1=a_0\frac{m_0}{b-1}+\cdots+(a_{i-1}+b)\frac{m_{i-1}}{b-1}+(a_i-1)\frac{m_i}{b-1}\cdots+a_{n-1}\frac{m_{n-1}}{b-1}$$

belongs to $S(b,n)$. □

Now we can prove the result announced above.

PROPOSITION 127. *Under the standing notation and $n\geq 2$. Then $S(b,n)$ is a numerical semigroups with a monotonic Apéry set.*

PROOF. In view of Corollary 119, we can deduce that $m(S(b,n)) = \frac{m_0}{b-1}$. To conclude the proof we only need to prove that $w(i) < w(i+1)$, where $w(i)$ is the least element of S congruent with i modulo $\frac{m_0}{b-1}$ for all $i \in \{1,\ldots,\mathrm{m}(S)-1\}$. Since $w(i+1) \in S(b,n)$ and $w(i+1) \not\equiv 0 \bmod \frac{m_0}{b-1}$, then by Lemma 126 we obtain that $w(i+1)-1 \in S(b,n)$. Thus $w(i) \leq w(i+1)-1$ because $w(i+1)-1 \equiv i \bmod \frac{m_0}{b-1}$. $\square$

Next we illustrate the previous results with an example.

EXAMPLE 128. Let us compute $\mathrm{Ap}\left(S(3,3), \frac{3^3-1}{3-1}\right) = \mathrm{Ap}(\langle 13,40,121\rangle, 13)$. It is easy to check that

$$R(3,3) = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),\\ (3,0),(3,1),(3,2),(0,3)\}.$$

Applying Corollary 125, we get that

$$\mathrm{Ap}(\langle 13,40,121\rangle, 13) = \{0,121,242,40,161,282,80,201,322,120,241,362,363\}.$$

Furthermore, by using Proposition 127, we get that $w(0) = 0, w(1) = 40, w(2) = 80, w(3) = 120, w(4) = 121, w(5) = 161, w(6) = 201, w(7) = 241, w(8) = 242, w(9) = 282, w(10) = 322, w(11) = 362$ and $w(12) = 363$.

From Remark 7 and using the previous example, it easily follows that $265 \in S(3,3)$ and $270 \notin S(3,3)$, because $265 \geq w(265 \bmod 13) = w(5) = 161$ and $270 < w(270 \bmod 13) = w(10) = 322$.

3.3. The Frobenius problem. Our next purpose is to give a formula for the Frobenius number of $S(b,n)$. The next result has an immediate proof.

LEMMA 129. *Let n be an integer greater than or equal to three. Then the maximal elements (with respect to the product order) in $R(b,n)$ are $(b,b-1,\ldots,b-1)$, $(0,b,b-1,\ldots,b-1)$ and $(0,\ldots,0,b)$.*

The next result shows that $bm_1+(b-1)m_2+\cdots+(b-1)m_{n-1}, bm_2+(b-1)m_3+\cdots+(b-1)m_{n-1},\ldots,bm_{n-1}$ is a sequence of integers where each term is obtained from the previous by adding $b-1$.

LEMMA 130. *Let n be an integer greater than or equal to three and let $i \in \{1,\ldots,n-2\}$, then $bm_i+b-1=m_{i+1}$.*

PROOF. In fact, $bm_i+b-1=b\left(b^{n+i}-1\right)+b-1=b^{n+i+1}-1=m_{i+1}$. □

Now we can state the result announced previously.

THEOREM 131. *Let n be an integer greater than or equal to two. Then $\mathrm{F}(S(b,n))=\frac{b^n-1}{b-1}b^n-1$.*

PROOF. From Corollary 125 and Lemmas 129 and 130, we deduce that $\max\left(\mathrm{Ap}\left(S(b,n),\frac{m_0}{b-1}\right)\right)=b\frac{m_{n-1}}{b-1}$. By applying Proposition 10 we obtain that $\mathrm{F}(S(b,n))=b\frac{m_{n-1}}{b-1}-\frac{m_0}{b-1}=\frac{b(b^{2n-1}-1)}{b-1}-\frac{b^n-1}{b-1}=\frac{b^n-1}{b-1}b^n-1$. □

Note that for $n=1$ the previous formula is not true, because $S(b,1)=\mathbb{N}$ and $\mathrm{F}(\mathbb{N})=-1\neq\frac{b-1}{b-1}b-1=b-1$.

EXAMPLE 132. Let us compute the Frobenius number of the numerical semigroup $S(3,4)=\langle\frac{3^4-1}{3-1},\frac{3^5-1}{3-1},\frac{3^6-1}{3-1},\frac{3^7-1}{3-1}\rangle=\langle 40,121,364,1093\rangle$. By using Theorem 131 we obtain that $\mathrm{F}(S(3,4))=\frac{3^4-1}{3-1}3^4-1=3239$.

Our next goal is to determine the set of all pseudo-Frobenius number and the type of $S(b,n)$.

Note that, as a consequence of Lemma 11, if we want to compute the pseudo-Frobenius number of $S(b,n)$ it suffices to determine the set of $\text{maximals}_{\leq_{S(b,n)}}\left(\mathrm{Ap}\left(S(b,n),\frac{m_0}{b-1}\right)\right)$.

THEOREM 133. *Let n be an integer greater than or equal to two. Then $\mathrm{t}(S(b,n))=n-1$. Moreover $\mathrm{PF}(S(b,n))=\{\mathrm{F}(S(b,n))-i\mid i\in\{0,\ldots,n-2\}\}$.*

PROOF. Assume that A is the set of maximal elements in $R(b,n)$ (with respect to the product order) and $B = \left\{a_1\frac{m_1}{b-1} + \cdots + a_{n-1}\frac{m_{n-1}}{b-1} \mid (a_1,\ldots,a_{n-1}) \in A\right\}$. From Corollary 125, we deduce that $\text{maximals}_{\leq_{S(b,n)}}\left(\text{Ap}\left(S(b,n), \frac{m_0}{b-1}\right)\right) = \text{maximals}_{\leq_{S(b,n)}} B$. Then, by applying Lemmas 129 and 130, the set B is formed by $n-1$ consecutive positive integers. Hence the difference between any two elements in B is smaller than or equal to $n-2$. As $\frac{m_0}{b-1} = \frac{b^n-1}{b-1}$ is the smallest positive integer in $S(b,n)$ and $\frac{b^n-1}{b-1} > n-2$, we can conclude that B is a set of incomparable elements with respect to the $\leq_{S(b,n)}$ order and thus $\text{maximals}_{\leq_{S(b,n)}} B = B$.

Now by using Lemma 11 we obtain that $\text{PF}(S(b,n)) = \left\{w - \frac{m_0}{b-1} \mid w \in B\right\}$. From the proof of Theorem 131 we have that $\max(B) = \text{F}(S(b,n)) + \frac{m_0}{b-1}$ and consequently $\text{PF}(S(b,n)) = \{\text{F}(S(b,n)) - i \mid i \in \{0,\ldots,n-2\}\}$. $\square$

Note that for $n = 1$ the previous theorem is not true, because $S(b,1) = \mathbb{N}$, $\text{PF}(\mathbb{N}) = \{-1\}$ and so $\text{t}(\mathbb{N}) = 1$. Notice also that for each positive integer n there are infinitely many Repunit numerical semigroups of type n. Specifically, this set is equal to $\{S(b,n+1) \mid b \in \mathbb{N}\backslash\{0,1\}\}$ coincides with the set of all Repunit numerical semigroups with embedding dimension $n+1$.

EXAMPLE 134. Let us compute the Pseudo-Frobenius numbers of the numerical semigroup $S(3,4)$. From Example 132 we know that $\text{F}(S(3,4)) = 3239$. By applying Theorem 133 we have that $\text{PF}(S(b,n)) = \{3239, 3238, 3237\}$.

The next result gives a formula for the genus of a Repunit numerical semigroup.

THEOREM 135. *Let n be a positive integer. Then* $\text{g}(S(b,n)) = \frac{b^n}{2}\left(\frac{b^n-b}{b-1} + n - 1\right)$.

PROOF. For $n = 1$ the result is trivial. Now assume that $n \geq 2$ and for each $i \in \{0,\ldots,n-1\}$ define $s_i = \frac{m_i}{b-1}$. By using Proposition 10 and Corollaries 124 and 125 we have that

$$\text{g}(S(b,n)) = \frac{1}{s_0}\left(\sum_{(a_1,\ldots,a_{n-1}) \in R(b,n)} a_1 s_1 + \cdots + a_{n-1}s_{n-1}\right) - \frac{s_0 - 1}{2}.$$

Clearly,

$$\sum_{(a_1,\dots,a_{n-1})\in R(b,n)} a_1s_1+\cdots+a_{n-1}s_{n-1}=$$
$$=\sum_{(a_1,\dots,a_{n-1})\in R(b,n),\ a_1=1} s_1+\cdots+\sum_{(a_1,\dots,a_{n-1})\in R(b,n),\ a_1=b} bs_1+\cdots$$
$$\cdots+\sum_{(a_1,\dots,a_{n-1})\in R(b,n),\ a_{n-1}=1} s_{n-1}+\cdots+\sum_{(a_1,\dots,a_{n-1})\in R(b,n),\ a_{n-1}=b} bs_{n-1}.$$

Let $i\in\{1,\cdots,n-1\}$. The reader can prove that:

- the cardinality of $\{(a_1,\dots,a_{n-1})\in R(b,n)\mid a_i=b\}$ is b^{n-1-i};
- if $x\in\{1,\dots,b-1\}$, then the cardinality of $\{(a_1,\dots,a_{n-1})\in R(b,n)\mid a_i=x \text{ and } b\notin\{a_1,\dots,a_{i-1}\}\}$ is b^{n-2};
- if $1\le j<i$ then the cardinality of $\{(a_1,\dots,a_{n-1})\in R(b,n)\mid a_i=x \text{ and } a_j=b\}$ is b^{n-j-2}.

Therefore, applying the previous results, we have that

$$
\begin{aligned}
&\sum_{(a_1,\dots,a_{n-1})\in R(b,n)} a_1s_1+\cdots+a_{n-1}s_{n-1} = \\
&= \sum_{x=1}^{b-1}\left(x\sum_{i=1}^{n-1}\left(b^{n-2}+b^{n-3}+\cdots+b^{n-1-i}\right)s_i\right)+b\sum_{i=1}^{n-1}b^{n-1-i}bs_i = \\
&= \frac{b(b-1)}{2}\sum_{i=1}^{n-1}\left(b^{n-2}+\cdots+b^{n-1-i}\right)s_i+b\sum_{i=1}^{n-1}b^{n-i-1}s_i = \\
&= \frac{b(b-1)}{2}\sum_{i=1}^{n-1}\frac{b^{n-1}-b^{n-i-1}}{b-1}s_i+b\sum_{i=1}^{n-1}b^{n-i-1}s_i = \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\left(b^n-b^{n-i}\right)s_i+\sum_{i=1}^{n-1}b^{n-i}s_i=\frac{1}{2}\sum_{i=1}^{n-1}\left(b^n+b^{n-i}\right)s_i = \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\left(b^n+b^{n-i}\right)\left(\frac{b^{n+i}-1}{b-1}\right)=\frac{1}{2(b-1)}\sum_{i=1}^{n-1}\left(b^{2n+i}-b^n+b^{2n}-b^{n-i}\right) = \\
&= \frac{1}{2(b-1)}\left(\frac{b^{3n}-b^{2n+1}}{b-1}-(n-1)b^n+(n-1)b^{2n}-\left(\frac{b^n-b}{b-1}\right)\right) = \\
&= \frac{1}{2(b-1)}\left(\frac{\left(b^n-1\right)\left(b^{2n}-b^{n+1}+b^n-b\right)}{b-1}+(n-1)b^n\left(b^n-1\right)\right) = \\
&= \frac{b^n-1}{2(b-1)}\left(\frac{b^{2n}-b^{n+1}+b^n-b}{b-1}+(n-1)b^n\right).
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
\mathrm{g}(S(b,n)) &= \frac{1}{2}\left(\frac{b^{2n}-b^{n+1}+b^n-b}{b-1}+(n-1)b^n\right)-\frac{\frac{b^n-1}{b-1}-1}{2} = \\
&= \frac{1}{2}\left(\frac{b^{2n}-b^{n+1}+b^n-b}{b-1}+(n-1)b^n-\frac{b^n-b}{b-1}\right) = \\
&= \frac{1}{2}\left(\frac{b^{2n}-b^{n+1}}{b-1}+(n-1)b^n\right)=\frac{b^n}{2}\left(\frac{b^n-b}{b-1}+n-1\right).
\end{aligned}
$$

□

EXAMPLE 136. Let us compute the genus of the numerical semigroup $S(3,4)$. By applying Theorem 135 we have that $\mathrm{g}(S(3,4))=\frac{3^4}{2}\left(\frac{3^4-3}{3-1}+4-1\right)=1701$.

