UNIVERSIDADE DE ÉVORA

Optimization with flexible objectives and constraints

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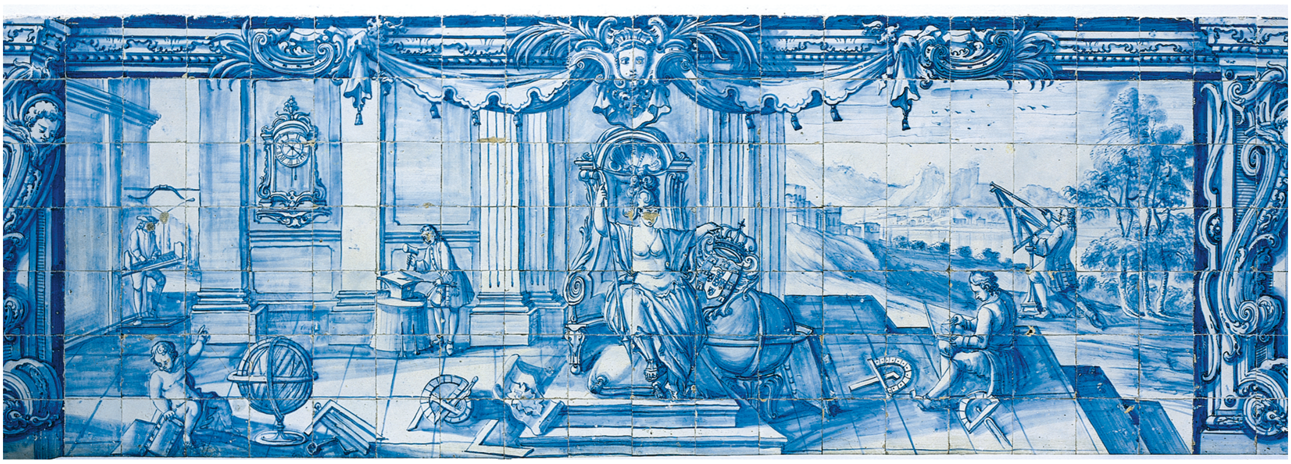
Tese apresentada à Universidade de Évora
para obtenção do Grau de Doutor em Matemática
na especialidade Matemática e Aplicações

Orientador: *Imme van den Berg*

January 23, 2018



INSTITUTO DE INVESTIGAÇÃO E FORMAÇÃO AVANÇADA

UNIVERSIDADE DE ÉVORA

Instituto de Investigação e Formação Avançada

Departamento de Matemática

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Matemática

Dissertação

January 23, 2018

To my family.

Acknowledgements

This thesis has been completed with the great help of so many people, I am very grateful for all of their help.

First and foremost, I am extremely thankful to my supervisor Prof. Imme van den Berg. I thank him for his kindness, help, being so patient with me during the course, spending a lot of time to help me improving English, to read throughout the manuscript and for giving me many invaluable suggestions and corrections. I also thank him for his guidance, encouragement and for teaching me the non-standard analysis.

I would like to thank Dr. Júlia Maria da Rocha Vlaverde Justino, Prof. Francince Diener, Prof. Mack Diener, Prof. Claude Lobry for the kindness of spending the time to discuss with me and in particular, I thank Dr. Bruno Dinis for some cooperation, discussions, suggestions.

Personal thanks are due to my friends in Évora, especially Kanchalika and Puthnith among other friends from Erasmus-mundus programme, for helping me to integrate with life in Évora and improve my English, for asking me to join many activities and a good time we have together in Évora. Special thanks to my friend Huynh Minh Hien for his encouragement and support, for helping me to read the manuscript and correct this work, and I also thank the secretary Ms. Gabriela for her administrative support.

I would like to express my gratitude to Department of Mathematics, University of Évora; University of Danang, Campus in Kontum for providing me the good conditions to complete this work. In particular, I am deeply indebted to the Erasmus-mundus programme Emmasia 2014 for financial support. This work is not possible to be done without this.

Finally, I am eternally grateful to my parents and my big family for their enormous love, encouragement, moral support. Especially, I thank my wife: Nguyen Thi Vy Vy because you sacrifice a lot to take care our family, look after our son so that I can concentrate on doing research, and my son: Tran Nguyen Nguyen Khang, because of you, I have more motivation, more energies to work. This work is dedicated to all of you.