CHAPTER 4

Combinatory optimization problems

In this chapter we will study the digital semigroups and the bracelet monoids. This study is fullfilled in sections 1 and 2, respectively, and were published in [**33**] and [**30**].

A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\},\cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$ with $\ell(n)$ the number of digits of n written in decimal expansion. In Section 1, we compute the smallest digital semigroup containing a set of positive integers. For this, we establish a connection between the digital semigroups and a class of numerical semigroups called LD-semigroups.

Given positive integers $n_1,\ldots,n_p$, we say that a submonoid M of $(\mathbb{N},+)$ is a $(n_1,\ldots,n_p)$-bracelet if $a+b+\left\{n_1,\ldots,n_p\right\} \subseteq M$ for every $a,b \in M\backslash\{0\}$. In Section 2, we explicitly describe the smallest $(n_1,\ldots,n_p)$-bracelet that contains a finite subset X of $\mathbb{N}$. We also present a recursive method that enables us to construct the whole set $\mathcal{B}(n_1,\ldots,n_p) = \left\{M \mid M \text{ is a } (n_1,\ldots,n_p)-\text{bracelet}\right\}$. Finally, we study $(n_1,\ldots,n_p)$-bracelets that cannot be expressed as the intersection of $(n_1,\ldots,n_p)$-bracelets properly containing it.

1. Sets of positive integers closed under product and the number of decimal digits

Given, A a subset of $\mathbb{N}\backslash\{0\}$, we denote by $L(A) = \{\ell(a) \mid a \in A\}$. We prove that if D a digital semigroup then $L(D)\cup\{0\}$ is a numerical semigroup. A numerical semigroup S is called LD-semigroup if there exist a digital semigroup D such that $S = L(D)\cup\{0\}$. The results presented in this section can be found in [**33**].

Denote by $\mathcal{D}$ (respectively $\mathcal{L}$) the set of all digital semigroups (respectively LD-semigroups). We see that the map $\varphi : \mathcal{D} \longrightarrow \mathcal{L}$ defined by $\varphi(D) = L(D)\cup\{0\}$ is bijective and its inverse is the map $\theta : \mathcal{L} \longrightarrow \mathcal{D}$ with $\theta(S) = \{a \in \mathbb{N}\backslash\{0\} \mid \ell(a) \in S\}$. From this it easily follows that if D is a digital semigroup then $\mathbb{N}\backslash D$ is finite.

This fact together with the results presented in Section 3 of the Preliminaries allows us to arrange the elements of $\mathcal{L}$ in a tree. We characterize the childs of any vertex of this tree and this will enable us to recursively construct the set $\mathcal{L}$ and consequently the set $\mathcal{D}$.

Given a set of positive integers X we denote by $\mathcal{D}(X)$ (respectively $\mathcal{L}(X)$) the smallest (with respect to the set inclusion order) digital semigroup containing X (respectively LD-semigroup). We prove that if X is a set of positive integers and S the smallest LD-semigroup containing $L(X)$ then $\theta(S)$ is the smallest digital semigroup containing X. As a first consequence of this we get that $\mathcal{D} = \{\mathcal{D}(X) \mid \text{ X is a nonempty finite subset of } \mathbb{N}\backslash\{0\}\}$ whence every digital semigroup can be described from a finite number of terms.

Given a finite set of positive integers X we describe an algorithmic procedure for computing the smallest LD-semigroup that contains X. As a consequence we have an algorithm that computes the smallest digital semigroup containing a finite set of positive integers.

1.1. LD-semigroups. The next result establish a relation between a digital semigroup and a LD-semigroup.

PROPOSITION 137. *If D is a digital semigroup, then $L(D) \cup \{0\}$ is a numerical semigroup.*

PROOF. Let $x, y \in L(D)$. Since $\ell(9 \times 10^{x-1}) = x$ and $\ell(9 \times 10^{y-1}) = y$ we get that $9 \times 10^{x-1}, 9 \times 10^{y-1} \in D$ and thus $81 \times 10^{x+y-2} \in D$. But we have that $81 \times 10^{x+y-2} = (8 \times 10 + 1)10^{x+y-2} = 8 \times 10^{x+y-1} + 10^{x+y-2}$ and so $\ell(81 \times 10^{x+y-2}) = x + y$. Therefore $x + y \in L(D)$ and consequently $L(D) \cup \{0\}$ is a submonoid of $(\mathbb{N}, +)$.

Let $d \in D$. Then $10^{\ell(d)-1} \in D$ and thus $10^{2\ell(d)-2} = 10^{\ell(d)-1} \times 10^{\ell(d)-1} \in D$. Since $\ell(10^{2\ell(d)-2}) = 2\ell(d) - 1$, obviously $\{\ell(d),\ 2\ell(d) - 1\} \subseteq L(D)$. Thus we conclude that $L(D) \cup \{0\}$ is a numerical semigroup. $\square$

Observe that not every numerical semigroup is of this form. In fact, by applying Proposition 137, we deduce that if S is a LD-semigroup and $x \in S\backslash\{0\}$ then $2x-1 \in S$. Then we have that $S = \langle 4,5\rangle$ is not a LD-semigroup, because $2\times 4-1 \notin S$.

Next we describe a characterization of LD-semigroups.

LEMMA 138. *Let x and y be positive integers. Then $\ell(xy) \in \{\ell(x)+\ell(y), \ell(x)+\ell(y)-1\}$.*

PROOF. As $10^{\ell(x)-1} \leq x < 10^{\ell(x)}$ and $10^{\ell(y)-1} \leq y < 10^{\ell(y)}$, we have that $10^{\ell(x)+\ell(y)-2} \leq xy < 10^{\ell(x)+\ell(y)}$. Therefore $\ell(x)+\ell(y)-1 \leq \ell(xy) < \ell(x)+\ell(y)+1$ and consequently $\ell(xy) \in \{\ell(x)+\ell(y)-1, \ell(x)+\ell(y)\}$. $\square$

THEOREM 139. *Let S be a numerical semigroup. The following conditions are equivalent.*

1) *S is a LD-semigroup.*
2) *If $a,b \in S\backslash\{0\}$ then $a+b-1 \in S$.*

PROOF. *1) implies 2).* Assume that S is a LD-semigroup, then there exists a digital semigroup D such that $S = L(D)\cup\{0\}$. If $a,b \in S\backslash\{0\}$ then $10^{a-1}, 10^{b-1} \in D$ and thus $10^{a+b-2} \in D$. Consequently $a+b-1 = \ell(10^{a+b-2}) \in L(D) \subseteq S$.

2) implies 1). Let $D = \{a \in \mathbb{N}\backslash\{0\} \mid \ell(a) \in S\}$. It is clear $S = L(D)\cup\{0\}$. In order to conclude the proof, it suffices to show that D is a digital semigroup. It is clear that if $d \in D$ then $\{a \in \mathbb{N}\backslash\{0\} \mid l(a) = l(d)\} \subseteq D$. Let us see that D is closed under product. In fact, if $a,b \in D$ then by Lemma 138 we deduce that $\ell(ab) \in \{\ell(a)+\ell(b), \ell(a)+\ell(b)-1\}$. This implies that $\ell(ab) \in S$, and consequently $ab \in D$. $\square$

Let $\mathcal{D} = \{D \mid D \text{ is a digital semigroup}\}$ and let $\mathcal{L} = \{S \mid S \text{ is a LD-semigroup}\}$. As a consequence of the proof of Theorem 139 we obtain the following result.

COROLLARY 140. *The correspondence $\varphi : \mathcal{D} \to \mathcal{L}$, defined by $\varphi(D) = L(D)\cup\{0\}$, is a bijective map. Furthermore its inverse is the map $\theta : \mathcal{L} \to \mathcal{D}$, $\theta(S) = \{a \in \mathbb{N}\backslash\{0\} \mid \ell(a) \in S\}$.*

From this result one easily deduces the following alternative characterization.

COROLLARY 141. *With the above notation, we have that*

$$\mathcal{D} = \{\theta(S) \mid S \text{ is a LD-semigroup}\}.$$

If $x_1, x_2, \ldots, x_k$ are integers, we denote by $\{x_1, x_2, \ldots, x_k, \rightarrow\}$ the set $\{x_1, x_2, \ldots, x_k\} \cup \{z \in \mathbb{Z} \mid z > x_k\}$. Given a positive integer n, we denote by $\triangle(n) = \{x \in \mathbb{N} \backslash \{0\} \mid \ell(x) = n\}$.

EXAMPLE 142. Let $S = \langle 3,5,7 \rangle = \{0,3,5,6,7,\rightarrow\}$. By using Theorem 139, we obtain that S is a LD-semigroup. Since $\mathbb{N}\backslash S = \{1,2,4\}$, in view of Corollary 141, we deduce that $\mathbb{N}\backslash(\triangle(1) \cup \triangle(2) \cup \triangle(4) \cup \{0\})$ is a digital semigroup.

COROLLARY 143. *If D is a digital semigroup, then $\mathbb{N}\backslash D$ is finite.*

PROOF. By Corollary 141 there exists a LD-semigroup S such that $D = \theta(S)$. Since $\left\{x \in \mathbb{N} \mid x \geq 10^{\mathrm{F}(S)}\right\} \subseteq D$ it follows that $\mathbb{N}\backslash D$ is finite. $\square$

COROLLARY 144. *Let S be a LD-semigroup not equal to $\mathbb{N}$ and let $\mathbb{N}\backslash S = \{h_1 = 1 < \cdots < h_t = \mathrm{F}(S)\}$. Then*

1) $\mathrm{F}(\theta(S)) = 10^{\mathrm{F}(S)} - 1$,
2) $\mathrm{g}(\theta(S)) = 9 \times \left(10^{h_1 - 1} + \cdots + 10^{h_t - 1}\right) + 1$.

PROOF. 1) It is enough to observe that $10^{\mathrm{F}(S)} - 1 = \max\{n \in \mathbb{N}\backslash\{0\} \mid \ell(n) = \mathrm{F}(S)\}$

2) Since $\theta(S) = \mathbb{N}\backslash(\triangle(h_1) \cup \cdots \cup \triangle(h_t) \cup \{0\})$, then $\mathrm{g}(\theta(S)) = \mathrm{cardinal}(\triangle(h_1) \cup \cdots \cup \triangle(h_t) \cup \{0\})$. In order to conclude the proof, it suffices to observe that if $i \in \{1, \ldots, t\}$ then $\triangle(h_i) = \left\{10^{h_i - 1}, 10^{h_i - 1} + 1, \ldots 10^{h_i} - 1\right\}$. Hence the cardinality of $\triangle(h_i) = 10^{h_i} - 1 - 10^{h_i - 1} + 1 = 10^{h_i} - 10^{h_i - 1} = 9 \times 10^{h_i - 1}$.

$\square$

EXAMPLE 145. Let $S = \langle 3,5,7\rangle$ the LD-semigroup of Example 142. Since $\mathrm{F}(S) = 4$ then $\mathrm{F}(\theta(S)) = 10^4 - 1 = 9999$, and as $\mathbb{N}\backslash S = \{1,2,4\}$ then $\mathrm{g}(\theta(S)) = 9\times\left(10^0+10^1+10^3\right)+1 = 9100$.

1.2. Frobenius Variety of LD-semigroups. We begin with the following result:

PROPOSITION 146. *Let $\mathcal{L} = \{S \mid S$ is a LD-semigroup$\}$. The set $\mathcal{L}$ is a Frobenius variety.*

PROOF. Clearly $\mathcal{L}$ is not empty, because $\mathbb{N} \in \mathcal{L}$. Assume that $S,T \in \mathcal{L}$ and let us show that $S\cap T \in \mathcal{L}$. If $a,b \in (S\cap T)\backslash\{0\}$ then $a,b \in S\backslash\{0\}$ and $a,b \in T\backslash\{0\}$. By using Theorem 139, we have that $a+b-1 \in (S\cap T)\backslash\{0\}$.

Now let us prove that if $S \in \mathcal{L}$ and $S \neq \mathbb{N}$, then $S\cup\{\mathrm{F}(S)\} \in \mathcal{L}$. To this end, we use again Theorem 139. Let $a,b \in (S\cup\{\mathrm{F}(S)\})\backslash\{0\}$.

. If $a,b \in S$, then $a+b-1 \in S \subseteq S\cup\{\mathrm{F}(S)\}$.

. If $\mathrm{F}(S) \in \{a,b\}$, then $a+b-1 \geq \mathrm{F}(S)$ and thus $a+b-1 \in S\cup\{\mathrm{F}(S)\}$.

□

Let $\mathcal{L} = \{S \mid S$ is a LD-semigroup$\}$. Recall that the graph $G(\mathcal{L})$ is the graph whose vertices are the elements of $\mathcal{L}$ and $(S,S') \in \mathcal{L}\times\mathcal{L}$ is an edge if $S' = S\cup\{\mathrm{F}(S)\}$.

Recall that every numerical semigroup S is finitely generated and therefore there exists a finite subset A of S such that $S = \langle A\rangle$. Furthermore, if no proper subset of A generates S, then we say that A is a minimal system of generators of S. Every numerical semigroup S admits a unique minimal system of generators, which will be denoted by $\mathrm{msg}(S)$. Besides, $\mathrm{msg}(S) = (S\backslash\{0\})\backslash(S\backslash\{0\}+S\backslash\{0\})$. We already know that if S is a numerical semigroup and $x \in S$ then $S\backslash\{x\}$ is a numerical semigroup if only if $x \in \mathrm{msg}(S)$.

As a consequence of Proposition 21 and Theorem 23 we obtain the following result.

THEOREM 147. *The graph* $G(\mathcal{L})$ *is a tree rooted in* $\mathbb{N}$. *Moreover, the childs of a vertex* $S \in \mathcal{L}$ *are* $S\setminus\{x_1\},\ldots,S\setminus\{x_l\}$ *with* $\{x_1,\ldots,x_l\}=\{x \in \mathrm{msg}(S) \mid x > \mathrm{F}(S) \text{ and } S\setminus\{x\} \in \mathcal{L}\}$

PROPOSITION 148. *Let* S *be a LD-semigroup not equal to* $\mathbb{N}$ *and let* $x \in \mathrm{msg}(S)$. *Then* $S\setminus\{x\}$ *is a LD-semigroup if and only if* $x+1 \in (\mathbb{N}\setminus S)\cup \mathrm{msg}(S)$.

PROOF. *Necessity*. If $x+1 \notin (\mathbb{N}\setminus S)\cup\mathrm{msg}(S)$, then clearly there exists $a,b \in S\setminus\{0\}$ such that $a+b = x+1$. Hence $a,b \in S\setminus\{0,x\}$ but $a+b-1 = x \notin S\setminus\{x\}$ and consequently $S\setminus\{x\}$ is not a LD-semigroup.

Sufficiency. Let $a,b \in S\setminus\{0,x\}$. Since by hypothesis S is a LD-semigroup, then we have that $a+b-1 \in S$. If $a+b-1 = x$ we obtain that $x+1 \notin (\mathbb{N}\setminus S)\cup\mathrm{msg}(S)$. Therefore $a+b-1 \neq x$ and thus $a+b-1 \in S\setminus\{x\}$. By using Theorem 139, we have that $S\setminus\{x\}$ is a LD-semigroup. □

From this result one easily deduces the following.

COROLLARY 149. *Let* S *be a LD-semigroup not equal to* $\mathbb{N}$ *and let* $x \in \mathrm{msg}(S)$ *such that* $x > \mathrm{F}(S)$. *Then* $S\setminus\{x\}$ *is a LD-semigroup if only if* $x+1 \in \mathrm{msg}(S)$.

These results allows us to construct recursively the tree, starting in $\mathbb{N}$, and compute the childs of each vertex (Figure 1). By using Theorem 147 and Corollary 149 we obtain the following.

. $\mathbb{N} = \langle 1\rangle$ has an only child that is $\mathbb{N}\setminus\{1\} = \langle 2,3\rangle$;
. $\langle 2,3\rangle$ has again an only child that is $\langle 2,3\rangle\setminus\{2\} = \langle 3,4,5\rangle$;
. $\langle 3,4,5\rangle$ has two childs that are $\langle 3,4,5\rangle\setminus\{3\} = \langle 4,5,6,7\rangle$ and $\langle 3,4,5\rangle\setminus\{4\} = \langle 3,5,7\rangle$;
. $\langle 3,5,7\rangle$ has no childs;
. $\langle 4,5,6,7\rangle$ has three childs that are $\langle 4,5,6,7\rangle\setminus\{4\} = \langle 5,6,7,8,9\rangle$, $\langle 4,5,6,7\rangle\setminus\{5\} = \langle 4,6,7,9\rangle$ and $\langle 4,5,6,7\rangle\setminus\{6\} = \langle 4,5,7\rangle$;
. $\cdots\ \cdots\ \cdots\ \cdots$

FIGURE 1. The tree of LD-numerical semigroups

$$
\begin{tikzcd}
 & & \mathbb{N}=\langle 1\rangle \arrow[d] & \\
 & & \langle 2,3\rangle \arrow[d] & \\
 & & \langle 3,4,5\rangle \arrow[dl] \arrow[dr] & \\
 & \langle 4,5,6,7\rangle \arrow[dl] \arrow[d] \arrow[dr] & & \langle 3,5,7\rangle \\
\langle 5,6,7,8,9\rangle & \langle 4,6,7,9\rangle & \langle 4,5,7\rangle &
\end{tikzcd}
$$

$\cdots \qquad \cdots\cdots \qquad \cdots \qquad \cdots \quad \cdots \quad \cdots \quad \cdots$

1.3. The smallest digital semigroup containing a set of positive integers. The following result has immediate proof.

LEMMA 150. *The intersection of digital semigroups is also a digital semigroup.*

This result motivates the following definition. Given $X \subseteq \mathbb{N}\backslash\{0\}$, we denote by $\mathcal{D}(X)$ the intersection of all digital semigroups containing X. Observe that as a consequence of previous result we obtain that $\mathcal{D}(X)$ is the smallest (with respect to set inclusion) digital semigroup containing X.

Nonfinite intersection of LD-semigroups is not in general a numerical semigroup. In fact, for every $n \in \mathbb{N}$ we have that $\{0, n, \rightarrow\}$ is a LD-semigroup and $\bigcap_{n\in\mathbb{N}}\{0, n, \rightarrow\} = \{0\}$ is not a numerical semigroup. Given $A \subseteq \mathbb{N}$, we denote by $\mathcal{L}(A)$ the intersection of all LD-semigroups containing A. It is clear that $\mathcal{L}(A)$ is a submonoid of $(\mathbb{N}, +)$.

PROPOSITION 151. *Let A be a nonempty subset of $\mathbb{N}\setminus\{0\}$. Then $\mathcal{L}(A)$ is a LD-semigroup.*

PROOF. Assume that S is a LD-semigroup containing A. If a is an element of A then, by applying Theorem 139, we have that $\{a, 2a-1\} \subseteq S$ and consequently $\{a, 2a-1\} \subseteq \mathcal{L}(A)$. It follows easily that $\mathcal{L}(A)$ is a numerical semigroup.

Let us see that $\mathcal{L}(A)$ is a LD-semigroup. Let $a, b \in \mathcal{L}(A)\setminus\{0\}$. If S is a LD-semigroup containing A, then $a, b \in S\setminus\{0\}$ and thus $a+b-1 \in S$. Hence $a+b-1 \in \mathcal{L}(A)$. By using Theorem 139, we can conclude that $\mathcal{L}(A)$ is a LD-semigroup. $\square$

As a consequence of previous proposition we have the following result.

COROLLARY 152. *Let A be a nonempty subset of $\mathbb{N}\setminus\{0\}$. Then $\mathcal{L}(A)$ is the smallest LD-semigroup containing A.*

Next, we see that for constructing the smallest digital semigroup containing a set X is equivalent to construct the smallest LD-semigroup containing $L(X)$.

PROPOSITION 153. *Let X be a nonempty subset of $\mathbb{N}\setminus\{0\}$. Then S is the smallest LD-semigroup containing $L(X)$ if and only if $\theta(S)$ is the smallest digital semigroup containing X.*

PROOF. *Necessity*. Let D be a digital semigroup that contains X. Then $L(D)\cup\{0\}$ is a LD-semigroup that contains $L(X)$. Hence $S \subseteq L(D)\cup\{0\}$ and consequently $\theta(S) \subseteq \theta(L(D)\cup\{0\}) = D$.

Sufficiency. Let T be a LD-semigroup that contains $L(X)$. Then $\theta(T)$ is a digital semigroup that contains X. Therefore $\theta(S) \subseteq \theta(T)$ and so we can assert that $S \subseteq T$. $\square$

Next we illustrate the previous proposition with an example.

EXAMPLE 154. We compute the smallest digital semigroup that contains $\{1235, 54321\}$. First we compute the smallest LD-semigroup that contains $L(\{1235, 54321\}) = \{4, 5\}$. From Theorem 139 we obtain that every LD-semigroup containing $\{4,5\}$ must contains the number 7 and $\langle 4,5,7\rangle$ is a LD-semigroup. Hence $\mathcal{L}(\{4,5\}) = \langle 4,5,7\rangle$. By applying Proposition 153, we get that $\mathcal{D}(\{1235, 54321\}) = \theta(\langle 4,5,7\rangle) = \mathbb{N}\backslash(\triangle(1)\cup\triangle(2)\cup\triangle(3)\cup\triangle(6)\cup\{0\})$.

Observe that every digital semigroup D is not finitely generated as a semigroup. In fact, if D is a digital semigroup, then by Corollary 143, we obtain that $\mathbb{N}\backslash D$ is finite. Whence D is a subsemigroup of $(\mathbb{N}\backslash\{0\},\cdot)$ that contains infinitely many primes and these belong to any system of generators of D.

Let D be a digital semigroup and $X \subseteq D$. We say that X is **$\mathcal{D}$-system of generators** of D if $\mathcal{D}(X) = D$. In the following result, we will see that every digital semigroup admits a finite $\mathcal{D}$-system of generators.

THEOREM 155. *With the above notation, we have that*

$$\mathcal{D} = \{\mathcal{D}(X) \mid X \textit{ is a nonempty finite subset of } \mathbb{N}\backslash\{0\}\}.$$

PROOF. It is clear that $\{\mathcal{D}(X) \mid X \text{ is a nonempty finite subset of } \mathbb{N}\backslash\{0\}\} \subseteq \mathcal{D}$.

For the other inclusion, take $D \in \mathcal{D}$ then $S = L(D)\cup\{0\}$ is a LD-semigroup. We have that every numerical semigroup is finitely generated, and therefore there exist positive integers $n_1,\ldots,n_p$ such that $S = \langle n_1,\ldots,n_p\rangle$. Let $x_1\ldots,x_p \in D$ with $l(x_1) = n_1,\ldots,l(x_p) = n_p$ and $X = \{x_1,\ldots,x_p\}$. In order to conclude the proof, it suffices to show that $D = \mathcal{D}(X)$. It is clear that S is the smallest LD-semigroup containing $L(X)$. Hence, by applying Proposition 153, we obtain that $D = \theta(S)$ is the smallest digital semigroup that contains X and consequently $D = \mathcal{D}(X)$. □

Let D be a digital semigroup and let X be a subset of D such that $\mathcal{D}(X) = D$. We say that X is a **minimal $\mathcal{D}$-system of generators** of D if no proper subset of X is a $\mathcal{D}$-system of generators of D. Now, we are interested to get an explicit description of the minimal $\mathcal{D}$-system of generators for a digital semigroup.