Contents

List of notations

$\mathcal{N}$	the set of all neutrices
$\oslash$	the set of all infinitesimals
£	the set of all limited numbers
$@$	the set of all positive appreciable numbers
$\not\infty$	the set of all positive unlimited numbers
$\oslash_N$	the set of all real absorbers of a neutrix N
$\not\infty_N$	the set of all real exploders of a neutrix N
$\mathbb{E}$	the set of all external numbers
$\mathbb{R}$	the set of all non-standard real numbers
$\mathbb{N}$	the set of all non-standard natural numbers
$\mathbb{R}_-$	the set of all non-positive real numbers
$\mathbb{R}_+$	the set of all non-negative real numbers
$\mathcal{P}(\mathbb{R})$	the collection of all subsets of $\mathbb{R}$
$\forall^{stfin} Z$	the standard finite set Z
$L(\mathbb{R}^n)$	the set of all linear operator over $\mathbb{R}^n$
st	standard
$N(\alpha)$	the neutrix part of an external number α
${}^{N}_{M}DF$	the $M \times N$-derivative of a flexible function F
$\dfrac{d_N F}{d_M x}$	the $M \times N$-derivative of a flexible function F
$\mathbb{N}^\sigma$	the set of all standard natural numbers
$\mathcal{M}_{m,n}(\mathbb{K})$	the set of all $m \times n$ matrices over $\mathbb{K}$
$\mathcal{M}_n(\mathbb{K})$	the set of all square matrices of order n over $\mathbb{K}$
$\underline{\text{conv}}(S)$	the lower convexification of S
$\overline{\text{conv}}(S)$	the upper convexification of S
$P(S)$	the projection of S on the non-standard real line
sup	the external supremum
inf	the external infimum
$\\|\cdot\\|$	the norm on the set of external numbers or $\mathbb{R}^n$
$u_n \hookrightarrow \alpha$	the flexible sequence u_n is strongly convergent to α

x_{opt}	optimal solution
$\subseteq$	the inclusion relation
$\subset$	the strict inclusion relation
$\alpha \# \beta$	two external numbers α, β are separate
$B[x_0, r]$	the closed M-ball centred x_0 of radius $r > M$
$B(x_0, r)$	the open M-ball centred x_0 of radius $r > M$
$B_M(x_0, r)$	the open outer M-ball centred x_0 of radius $r > M$
$\langle \cdot \rangle$	the inner product
$\mathrm{mr}(\mathcal{A})$	the minor-rank of a matrix $\mathcal{A}$
$r(\mathcal{A})$	the row-rank of a matrix $\mathcal{A}$
$\mathrm{sr}(\mathcal{A})$	the strict rank of $\mathcal{A}$
I_A	a near identity matrix
$R(\mathcal{A}) = \dfrac{\overline{A}\overline{\alpha}^{n-1}}{\underline{\Delta}}$	the relative uncertainty of $\mathcal{A}$
$P(\mathcal{B}) = \underline{B}/\overline{\beta} \vee P(\mathcal{B}) = \underline{B} : \overline{B}$	the relative precision of $\mathcal{B}$
$\mathrm{Im}(F)$	the image of F
$\mathrm{epi}(F)$	the epigraph of F
$\mathrm{Gr}(F)$	the graph of F
$\mathcal{A}_{i_1 \dots i_k, j_1 \dots j_k}$	the $k \times k$ matrix formed by removing from $\mathcal{A}$ the rows whose indices do not belong to $\{i_1, \dots, i_k\}$ and columns whose indices do not belong to $\{j_1, \dots, j_k\}$
$M_{i_1 \dots i_k, j_1 \dots j_k} = \det(\mathcal{A}_{i_1 \dots i_k, j_1 \dots j_k})$	a $k \times k$ minor of $\mathcal{A}$
$M_{i,j}^{(k)} = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1k} & \alpha_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kk} & \alpha_{kj} \\ \alpha_{i1} & \cdots & \alpha_{ik} & \alpha_{ij} \end{bmatrix}$	
$M_{i,j}^{(0)} = \alpha_{ij}$ for all $1 \le i, j \le n$	
$M_{k,k}^{(k-1)} = M^{(k)}$	
$m_{i_1 \dots i_k, j_1 \dots j_k}$	the representatives of $M_{i_1 \dots i_k, j_1 \dots j_k}$
$m_{i,j}^{(k)}$	the representatives of $M_{i,j}^{(k)}$
$m^{(k)}$	the representatives of $M^{(k)}$
$\lvert\overline{\alpha}\rvert = \max\limits_{\substack{1 \le i \le m \\ 1 \le j \le n}} \lvert\alpha_{ij}\rvert$	
$\overline{A} = \max\limits_{\substack{1 \le i \le m \\ 1 \le j \le n}} A_{ij}$	
$\underline{A} = \min\limits_{\substack{1 \le i \le m \\ 1 \le j \le n}} A_{ij}$	
$\lvert\overline{\beta}\rvert = \max\limits_{1 \le i \le m} \lvert\beta_i\rvert$	
$\overline{B} = \max\limits_{1 \le i \le m} B_i$	
$\underline{B} = \min\limits_{1 \le i \le m} B_i$	
$\overline{a} \in \overline{\alpha}$	
$\overline{b} \in \overline{\beta}$	