Let S be a *LD*- semigroup and let $A \subseteq S$. We say that A is a **$\mathcal{L}$-system of generators** of S if $\mathcal{L}(A) = S$. Moreover, if no proper subset of A generates S, then we say that A is a

minimal $\mathcal{L}$-system of generators of S. As $\mathcal{L}$ is a Frobenius variety, then by Corollary 20 we obtain the following result.

PROPOSITION 156. *Every LD-semigroup admits a unique minimal $\mathcal{L}$-system of generators. This minimal $\mathcal{L}$-system of generators is finite.*

Using previous proposition it makes sense to define the **$\mathcal{L}$-rank** of a *LD*-semigroup S by the cardinality of its minimal *LD*-system of generators.

As an immediate consequence of the Proposition 153, we obtain the following.

PROPOSITION 157. *Let D be a digital semigroup and let $\{n_1,\ldots,n_p\}$ be the minimal $\mathcal{L}$-system of generators of $L(D)\cup\{0\}$. For each $i\in\{1,\ldots,p\}$ let $d_i\in D$ be such that $\ell(d_i)=n_i$. Then $\{d_1,\ldots,d_p\}$ is a minimal $\mathcal{D}$-system of generators for D. Furthermore every minimal $\mathcal{D}$-system of generators for D is of this form.*

From previous proposition we can conclude that not every digital semigroups admits a unique minimal $\mathcal{D}$-system of generators. But the cardinality of its minimal $\mathcal{D}$-system of generators is always the same. And this is precisely $\mathcal{L}$-rank of $L(D)\cup\{0\}$, which we will denote by $\mathcal{D}\text{-rank}(D)$.

As a consequence of Proposition 21 we obtain the following.

PROPOSITION 158. *Let S be a LD-semigroup and let $x\in S$. Then, the set $S\backslash\{x\}$ is a LD-semigroup if and only if x belongs to the minimal $\mathcal{L}$-system of generators of S.*

From Proposition 148 we can deduce the following result.

COROLLARY 159. *Let S be a LD-semigroup not equal to $\mathbb{N}$ and let $x\in S$. Then x belongs to the minimal $\mathcal{L}$-system of generators of S if and only if $x\in\operatorname{msg}(S)$ and $x+1\in(\mathbb{N}\backslash S)\cup\operatorname{msg}(S)$.*

We illustrate some of these results with an example.

EXAMPLE 160. Let $X = \{1234, 2341521, 1234567890\}$ and let $D = \mathcal{D}(X)$. We compute a minimal $\mathcal{D}$-system of generators of D. We will start by computing the smallest LD-semigroup that contains $L(X) = \{4,7,10\}$. From Theorem 139 we have that every LD-semigroup containing $\{4,7,10\}$ must contain 13 and thus $S = \langle 4,7,10,13\rangle = \{0,4,7,8,10,\rightarrow\}$ is a LD-semigroup. Therefore S is the smallest LD-semigroup that contains $L(X)$. By applying Corollary 159 the set $\{4\}$ is the minimal $\mathcal{L}$-system of generators for S. This implies by Proposition 157 that $\{1234\}$ is a minimal $\mathcal{D}$-system of generators of D. Notice that in general $D = \mathcal{D}(\{a\})$ with a a positive integer such that $\ell(a) = 4$ and $\{a\}$ is a minimal $\mathcal{D}$-system of generators D.

1.4. The smallest LD-semigroup containing a set of positive integers. Let $x_1, \ldots, x_t$ be positive integers. Denote by

$$S(x_1,\ldots,x_t) = \{x \in \mathbb{N}\setminus\{0\} \mid x = a_1x_1 + \cdots + a_tx_t - r \text{ with } r, a_1, \ldots, a_t \in \mathbb{N} \text{ and } r < a_1 + \cdots + a_t\} \cup \{0\}.$$

Our next goal is to prove that $S(x_1,\ldots,x_t)$ is the smallest LD-semigroup containing the set $\{x_1,\ldots,x_t\}$.

Let S be a numerical semigroup and let $\{n_1,\ldots,n_p\}$ be its minimal set of generators. For $s \in S$, denote by $\mathcal{P}(s) = \max\{a_1 + \cdots + a_p \mid s = a_1n_1 + \cdots + a_pn_p \text{ and } a_1,\ldots,a_p \in \mathbb{N}\}$.

The reader can easily verify the following result.

LEMMA 161. *Let S be a numerical semigroup minimally generated by $\{n_1,\ldots,n_p\}$ and let $s \in S$. Then*

(1) If $i \in \{1,\ldots,p\}$ and $s - n_i \in S$ then $\mathcal{P}(s-n_i) \leq \mathcal{P}(s) - 1$.
(2) If $s = a_1n_1 + \cdots + a_pn_p$ and $\mathcal{P}(s) = a_1 + \cdots + a_p$ with $a_i \neq 0$ for some $i \in \{1,\ldots,p\}$, then $\mathcal{P}(s-n_i) = \mathcal{P}(s) - 1$.

PROPOSITION 162. *Let S be a numerical semigroup minimally generated by $\{n_1,\ldots,n_p\}$. The following conditions are equivalent:*

(1) *S is a LD-semigroup;*

(2) *if $i,j \in \{1,\dots,p\}$ then $n_i+n_j-1 \in S$;*

(3) *if $s \in S\setminus\{0,n_1,\dots,n_p\}$ then $s-1 \in S$;*

(4) *if $s \in S\setminus\{0\}$ then $s-\{0,\dots,\mathcal{P}(s)-1\} \subseteq S$.*

PROOF. $1)$ *implies* $2)$. It is an immediate consequence of Theorem 139.

$2)$ *implies* $3)$. If $s \in S\setminus\{0,n_1,\dots,n_p\}$ then there exist $s' \in S$ and $i,j \in \{1,\dots,p\}$ such that $s = n_i+n_j+s'$. Hence we obtain that $s-1 = (n_i+n_j-1)+s' \in S$

$3)$ *implies* $4)$. We proceed by induction on $\mathcal{P}(s)$. For $\mathcal{P}(s) = 1$ the result is trivial. Now, assume that $\mathcal{P}(s) \geq 2$, $a_1,\dots,a_t$ are nonnegative integers such that $s = a_1n_1+\cdots+a_pn_p$ and $\mathcal{P}(s) = a_1+\cdots+a_p$ with $a_i \neq 0$ for some $i \in \{1,\dots,p\}$. From Lemma 161, we conclude that $\mathcal{P}(s-n_i) = \mathcal{P}(s)-1$ and by induction hypothesis $s-n_i-\{0,\dots,\mathcal{P}(s)-2\} \subseteq S$. Hence $s-\{0,\dots,\mathcal{P}(s)-2\} \subseteq S$.

From the preceding paragraph we have that $s-n_i-(\mathcal{P}(s)-2) \in S$. Let us prove that $s-n_i-(\mathcal{P}(s)-2) \neq 0$. In fact, since $s \in S\setminus\{0,n_1,\dots,n_p\}$, then $s-n_i = a_1n_1+\cdots+(a_i-1)n_i+\cdots+a_pn_p \neq 0$. This implies that either $a_i \geq 2$ or there exist $j \in \{1,\dots,p\}\setminus\{i\}$ such that $a_j \neq 0$. And thus we obtain that either $s = 2n_i+a_1n_1+\cdots+(a_i-2)n_i+\cdots+a_pn_p$ or $s = n_i+n_j+a_1n_1+\cdots+(a_i-1)n_i+\cdots+(a_j-1)n_j+\cdots+a_pn_p$. This leads to $s-n_i-(\mathcal{P}(s)-2) > 0$ and consequently $s-n_i-(\mathcal{P}(s)-2) \in S\setminus\{0\}$. From this we get that $(s-n_i-(\mathcal{P}(s)-2))+n_i \in S\setminus\{0,n_1,\dots,n_p\}$ and so $s-n_i-(\mathcal{P}(s)-2)+n_i-1 \in S$. Then we have that $s-(\mathcal{P}(s)-1) \in S$ and therefore $s-\{0,\dots,\mathcal{P}(s)-1\} \subseteq S$.

$4)$ *implies* $1)$. If $a,b \in S\setminus\{0\}$, then $\mathcal{P}(a+b) \geq 2$. From the hypothesis, we have that $a+b-1 \in S$. By applying Theorem 139, we obtain that S is a LD-semigroup. $\square$

Observe that the previous proposition (condition 2) gives a criterion to check whether or not a numerical semigroup is a LD-semigroup.

EXAMPLE 163. Let us see that $\langle 4,5,7\rangle$ is a LD-semigroup. In order to see this, applying Proposition 162, it is enough to see that

$$\{4+4-1=7, 4+5-1=8, 4+7-1=10, 5+5-1=9, 5+7-1=11,$$
$$7+7-1=13\}\subseteq S.$$

The next lemma is straightforward to prove.

LEMMA 164. *Let S be a numerical semigroup, $s_1,\dots,s_t\in S\backslash\{0\}$ and let $a_1,\dots,a_t$ be nonnegative integers. Then $\mathcal{P}(a_1s_1+\cdots+a_ts_t)\geq a_1+\cdots+a_t$.*

THEOREM 165. *Let $x_1,\dots,x_t$ be positive integers. Then $S(x_1,\dots,x_t)$ is the smallest LD-semigroup that contains $\{x_1,\dots,x_t\}$.*

PROOF.
(1) Let us see that if $x,y\in S(x_1,\dots,x_t)\backslash\{0\}$, then $\{x+y,x+y-1\}\subseteq S(x_1,\dots,x_t)$. In fact, there exist nonnegative integers $a_1,\dots,a_t,b_1,\dots,b_t,r,r'$ such that $x=a_1x_1+\cdots+a_tx_t-r, y=b_1x_1+\cdots+b_tx_t-r'$, $r<a_1+\cdots+a_t$ and $r'<b_1+\cdots+b_t$. And thus $x+y=(a_1+b_1)x_1+\cdots+(a_t+b_t)x_t-(r+r')$ with $r+r'+1<(a_1+b_1)+\cdots+(a_t+b_t)$. Consequently we conclude that $\{x+y,x+y-1\}\subseteq S(x_1,\dots,x_t)$.
(2) As a consequence of the previous proof, we can deduce that $S(x_1,\dots,x_t)$ is submonoid of $(\mathbb{N},+)$. Since $x_1=1x_1+\cdots+0x_t-0$ and $2x_1-1=2x_1+\cdots+0x_t-1$, we obtain that $\{x_1,2x_1-1\}\subseteq S(x_1,\dots,x_t)$ and so we get that $S(x_1,\dots,x_t)$ is a numerical semigroup. From Condition 1) we can assert that $S(x_1,\dots,x_t)$ is a LD-semigroup.
(3) Arguing in a similar way with $x_1\in S(x_1,\dots,x_t)$, we get $x_i\in S(x_1,\dots,x_t)$ for all $i\in\{1,\dots,t\}$. This proves that $S(x_1,\dots,x_t)$ is a LD-semigroup containing $\{x_1,\dots,x_t\}$.
(4) Let us prove that $S(x_1,\dots,x_t)$ is the smallest LD-semigroup containing $\{x_1,\dots,x_t\}$. Assuming that T is a LD-semigroup containing $\{x_1,\dots,x_t\}$, we

will prove that $S(x_1,\dots,x_t)\subseteq T$. In fact if $x\in S(x_1,\dots,x_t)\setminus\{0\}$ then there exist $a_1,\dots,a_t,r\in\mathbb{N}$ such that $x=a_1x_1+\cdots+a_tx_t-r$ and $r<a_1+\cdots+a_t$. Since $\{x_1,\dots,x_t\}\subseteq T$ then $a_1x_1+\cdots+a_tx_t\in T$. From Proposition 162 we obtain that $a_1x_1+\cdots+a_tx_t-\{0,\dots,\mathcal{P}(a_1x_1+\cdots+a_tx_t)-1\}\subseteq T$. By using now Lemma 164 we have that $r<\mathcal{P}(a_1x_1+\cdots+a_tx_t)$ and so x belongs to T. $\square$

We conclude this section by giving an algorithm that allows to determine the smallest LD-semigroup containing a set of positive integers A.

ALGORITHM 166. Input: A set of positive integers A.

Output: The minimal system of generators of the smallest LD-semigroup containing the set A.

1) $B=\operatorname{msg}(\langle A\rangle)$
2) if $a+b-1\in\langle B\rangle$ for all $a,b\in B$, then return B.
3) $A=B\cup\{a+b-1\mid a,b\in B \text{ and } a+b-1\notin\langle B\rangle\}$ and go to 1).

Next we justify how the algorithm behaves. Let $B_1,B_2,\dots$ be the possible values of B arising from the algorithm. It is clear that $\langle B_1\rangle\subsetneq\langle B_2\rangle\subsetneq\dots$. Note that if $a\in B_1$ then $\{a,2a-1\}\subset\langle B_2\rangle$ and then we have that $\langle B_2\rangle$ is a numerical semigroup. Then $\mathbb{N}\setminus\langle B_2\rangle$ is finite and thus this chain $\langle B_1\rangle\subsetneq\langle B_2\rangle\subsetneq\dots$ is finite. Consequently the algorithm stops in a finite number of steps and gives us $\langle B_1\rangle\subsetneq\langle B_2\rangle\subsetneq\dots\subsetneq\langle B_n\rangle$. Follows from Proposition 162 that $\langle B_n\rangle$ is a LD-semigroup. Furthermore, as a consequence of Theorem 139 and by the way we compute B_n, we conclude that every LD-semigroup containing the set A must contain $\langle B_n\rangle$.

EXAMPLE 167. Let us compute the smallest LD-semigroup that contain $\{5\}$. To this end we use the Algorithm 166. The values arising for A and B are:

. $A=\{5\}$;
. $B=\{5\}$;
. $A=\{5,9\}$;

. $B = \{5, 9\}$;

. $A = \{5, 9, 13, 17\}$;

. $B = \{5, 9, 13, 17\}$;

. $A = \{5, 9, 13, 17, 21\}$;

. $B = \{5, 9, 13, 17, 21\}$.

Therefore the smallest LD-semigroup containing $\{5\}$ is $\langle 5, 9, 13, 17, 21\rangle = \{0, 5, 9, 10, 13, 14, 15, 17, \rightarrow\}$.

2. Bracelet Monoids and Numerical Semigroups

This section is devoted to the study of the $(n_1, \ldots, n_p)$-bracelets and its content is organized as follows. In Theorem 173 we explicitly describe the smallest $(n_1, \ldots, n_p)$-bracelet containing a set of positive integers. Denote by $\mathcal{B}(n_1, \ldots, n_p) = \{M \mid M \text{ is a } (n_1, \ldots, n_p) - \text{bracelet}\}$ and by $\mathcal{N}(n_1, \ldots, n_p) = \{S \mid S \text{ is a numerical } (n_1, \ldots, n_p) - \text{bracelet}\}$. In Theorem 179 we show that, if D is the set of all positive divisors of $\gcd(n_1, \ldots, n_p)$ then $\mathcal{B}(n_1, \ldots, n_p) = \left(\bigcup_{d \in D} \{dS \mid S \in \mathcal{N}(\frac{n_1}{d}, \ldots, \frac{n_p}{d})\}\right) \cup \{\{0\}\}$. The results presented in this section can be found in [**30**].

We will prove that $\mathcal{N}(n_1, \ldots, n_p)$ is a Frobenius variety. This fact together with the results presented in Section 3 of the Preliminaries allows us to arrange the elements of $\mathcal{N}(n_1, \ldots, n_p)$ in a tree rooted in $\mathbb{N}$. We describe the childs of any vertex of this tree and this will enable us to recursively construct the set $\mathcal{N}(n_1, \ldots, n_p)$.

The intersection of $(n_1, \ldots, n_p)$-bracelets is again a $(n_1, \ldots, n_p)$-bracelet. As a consequence of this result we will introduce the concepts of $(n_1, \ldots, n_p)$-system of generators and minimal $(n_1, \ldots, n_p)$-system of generators of a $(n_1, \ldots, n_p)$-bracelet. In Theorem 186 we will show that M is a $(n_1, \ldots, n_p)$-bracelet if and only if M is the intersection of numerical $(n_1, \ldots, n_p)$-bracelets. Using this result together with the results presented in Section 3 of the Preliminaries we obtain that every $(n_1, \ldots, n_p)$-bracelet has a unique minimal $(n_1, \ldots, n_p)$-system of generators. We will also characterize the elements in this minimal $(n_1, \ldots, n_p)$-system of generators.

A $(n_1,\ldots,n_p)$-bracelet is indecomposable if it cannot be expressed as the intersection of $(n_1,\ldots,n_p)$-bracelets properly containing it. In Corollary 206 we give an algorithm procedure which allows us to determine whether or not a $(n_1,\ldots,n_p)$-bracelet is indecomposable.

2.1. Characterization of the $(n_1,\ldots,n_p)$-bracelets. Once more, given M a submonoid of $(\mathbb{N},+)$, we denote by $\mathrm{msg}(M)$ the minimal system of generators of M. We already saw that $\mathrm{msg}(M) = (M\setminus\{0\})\setminus(M\setminus\{0\}+M\setminus\{0\})$.

PROPOSITION 168. *Let $m_1,\ldots,m_q$ and $n_1,\ldots,n_p$ be positive integers and let M be a submonoid of $(\mathbb{N},+)$ generated by $\{m_1,\ldots,m_q\}$. The following conditions are equivalent.*

(1) M is a $(n_1,\ldots,n_p)$-bracelet.

(2) If $i,j\in\{1,\ldots,q\}$ then $m_i+m_j+\{n_1,\ldots,n_p\}\subseteq M$.

PROOF. 1) *implies* 2). Trivial.

2) *implies* 1). If $a,b\in M\setminus\{0\}$ then there exist $i,j\in\{1,\ldots,q\}$ and $m,m'\in M$ such that $a=m_i+m$ and $b=m_j+m'$. Since $m_i+m_j+\{n_1,\ldots,n_p\}\subseteq M$, we have that $m_i+m_j+m+m'+\{n_1,\ldots,n_p\}\subseteq M$ and thus $a+b+\{n_1,\ldots,n_p\}\subseteq M$. □

The previous result allow us to determine whether or not a submonoid of $(\mathbb{N},+)$ is a $(n_1,\ldots,n_p)$-bracelet.

EXAMPLE 169. Let $M=\langle\{4,6\}\rangle=\{0,4,6,8,10,12,\ldots\}$. We prove that M is a $(2,4)$-bracelet. As $4+4+\{2,4\}\subseteq M$, $4+6+\{2,4\}\subseteq M$ and $6+6+\{2,4\}\subseteq M$, by applying Proposition 168, we obtain that M is a $(2,4)$-bracelet.

The following result is easy to prove.

LEMMA 170. *Let $n_1,\ldots,n_p$ be positive integers. The intersection of $(n_1,\ldots,n_p)$-bracelets is a $(n_1,\ldots,n_p)$-bracelet.*

The previous result motivates the following definition. Given $X\subseteq\mathbb{N}$ we define the $(n_1,\ldots,n_p)$-bracelet generated by X as the intersection of all $(n_1,\ldots,n_p)$-bracelet containing X. We will denote it by $L_{\{n_1,\dots,n_p\}}(X)$, and is the smallest (with respect to the set inclusion order) $(n_1,\dots,n_p)$-bracelet containing X. If $M = L_{\{n_1,\dots,n_p\}}(X)$ we say that X is a $(n_1,\dots,n_p)$**-system of generators** of M. Moreover, if no proper subset of X generates M, then we say that X is a **minimal** $(n_1,\dots,n_p)$**-system of generators**. The next result shows that every $(n_1,\dots,n_p)$-bracelet admits a finite $(n_1,\dots,n_p)$-system of generators.

PROPOSITION 171. *Let $n_1,\dots,n_p$ be positive integers. Then* $\mathcal{B}(n_1,\dots,n_p) = \left\{L_{\{n_1,\dots,n_p\}}(X) \mid X \text{ is a finite subset of } \mathbb{N}\right\}$

PROOF. It is clear that $\left\{L_{\{n_1,\dots,n_p\}}(X) \mid X \text{ is a finite subset of } \mathbb{N}\right\} \subseteq \mathcal{B}(n_1,\dots,n_p)$. Let us prove the other inclusion. If $M \in \mathcal{B}(n_1,\dots,n_p)$ then M is a submonoid of $(\mathbb{N},+)$. By Corollary 9 we deduce that there exists a finite subset X of $\mathbb{N}$ such that $M = \langle X\rangle$. Hence $M = L_{\{n_1,\dots,n_p\}}(X)$. $\square$

Observe that, in view of the proof of Proposition 171, we obtain that if M is a $(n_1,\dots,n_p)$-bracelet and $M = \langle X\rangle$ then $M = L_{\{n_1,\dots,n_p\}}(X)$. The next example shows that X can be the minimal system of generators of M but X cannot be a minimal $(n_1,\dots,n_p)$-system of generators of M.

EXAMPLE 172. Let $M = \langle\{3,8,13\}\rangle$. Clearly $\{3,8,13\}$ is the minimal system of generators of M. Using the Proposition 168 we have that M is a $(2,3)$-bracelet. Observe that every $(2,3)$-bracelet containing $\{3\}$ must contain $3+3+2 = 8$ and $3+8+2 = 13$. Therefore $M = \langle\{3,8,13\}\rangle \subseteq L_{\{2,3\}}(\{3\})$. Since $L_{\{2,3\}}(\{3\})$ is the smallest $(2,3)$-bracelet containing $\{3\}$ we deduce that $M = L_{\{2,3\}}(\{3\})$. Thus the set $\{3\}$ is a minimal $(2,3)$-system of generators of M.

The following result gives an explicit description of the smallest $(n_1,\dots,n_p)$-bracelet that contain a finite subset X of $\mathbb{N}$.

THEOREM 173. *Let* $X=\{x_1,\dots,x_t\}\subseteq\mathbb{N}\backslash\{0\}$ *and let* $\{n_1,\dots,n_p\}\subseteq\mathbb{N}\backslash\{0\}$. *Then*

$$L_{\{n_1,\dots,n_p\}}(X)=\big\{a_1x_1+\cdots+a_tx_t+b_1n_1+\cdots+b_pn_p\mid a_1,\dots,a_t,b_1,\dots,b_p\in\mathbb{N}\text{ and }a_1+\cdots+a_t>b_1+\cdots+b_p\big\}\cup\{0\}.$$

PROOF. Let $A=\big\{a_1x_1+\cdots+a_tx_t+b_1n_1+\cdots+b_pn_p\mid a_1,\dots,a_t,$ $b_1,\dots,b_p\in\mathbb{N}$ and $a_1+\cdots+a_t>b_1+\cdots+b_p\big\}\cup\{0\}$. Clearly A is a subset of $\mathbb{N}$ that is closed under addition, contains the zero element, and if $x,y\in A\backslash\{0\}$ then $x+y+\{n_1,\dots,n_p\}\subseteq A$. Hence A is a $(n_1,\dots,n_p)$-bracelet. Since X is a subset of A it follows that $L_{\{n_1,\dots,n_p\}}(X)\subseteq A$. For the other inclusion, take $x=a_1x_1+\cdots+a_tx_t+b_1n_1+\cdots+b_pn_p\in A$. The proof follows using induction on $a_1+\cdots+a_t$. If $a_1+\cdots+a_t=1$ then $b_1=\cdots=b_p=0$ and thus $x=a_1x_1+\cdots+a_tx_t\in L_{\{n_1,\dots,n_p\}}(X)$. Suppose now that $a_1+\cdots+a_t\geq 2$ and $b_1+\cdots+b_p\geq 1$. Then there exist $i\in\{1,\dots,t\}$ and $j\in\{1,\dots,p\}$ such that $a_i\neq 0$ and $b_j\neq 0$. By induction hypothesis we deduce that $x-x_i-n_j\in L_{\{n_1,\dots,n_p\}}(X)$. Since $a_1+\cdots+a_t\geq 2$ we get that $x-x_i-n_j\neq 0$. Applying that $L_{\{n_1,\dots,n_p\}}(X)$ is a $(n_1,\dots,n_p)$-bracelet we have that $x-x_i-n_j+x_i+\{n_1,\dots,n_p\}\subseteq L_{\{n_1,\dots,n_p\}}(X)$. Hence $x\in L_{\{n_1,\dots,n_p\}}(X)$. □

Next we illustrate this result with an example.