Lista de Acrónimos

URL *Uniform Resource Locator*

IIFA Instituto de Investigação e Formação Avançada

ECT Escola de Ciências e Tecnologia

ECS Escola de Ciências Sociais

EA Escola de Artes

ESESJD Escola Superior de Enfermagem S. João de Deus

UE Universidade de Évora

Sumário

Otimização com objetivos e restrições flexíveis

A programação linear e a otimização não linear são estudadas do ponto de vista da análise não-standard, nos casos em que a função objetivo e/ou as restrições não são totalmente especificadas, permitindo de facto alguma imprecisão ou flexibilidade em termos de pequenas variações.

A ordem de grandeza de tais variações será modelada por neutrizes, que são subgrupos convexos aditivos da reta real não-standard, e por números externos, que são a soma de um número real com uma neutrix. Esta abordagem preserva as características essenciais de imprecisão, mantendo regras de cálculo bastante fortes e eficazes.

Funções, sequências e equações que envolvem números externos são designadas de *flexíveis*. Consideram-se problemas de otimização com funções objetivo e/ou restrições flexíveis em que são dadas as condições necessárias e suficientes para a existência de soluções ótimas ou aproximadamente ótimas, tato para problemas de otimização linear como não linear.

Para exemplificar a programação linear nesta configuração são estudados, sistemas flexíveis de equações lineares. As condições para a solubilidade de um sistema flexível por métodos usuais tais como a regra de Cramer e o mo todo de eliminação de Gauss-Jordan são estabelecidas. Além disso, é considerado um método de parâmetros para resolver sistemas flexíveis onde são apresentadas fórmulas de soluções dependendo dos parâmetros. O conjunto de soluções de um sistema flexível é expresso em termos de vetores externos e neutrizes.

Para estudar a otimização não linear com objetivos e restrições flexíveis, são desenvolvidas ferramentas de análise para sucessões e funções flexíveis.

Palavras chave: Otimização, incerteza, número externo, sistema flexível, função flexível, análise não-standard

Abstract

Optimization with flexible objectives and constraints

Both linear programming and non-linear optimization are studied from the point of view of non-standard analysis, in cases where the objective function and/or the constraints are not fully specified, indeed allow for some imprecision or flexibility in terms of some limited shifts.

The order of magnitude of such shifts will be modelled by neutrices, additive convex subgroups of the non-standard real line and external numbers, sums of a neutrix and a non-standard real number. This approach captures essential features of imprecision, maintaining rather strong and effective rules of calculation.

Functions, sequences and equations which involve external numbers are called *flexible*. We consider optimization problems with flexible objective functions and/or constraints.

Necessary and sufficient conditions for the existence of optimal or approximate optimal solutions are given for both linear and non-linear optimization problems with flexible objective functions and constraints.