EXAMPLE 174. Let us calculate $L_{\{2,3\}}(\{4\})$. From Theorem 173, we have that $L_{\{2,3\}}(\{4\})=\{a_14+b_12+b_23\mid a_1,b_1,b_2\in\mathbb{N}\text{ and }a_1>b_1+b_2\}\cup\{0\}$. Therefore $L_{\{2,3\}}(\{4\})=\{0,4,8,10,11,12,14,15,16,17,18,\to\}=\langle 4,10,11,17\rangle$.

2.2. The numerical $(n_1,\dots,n_p)$**-bracelets.** A numerical semigroup is a submonoid S of $(\mathbb{N},+)$ such that $\mathbb{N}\backslash S$ is finite. As we saw before a submonoid M of $(\mathbb{N},+)$ is a numerical semigroup if and only if $\gcd(M)=1$. Furthermore, it is easy to prove that if M is submonoid of $(\mathbb{N},+)$ such that $M\neq\{0\}$ and $\gcd(M)=d$, then $\frac{M}{d}=\left\{\frac{m}{d}\mid m\in M\right\}$ is a numerical semigroup. As a consequence, we deduce that the numerical semigroups classify, up to isomorphism, the set of all submonoids of $(\mathbb{N},+)$ not equal to $\{0\}$ (see Proposition 1).

PROPOSITION 175. *Let X be a nonempty subset of $\mathbb{N}\backslash\{0\}$ and let $n_1,\ldots,n_p$ be positive integers. Then $L_{\{n_1,\ldots,n_p\}}(X)$ is a numerical semigroup if and only if* $\gcd\left(X\cup\left\{n_1,\ldots,n_p\right\}\right)=1$.

PROOF. *Necessity*. Suppose that $\gcd\left(X\cup\left\{n_1,\ldots,n_p\right\}\right)=d\neq 1$. Then we deduce that $\langle\{d\}\rangle$ is a $(n_1,\ldots,n_p)$-bracelet that contain X and consequently $L_{\{n_1,\ldots,n_p\}}(X)\subseteq\langle\{d\}\rangle$. Hence $L_{\{n_1,\ldots,n_p\}}(X)$ is not a numerical semigroup.

Sufficiency. Let $A=X\cup\left(2X+\left\{n_1,\ldots,n_p\right\}\right)$. It is clear that $A\subseteq L_{\{n_1,\ldots,n_p\}}(X)$. If x belongs to X then $\gcd\left\{x,2x+n_1,\ldots,2x+n_p\right\}=\gcd\left\{x,n_1,\ldots,n_p\right\}$. Therefore $\gcd(A)=\gcd\left(X\cup\left\{n_1,\ldots,n_p\right\}\right)=1$ and thus $\gcd\left(L_{\{n_1,\ldots,n_p\}}(X)\right)=1$. This proves that $L_{\{n_1,\ldots,n_p\}}(X)$ is a numerical semigroup. □

We say that a $(n_1,\ldots,n_p)$-bracelet M is a numerical $(n_1,\ldots,n_p)$-bracelet if $\gcd(M)=1$ (i.e. $\mathbb{N}\backslash M$ is finite). Recall that we denote by

$$\mathcal{N}(n_1,\ldots,n_p)=\left\{M\in\mathcal{B}(n_1,\ldots,n_p)\mid M\text{ is a numerical }(n_1,\ldots,n_p)-\text{bracelet}\right\}.$$

As a consequence of Proposition 171 and 175 we obtain the following corollary.

COROLLARY 176. *Let $n_1,\ldots,n_p$ be positive integers such that* $\gcd\left\{n_1,\ldots,n_p\right\}=1$. *Then $\mathcal{B}(n_1,\ldots,n_p)=\mathcal{N}(n_1,\ldots,n_p)\cup\{\{0\}\}$.*

Our next goal is to study the case $\gcd\left\{n_1,\ldots,n_p\right\}\neq 1$.

LEMMA 177. *Let $n_1,\ldots,n_p$ be positive integers. If M is a $(n_1,\ldots,n_p)$-bracelet such that $M\neq\{0\}$ then* $\gcd(M)\mid\gcd\left\{n_1,\ldots,n_p\right\}$.

PROOF. Let $x\in M\backslash\{0\}$. Then we have that $\left\{x,2x+n_1,\ldots,2x+n_p\right\}\subseteq M$ and thus

$$\gcd(M)\mid\gcd\left\{x,2x+n_1,\ldots,2x+n_p\right\}.$$

Since

$$\gcd\left\{x,2x+n_1,\ldots,2x+n_p\right\}=\gcd\left\{x,n_1,\ldots,n_p\right\}$$

and

$$\gcd\left\{x, n_1, \ldots, n_p\right\} \mid \gcd\left\{n_1, \ldots, n_p\right\},$$

we can conclude that $\gcd(M) \mid \gcd\left\{n_1, \ldots, n_p\right\}$. $\square$

LEMMA 178. *Let M be a submonoid of $(\mathbb{N}, +)$ such that $M \neq \{0\}$ and $\gcd(M) = d$. Then M is a $(n_1, \ldots, n_p)$-bracelet if and only if $\frac{M}{d}$ is a $\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right)$-bracelet.*

PROOF. *Necessity.* If $a, b \in \frac{M}{d} \setminus \{0\}$ then there exist $x, y \in M$ such that $a = \frac{x}{d}$ and $b = \frac{y}{d}$. Since by hypothesis M is a $(n_1, \ldots, n_p)$-bracelet we have that $x + y + \left\{n_1 \ldots, n_p\right\} \subseteq M$. In view of Lemma 177, we know that $d \mid \gcd\left\{n_1, \ldots, n_p\right\}$ and so $\frac{x}{d} + \frac{y}{d} + \left\{\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right\} \subseteq \frac{M}{d}$. This proves that $\frac{M}{d}$ is a $\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right)$-bracelet.

Sufficiency. If $a, b \in M \setminus \{0\}$ then $\frac{a}{d}, \frac{b}{d} \in \frac{M}{d}$. Since $\frac{M}{d}$ is a $\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right)$-bracelet, we deduce that $\frac{a}{d} + \frac{b}{d} + \left\{\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right\} \subseteq \frac{M}{d}$. Hence $a + b + \left\{n_1 \ldots, n_p\right\} \subseteq M$ and thus M is a $(n_1, \ldots, n_p)$-bracelet. $\square$

THEOREM 179. *Let $n_1, \ldots, n_p$ be positive integers and let D be the set of all positive divisors of $\gcd\left\{n_1, \ldots, n_p\right\}$. Then*

$$\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\} = \bigcup_{d \in D} \left\{ dS \mid S \in \mathcal{N}\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right) \right\}.$$

PROOF. Let $M \in \mathcal{B}(n_1, \ldots, n_p)$ such that $M \neq \{0\}$ and $\gcd(M) = d$. Thus, by applying Lemma 177 and 178, we get that d is an element of D and $\frac{M}{d} \in \mathcal{N}\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right)$. For the other inclusion, take $d \in D$ and $S \in \mathcal{N}\left(\frac{n_1}{d}, \ldots, \frac{n_p}{d}\right)$. Then by Lemma 178 we have that $dS \in \mathcal{B}(n_1, \ldots, n_p)$. $\square$

We define in $\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\}$ the following equivalence relation $\mathcal{R}$:

$$M \;\mathcal{R}\; M' \text{ if } \gcd(M) = \gcd(M').$$

The set of classes of elements of $\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\}$ modulo $\mathcal{R}$ is denoted by $\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\} / \mathcal{R}$, and as a consequence of Theorem 179 it is equal to $\left\{\left\{dS \mid S \in \mathcal{N}(\frac{n_1}{d}, \ldots, \frac{n_p}{d})\right\} \mid d \in D\right\}$. In particular the previous set is a partition of $\mathcal{B}(n_1, \ldots, n_p) \setminus \{\{0\}\}$.

EXAMPLE 180. Let us compute $\mathcal{B}(4,6)\setminus\{\{0\}\}$. By using Theorem 179, we have that $\mathcal{B}(4,6)\setminus\{\{0\}\} = \{S \mid S\in\mathcal{N}(4,6)\}\cup\{2S \mid S\in\mathcal{N}(2,3)\}$. Hence, in order to compute $\mathcal{B}(4,6)\setminus\{\{0\}\}$ it is sufficient to compute the sets $\mathcal{N}(4,6)$ and $\mathcal{N}(2,3)$.

2.3. The Frobenius variety of the numerical $(n_1,\ldots,n_p)$-bracelet. We begin with the following result.

PROPOSITION 181. *Let $n_1,\ldots,n_p$ be positive integers. Then $\mathcal{N}(n_1,\ldots,n_p)$ is a Frobenius variety.*

PROOF. First we have that $\mathcal{N}(n_1,\ldots,n_p)$ is not empty, because $\mathbb{N}$ is a numerical $(n_1,\ldots,n_p)$-bracelet.

If S and T are in $\mathcal{N}(n_1,\ldots,n_p)$ with a and b elements of $(S\cap T)\setminus\{0\}$, then $a+b+\{n_1,\ldots,n_p\}\in S\cap T$. Therefore $S\cap T\in\mathcal{N}(n_1,\ldots,n_p)$.

Let $S\in\mathcal{N}(n_1,\ldots,n_p)$ such that $S\neq\mathbb{N}$ and let $a,b\in(S\cup\{\mathrm{F}(S)\})\setminus\{0\}$. If $a,b\in S$ then $a+b+\{n_1,\ldots,n_p\}\subseteq S\subseteq S\cup\{\mathrm{F}(S)\}$. If $\mathrm{F}(S)\in\{a,b\}$ then $a+b+\{n_1,\ldots,n_p\}\subseteq\{\mathrm{F}(S)+1,\rightarrow\}\subseteq S\subseteq S\cup\{\mathrm{F}(S)\}$. Hence $S\cup\{\mathrm{F}(S)\}\in\mathcal{N}(n_1,\ldots,n_p)$. $\square$

We define the graph $G(\mathcal{N}(n_1,\ldots,n_p))$ as follows:

(1) the vertices are the elements of $\mathcal{N}(n_1,\ldots,n_p)$;
(2) an element $(S,S')\in\mathcal{N}(n_1,\ldots,n_p)\times\mathcal{N}(n_1,\ldots,n_p)$ is an edge if $S\cup\{\mathrm{F}(S)\}=S'$.

As a consequence of Proposition 21 and Theorem 23 we have the following result.

THEOREM 182. *The graph $G(\mathcal{N}(n_1,\ldots,n_p))$ is a tree rooted in $\mathbb{N}$. Moreover, the childs of $S\in\mathcal{N}(n_1,\ldots,n_p)$ are the elements of the set $\{S\setminus\{x\}\mid x\in\mathrm{msg}(S),\ x>F(S) \text{ and } S\setminus\{x\}\in\mathcal{N}(n_1,\ldots,n_p)\}$.*

The next result is well known.

LEMMA 183. *Let M be a submonoid of $(\mathbb{N},+)$ such that $M\neq\{0\}$ and $x\in M$. Then $M\setminus\{x\}$ is a submonoid of $(\mathbb{N},+)$ if and only if $x\in\mathrm{msg}(M)$.*

PROPOSITION 184. *Let M be a $(n_1,\ldots,n_p)$-bracelet and $x \in \operatorname{msg}(M)$. Then $M\backslash\{x\}$ is a $(n_1,\ldots,n_p)$-bracelet if and only if $x - \{n_1,\ldots,n_p\} \subseteq (\mathbb{Z}\backslash M) \cup \operatorname{msg}(M) \cup \{0\}$.*

PROOF. *Necessity*. If there exists $i \in \{1,\ldots,p\}$ such that $x - n_i \notin (\mathbb{Z}\backslash M) \cup \operatorname{msg}(M) \cup \{0\}$ then we deduce that $x - n_i = a + b$ for some $a, b \in M\backslash\{0\}$. Hence $x = a + b + n_i \notin M\backslash\{x\}$. Since $x \notin \{a,b\}$ because $x - n_i = a + b$ then we have that $a, b \in M\backslash\{x,0\}$ and $a + b + n_i \notin M\backslash\{x\}$. It follows that $M\backslash\{x\}$ is not a $(n_1,\ldots,n_p)$-bracelet.

Sufficiency. Let $a, b \in M\backslash\{x,0\}$. Then we have that $a + b + \{n_1,\ldots,n_p\} \subseteq M$. If there exists $i \in \{1,\ldots,p\}$ such that $a + b + n_i = x$ then we get that $x - n_i \notin (\mathbb{Z}\backslash M) \cup \operatorname{msg}(M) \cup \{0\}$, which is absurd. Therefore $a + b + \{n_1,\ldots,n_p\} \subseteq M\backslash\{x\}$ and thus $M\backslash\{x\}$ is a $(n_1,\ldots,n_p)$-bracelet. □

EXAMPLE 185. We now draw part of the tree associated to the numerical $(2,3)$-bracelets.

By using Theorem 182 and Proposition 184 we obtain the following:

. $\mathbb{N}$ has an only child $\mathbb{N}\backslash\{1\} = \langle 2,3\rangle$,
. $\langle 2,3\rangle$ has two childs $\langle 2,3\rangle\backslash\{2\} = \langle 3,4,5\rangle$ and $\langle 2,3\rangle\backslash\{3\} = \langle 2,5\rangle$,
. $\langle 2,5\rangle$ has an only child $\langle 2,5\rangle\backslash\{5\} = \langle 2,7\rangle$,
. $\langle 2,7\rangle$ has no childs,
. $\langle 3,4,5\rangle$ has tree childs $\langle 3,4,5\rangle\backslash\{3\} = \langle 4,5,6,7\rangle$, $\langle 3,4,5\rangle\backslash\{4\} = \langle 3,5,7\rangle$ and $\langle 3,4,5\rangle\backslash\{5\} = \langle 3,4\rangle$,
. $\langle 3,4\rangle$ has no childs,
. $\langle 3,5,7\rangle$ has two childs $\langle 3,5,7\rangle\backslash\{5\} = \langle 3,7,8\rangle$ and $\langle 3,5,7\rangle\backslash\{7\} = \langle 3,5\rangle$,
. $\langle 3,5\rangle$ has no childs,
. $\langle 3,7,8\rangle$ has an only child $\langle 3,7,8\rangle\backslash\{7\} = \langle 3,8,10\rangle$,
. $\langle 3,8,10\rangle$ has an only child $\langle 3,8,10\rangle\backslash\{10\} = \langle 3,8,13\rangle$,
. $\langle 3,8,13\rangle$ has no childs,
. $\langle 4,5,6,7\rangle$ has four childs ... ...

$$\begin{tikzcd}[column sep=small]
 & & & & & \langle 1\rangle=\mathbb{N} & & \\
 & & & & & \langle 2,3\rangle \arrow[u] & & \\
 & & & & \langle 3,4,5\rangle \arrow[ur] & & \langle 2,5\rangle \arrow[ul] & \\
 & & & \langle 4,5,6,7\rangle \arrow[ur] & \langle 3,5,7\rangle \arrow[u] & \langle 3,4\rangle \arrow[ul] & & \langle 2,7\rangle \arrow[ul] \\
\cdots \arrow[urrr] & \cdots \arrow[urr] & \cdots \arrow[ur] & \cdots \arrow[u] & \langle 3,7,8\rangle \arrow[u] & \langle 3,5\rangle \arrow[ul] & & \\
 & & & & \langle 3,8,10\rangle \arrow[u] & & & \\
 & & & & \langle 3,8,13\rangle \arrow[u] & & &
\end{tikzcd}$$

2.4. Minimal $(n_1,\ldots,n_p)$-system of generators. Observe that the (infinite) intersection of elements in $\mathcal{N}(n_1,\ldots,n_p)$ is not in general a numerical semigroup because, as we already saw, $\bigcap_{n\in\mathbb{N}}\{0,n,\rightarrow\}=\{0\}$. On the other hand the intersection of numerical semigroups is always a submonoid of $(\mathbb{N},+)$.

THEOREM 186. *Let M be a submonoid of* $(\mathbb{N},+)$. *The following conditions are equivalent.*

(1) M is a $(n_1,\ldots,n_p)$-bracelet;

(2) M is an intersection of numerical $(n_1,\ldots,n_p)$-bracelets.

PROOF. *1) implies 2).* For each positive integer k, we define $S_k=M\cup\{k,\rightarrow\}$. It is clear that S_k is a numerical $(n_1,\ldots,n_p)$-bracelet and $M=\cap_{k\in\mathbb{N}\setminus\{0\}}S_k$.

2) implies 1). Suppose that $M = \cap_{i \in I} S_i$ such that S_i a numerical $(n_1, \ldots, n_p)$-bracelet, for every $i \in I$. If $a, b \in M \backslash \{0\}$ then $a, b \in S_i \backslash \{0\}$ and thus $a + b + \{n_1, \ldots, n_p\} \subseteq S_i$, for every $i \in I$. Hence $a + b + \{n_1, \ldots, n_p\} \subseteq M$ and consequently M is a $(n_1, \ldots, n_p)$-bracelet. □

If we apply Proposition 19 and Theorem 20 to the Frobenius variety $\mathcal{N}(n_1, \ldots, n_p)$ together with Theorem 186, we obtain the following result.

COROLLARY 187. *Every $(n_1, \ldots, n_p)$-bracelet has a unique minimal $(n_1, \ldots, n_p)$-system of generators and this set is finite.*

As a consequence of Proposition 21 we have the following.

COROLLARY 188. *Let M be a $(n_1, \ldots, n_p)$-bracelet and $x \in M$. The set $M \backslash \{x\}$ is a $(n_1, \ldots, n_p)$-bracelet if and only if x belongs to the minimal $(n_1, \ldots, n_p)$-system of generators of M.*

Using Corollary 187 it makes sense to define the $(n_1, \ldots, n_p)$**-rank** of a $(n_1, \ldots, n_p)$-bracelet M by the cardinality of its minimal $(n_1, \ldots, n_p)$-system of generators, which we will denote by $(n_1, \ldots, n_p)$-rank(M).

We illustrate the previous results with an example.

EXAMPLE 189. Let $S = \langle 3, 7, 8 \rangle$. From Example 185 we have that S is a $(2,3)$-bracelet. By applying Proposition 184 and Corollary 188 we obtain that $\{3, 7\}$ is the minimal $(2,3)$-system of generators of S and thus $(2,3)$-rank$(S) = 2$.

We finish this section studding the $(2,3)$-bracelet S with $(2,3)$-rank(S) equal to 1.

The next result is easy to prove by induction on k.

LEMMA 190. *If $k \in \mathbb{N}$ then $\{\lambda 2 + \mu 3 \mid \lambda, \mu \in \mathbb{N} \text{ and } \lambda + \mu \leq k\} = \{x \in \mathbb{N} \mid 2 \leq x \leq 3k\} \cup \{0\}$.*

As an immediate consequence of Theorem 173 and Lemma 190 we have the following.

PROPOSITION 191. *If m is a positive integer, then*

$$L_{\{2,3\}}(\{m\}) = \{km+i \mid k \in \mathbb{N}\backslash\{0\},\ i \in \{0,2,3,\ldots,3(k-1)\}\} \cup \{0\}.$$

The following result is straightforward to prove.

LEMMA 192. *Let S be a numerical semigroup and $m \in S\backslash\{0\}$. If $\{a, a+1, \ldots, a+m-1\} \subseteq S$ then $\{a, \rightarrow\} \subseteq S$.*

The next result gives a formula for the Frobenius number of $(2,3)$-bracelet S with $(2,3)$-rank$(S) = 1$.

COROLLARY 193. *If m is a positive integer then* $\mathrm{F}\left(L_{\{2,3\}}(\{m\})\right) = \left(\lfloor \frac{m}{3} \rfloor + 2\right) m + 1$.

PROOF.

(1) Let k be a positive integer. By applying Proposition 191 we deduce that $km+1 \in L_{\{2,3\}}(\{m\})$ if and only if $km+1 = (k-1)m+i$ for some $i \in \{0,2,3,\ldots,3(k-2)\}$. This is equivalent to $m+1 \in \{0,2,3,\ldots,3(k-2)\}$.
(2) Next we show that if $km+1 \in L_{\{2,3\}}(\{m\})$, then $\{km+1, \rightarrow\} \subseteq L_{\{2,3\}}(\{m\})$. In fact, if $km+1 \in L_{\{2,3\}}(\{m\})$ then by 1) we deduce that $m+1 \leq 3(k-2)$. Using a Proposition 191 we obtain that $\{(k-1)m+2, (k-1)m+3, \ldots, (k-1)m+m+1\} \subseteq L_{\{2,3\}}(\{m\})$. Follows from Lemma 192 $\{(k-1)m+2, \rightarrow\} \subseteq L_{\{2,3\}}(\{m\})$ and thus $\{km+1, \rightarrow\} \subseteq L_{\{2,3\}}(\{m\})$.
(3) Observe that from 1) we get that $km+1 \notin L_{\{2,3\}}(\{m\})$ if and only if $m+1 > 3(k-2)$. This is equivalent to $m \geq 3(k-2)$. Thus it proves that $km+1 \notin L_{\{2,3\}}(\{m\})$ if and only if $k \leq \lfloor \frac{m}{3} \rfloor + 2$.
(4) Now as a consequence of previous items we obtain that $\mathrm{F}\left(L_{\{2,3\}}(\{m\})\right) = \left(\lfloor \frac{m}{3} \rfloor + 2\right) m + 1$.

$\square$

We illustrate the preceding results with an example.

EXAMPLE 194. Let us calculate the set of elements in $L_{\{2,3\}}(\{7\})$. In view of Corollary 193 we obtain that $\mathrm{F}\left(L_{\{2,3\}}(\{7\})\right) = 29$. By using Proposition 191 we have that $L_{\{2,3\}}(\{7\}) = \{0\} \cup \{7\} \cup (14+\{0,2,3\}) \cup (21+\{0,2,3,4,5,6\}) \cup (28+\{0,2,3,4,5,6,7,8,9\}) \cup \{30,\rightarrow\}$ and thus $L_{\{2,3\}}(\{7\}) = \{0,7,14,16,17,21,23,24,25,26,27,28,30,\rightarrow\} = \langle 7,16,17,25,26,27,36\rangle$.

2.5. Indecomposable $(n_1,\ldots,n_p)$-bracelets. We say that a $(n_1,\ldots,n_p)$-bracelet is **indecomposable** if it can not be expressed as an intersection of $(n_1,\ldots,n_p)$-bracelets that contain it properly. As an immediate consequence of Theorem 186 we have the following result.

LEMMA 195. *Every indecomposable $(n_1,\ldots,n_p)$-bracelet is a numerical $(n_1,\ldots,n_p)$-bracelet.*

Observe that if S is a numerical semigroup, then $\mathbb{N}\backslash S$ is finite and thus the set of numerical semigroups containing S is also finite. Hence S can be expressed as an intersection of numerical semigroups containing it properly if and only if S is a intersection of finitely many numerical semigroups containing it properly.

LEMMA 196. *A numerical $(n_1,\ldots,n_p)$-bracelet is indecomposable if it can not be expressed as the intersection of two numerical $(n_1,\ldots,n_p)$-bracelets containing it properly.*

PROOF. *Necessity*. Trivial.

Sufficiency. Let S be a numerical $(n_1,\ldots,n_p)$-bracelet. By applying the comment preceding Lemma 196, if S is not indecomposable then there exist $S_1,\ldots,S_k$ numerical $(n_1,\ldots,n_p)$-bracelets that contain S properly and $S = S_1\cap\cdots\cap S_k$. We can assume that this decomposition is minimal in the sense of minimal number of numerical $(n_1,\ldots,n_p)$-bracelets involved, that is, if $j\in\{1,\ldots,k\}$ then $\bigcap_{i=1,i\neq j}^{k} S_i \neq S$. Let $S_a = S_1\cap\cdots\cap S_{k-1}$. From Proposition 181 and by minimality of decomposition of S, we have that S_a is a numerical $(n_1,\ldots,n_p)$-bracelet such that $S \subsetneq S_a$. Hence S_a and S_k are numerical $(n_1,\ldots,n_p)$-bracelets that contain S properly and $S = S_a \cap S_k$. $\square$

The next result is an adaptation of Theorem 12 to the Frobenius variety $\mathcal{N}(n_1 \ldots, n_p)$.

PROPOSITION 197. *Let $S \in \mathcal{N}(n_1 \ldots, n_p)$. The following conditions are equivalent:*

(1) S is an indecomposable $(n_1,\ldots,n_p)$-bracelet;
(2) S is maximal in the set of all numerical $(n_1,\ldots,n_p)$-bracelets with Frobenius number $\mathrm{F}(S)$*;*
(3) S is maximal in the set of all numerical $(n_1,\ldots,n_p)$-bracelets that do not contain $\mathrm{F}(S)$.

PROOF. 1) *implies* 2). Let $\overline{S}$ a numerical $(n_1,\ldots,n_p)$-bracelet such that $S \subseteq \overline{S}$ and $\mathrm{F}(S) = \mathrm{F}(\overline{S})$. It is clear that $S = (S \cup \{\mathrm{F}(S)\}) \cap \overline{S}$ and, by Proposition 181, we have that $S \cup \{\mathrm{F}(S)\}$ is a numerical $(n_1,\ldots,n_p)$-bracelet. Hence, by Lemma 196, we conclude that $S = \overline{S}$.

2) *implies* 3). Let $\overline{S}$ be a numerical $(n_1,\ldots,n_p)$-bracelet fulfilling that $S \subseteq \overline{S}$ and $\mathrm{F}(S) \notin \overline{S}$. Applying Proposition 181, we deduce that $S' = \overline{S} \cup \{\mathrm{F}(S)+1, \rightarrow\}$ is a numerical $(n_1,\ldots,n_p)$-bracelet. As $S \subseteq S'$ and $\mathrm{F}(S) = \mathrm{F}(S')$ we obtain that $S = S'$. Therefore $S = \overline{S}$.

3) *implies* 1). Let S_1 and S_2 be two numerical $(n_1,\ldots,n_p)$-bracelets that contain S properly. Then, by hypothesis, $\mathrm{F}(S) \in S_1$ and $\mathrm{F}(S) \in S_2$ this implies that $\mathrm{F}(S) \in S_1 \cap S_2$ and consequently $S \neq S_1 \cap S_2$. $\square$

The following result is easy to prove.

LEMMA 198. *Let S and $\overline{S}$ be two numerical semigroups such that $S \subsetneq \overline{S}$ and let $x = \max\left(\overline{S} \backslash S\right)$. Then $S \cup \{x\}$ is a numerical semigroup.*

The next result shows that Lemma 198 is also true for the numerical $(n_1,\ldots,n_p)$-bracelets.

LEMMA 199. *Let S and $\overline{S}$ be two numerical $(n_1,\dots,n_p)$-bracelets such that $S \subsetneq \overline{S}$ and let $x = \max\left(\overline{S}\backslash S\right)$. Then $S \cup \{x\}$ is a numerical $(n_1,\dots,n_p)$-bracelet.*

PROOF. By using Lemma 198, we know that $S \cup \{x\}$ is a numerical semigroup. To conclude the proof it suffices to show that, if $a, b \in (S \cup \{x\}) \backslash \{0\}$ then $a + b + \left\{n_1,\dots,n_p\right\} \subseteq S \cup \{x\}$. We consider the following cases.

. Since $2x + \left\{n_1,\dots,n_p\right\} \subseteq \overline{S}$ we get that $2x + \left\{n_1,\dots,n_p\right\} \subseteq S \cup \{x\}$.

. If $a \in S\backslash\{0\}$ then $a + x + \left\{n_1,\dots,n_p\right\} \subseteq \overline{S}$ and so $a + x + \left\{n_1,\dots,n_p\right\} \subseteq S \cup \{x\}$.

. If $a, b \in S\backslash\{0\}$ then $a + b + \left\{n_1,\dots,n_p\right\} \subseteq S \subseteq S \cup \{x\}$.

$\square$

THEOREM 200. *Let S be a numerical $(n_1,\dots,n_p)$-bracelet. Then S is an indecomposable $(n_1,\dots,n_p)$-bracelet if and only if for every $x \in \mathbb{N} \backslash (S \cup \{\mathrm{F}(S)\})$ we have that $S \cup \{x\}$ is not a $(n_1,\dots,n_p)$-bracelet.*

PROOF. *Necessity*. Assume that $S \cup \{x\}$ is a numerical $(n_1,\dots,n_p)$-bracelet. As $S = (S \cup \{\mathrm{F}(S)\}) \cap (S \cup \{x\})$ we get that S can be expressed as the intersection of two numerical $(n_1,\dots,n_p)$-bracelet properly containing it. Consequently S is not an indecomposable $(n_1,\dots,n_p)$-bracelet.

Sufficiency. If S is not an indecomposable $(n_1,\dots,n_p)$-bracelet then, by Proposition 197, there exists a numerical $(n_1,\dots,n_p)$-bracelet $\overline{S}$ such that $S \subsetneq \overline{S}$ and $\mathrm{F}(S) = \mathrm{F}(\overline{S})$. Let $x = \max\left(\overline{S}\backslash S\right)$ and so $x \in \mathbb{N} \backslash (S \cup \{\mathrm{F}(S)\})$. In view of Lemma 199 we deduce that $S \cup \{x\}$ a numerical $(n_1,\dots,n_p)$-bracelet. $\square$

We illustrate the preceding theorem with an example.

EXAMPLE 201. Let us show that $S = \langle 4, 9, 10, 15 \rangle$ is an indecomposable numerical (1)-bracelet. Since $S = \{0, 4, 8, 9, 10, 12, \rightarrow\}$, then we get that $\mathrm{F}(S) = 11$. By applying Proposition 168 we deduce that S is a numerical (1)-bracelet. Note that $\mathbb{N} \backslash (S \cup \{\mathrm{F}(S)\}) = \{1, 2, 3, 5, 6, 7\}$. It is clear that the sets $S \cup \{1\}$, $S \cup \{2\}$, $S \cup \{3\}$

and $S\cup\{7\}$ are not closed under addition and thus these sets are not numerical (1)-bracelet. Nevertheless the sets $S\cup\{5\}$ and $S\cup\{6\}$ are numerical semigroups. Since $5+5+1=11\notin S\cup\{5\}$ and $4+6+1=11\notin S\cup\{6\}$ these numerical semigroups are not (1)-bracelet. In view of Theorem 200 we can conclude that S is an indecomposable numerical (1)-bracelet.

Following the notation introduced in the Preliminaries, we say that a numerical semigroup is **irreducible** if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Clearly, if a numerical $(n_1,\dots,n_p)$-bracelet is irreducible then it is indecomposable. The Example 201 show us that the converse is not true. In fact $S=\langle 4,9,10,15\rangle$ is an indecomposable (1)-bracelet and S is not an irreducible numerical semigroup because $S=(S\cup\{5\})\cap(S\cup\{6\})$.

The Theorem 200 allows to algorithmically determine, whether or not a given numerical $(n_1,\dots,n_p)$-bracelet is indecomposable. Our next goal of this section is to improve this result. To this purpose we introduce some concepts and results.

From Lemma 11 we easily deduce the next result.

PROPOSITION 202. *Let S be a numerical semigroup and let $n\in S\backslash\{0\}$. Then*

$$PF(S)=\left\{w-n \mid w\in Ap(S,n) \text{ and } w'-w\notin S \text{ for all } w'\in Ap(S,n)\backslash\{w\}\right\}.$$

Given a numerical semigroup S, denote by $\mathrm{SG}(S)=\{x\in \mathrm{PF}(S)\mid 2x\in S\}$. Its elements will be called the **special gaps** of S. The following result is easy to prove.

LEMMA 203. *Let S be a numerical semigroup and let $x\in\mathbb{N}\backslash S$. Then $x\in SG(S)$ if and only if $S\cup\{x\}$ is a numerical semigroup.*

PROPOSITION 204. *Let $m_1,\dots,m_q$ be positive integers such that $S=\langle m_1,\dots,m_q\rangle$ is a numerical $(n_1,\dots,n_p)$-bracelet and let $x\in SG(S)$. Then $S\cup\{x\}$ is a $(n_1,\dots,n_p)$-bracelet if and only if $x+\left\{x,m_1,\dots,m_q\right\}+\left\{n_1,\dots,n_p\right\}\subseteq S$.*

PROOF. *Necessity*. Trivial.

Sufficiency. Take $a,b \in (S\cup\{x\})\setminus\{0\}$, and let us prove that $a+b+\{n_1,\ldots,n_p\} \subseteq S\cup\{x\}$. We distinguish three different cases.

. If $a,b \in S$ then $a+b+\{n_1,\ldots,n_p\} \subseteq S \subseteq S\cup\{x\}$.

. If $a=b=x$ then $a+b+\{n_1,\ldots,n_p\} = 2x+\{n_1,\ldots,n_p\} \subseteq S \subseteq S\cup\{x\}$.

. If $a=x$ and $b\in S$, then there exist $s\in S$ and $i\in\{1,\ldots,q\}$ such that $b=m_i+s$, because $b\neq 0$. Therefore $a+b+\{n_1,\ldots,n_p\} = x+m_i+s+\{n_1,\ldots,n_p\} \subseteq S \subseteq S\cup\{x\}$.

□

Next we illustrate some of these results with an example

EXAMPLE 205. Let $S = \langle 5,12,19,26,33\rangle$. Then $S = \{0,5,10,12,15,17,19,20,22,24,25,26,27,29,30,31,32,33,\rightarrow\}$ and thus $\mathrm{F}(S)=28$. It easy clear that S is a numerical (2)-bracelet such that $\mathrm{Ap}(S,5)=\{0,12,19,26,33\}$. By Proposition 202, we have that $\mathrm{PF}(S)=\{7,14,21,28\}$ and thus $\mathrm{SG}(S)=\{21,28\}$. Applying Lemma 203 we obtain that $S\cup\{21\}$ and $S\cup\{28\}$ are numerical semigroups. Since $21+5+2=28\notin S$ and $28+\{28,5,12,19,26,33\}+\{2\}\subseteq S$, by Proposition 204, we get that $S\cup\{21\}$ is not a numerical (2)-bracelet and $S\cup\{28\}$ is a numerical (2)-bracelet.

As an immediate consequence of Theorem 200 and Proposition 204, we obtain the following result.

COROLLARY 206. *Let $m_1,\ldots,m_q$ be positive integers such that $S=\langle m_1,\ldots,m_q\rangle$ is a numerical $(n_1,\ldots,n_p)$-bracelet. Then S is an indecomposable $(n_1,\ldots,n_p)$-bracelet if and only if for every $x\in SG(S)\setminus\{\mathrm{F}(S)\}$ we have that $x+\{x,m_1,\ldots,m_q\}+\{n_1,\ldots,n_p\}\nsubseteq S$.*

Observe that as a consequence of previous corollary we can conclude that the numerical (2)-bracelet $S=\langle 5,12,19,26,33\rangle$ given in Example 205 is indecomposable (2)-bracelet.

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Combinatory Problems in Numerical Semigroups

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