To deal with linear programming in this setting, flexible systems of linear equations are studied. Conditions for the solvability of a flexible system by usual methods such as Cramer's rule and Gauss-Jordan elimination are established. Also, a parameter method is considered to solve flexible systems. Formulas of solutions depending on parameters are presented. The set of solutions of a flexible system is expressed in terms of external vectors and neutrices.

In order to investigate non-linear optimization with flexible objectives and constraints, we develop tools of analysis for both flexible sequences and functions.

Keywords: Optimization, uncertainty, external number, flexible system, flexible function, non-standard analysis

1

Introduction

The main purpose of this work is to study optimization problems, where firstly the objective function and/or the constraints are not fully specified and secondly the processing of data involves approximations and propagations of errors.

Mathematical models may not transmit completely information and some factors maybe lacking. For instance, in many situations we do not fix the amount we will spend for what we want buy, but only approximately. Some goods, like houses, cars do only have approximate values which are subject to negotiations. Imprecisions may also be subjective: usually the seller knows the market better than customers. Or in the process of producing goods we may not know all factors affecting this process which will be reflected in the definite price. In addition, data may come from physical measuring and statistical testing will never give precise outcomes. We may only be able to estimate upper and lower bounds of unknown qualities, or sometimes it is difficult to find the probability distribution. As a consequence, it may be more natural if variables range over a subset of $\mathbb{R}$, instead of representing precise real numbers.

Also functions used in mathematics models tend to be complicated and then we commonly make approximations to implement more easily. All these imprecisions and uncertainties are present when we are processing the data, to which we add also errors of calculations and rounding-off. As a result, in practice, outcomes of these models represent reality only in approximations.

In classical mathematics, uncertainties can be expressed by either the functional $o(\cdot)$ and $O(\cdot)$, by interval calculus [24, 17], by parameters or by using random variables like in statistics and stochastic processes [30].

All these methods are not very effective in dealing with algebraic operations and thus with error propagation. The functional notions $o(\cdot), O(\cdot)$ do not satisfy all algebraical properties and moreover lack total order which causes complexities and inefficiency. The situation is even worse for interval calculus and calculus based on random variables, moreover there are some difficulties to implement these operations in practice.

In this work we will use neutrices - convex additive groups of non-standard real line, and external numbers -the sum of a real number and a neutrix to model uncertainties.

An external number is an external set of real numbers relatively close to a given number (see [22, 23, 11, 19, 20]). These numbers capture essential features of imprecisions. In fact, they are stable under some shifts, additions and multiplications. For example $1 + \pounds = \pounds + \pounds = \pounds \cdot \pounds = \pounds$, where £ is the set of all limited numbers of non-standard real line. These properties of invariance induce flexibility and possibilities of neglecting and simplification to external numbers and operations on them. In fact, the term neutrix is borrowed from Van der Corput [8] who introduced it in the form of rings of functions, with the same objective of neglecting and simplification. We observe that Van der Corput's calculus, like the calculus of $o's$ and $O's$ does not respect all algebraic operations and total order.

Operations on external numbers and their rules of calculation respect more algebraic operations and also total order. In this work we exploit several advantages which lead to simplifications and efficiency in calculations. External numbers were introduced by Van den Berg and Koudjeti, [22, 23] and further developed by Van den Berg, Dinis and Julia Justino [11, 13, 12, 19].

Functions and sequences with external numbers are called *flexible*. An optimization problem such that the values of the objective function and/or variables are external numbers is called an *optimization problem with flexible objective and constraints*. We will consider both linear programming and non-linear optimization with flexible objective and constraints. As for both cases, we study necessary and sufficient conditions for the existence of optimal solutions.

The theory of linear programming is based on the theory of linear systems and matrix calculus. Similarly, in order to study linear programming with flexible objective and constraints, we first need to investigate matrices, determinants and also systems of linear equations in which coefficients are not real numbers but external numbers, called *flexible systems*. In our setting equalities become inclusions.

To study non-linear optimization problems and determine optimality conditions, tools of analysis of both flexible functions and sequences are needed such as convergence, continuity and differentiation.

The thesis has the following structure.

Chapter 2 is devoted to neutrices and external numbers. We will prove some properties related to the order relations, the external infimum and supremum, absolute values and norm of external numbers, and n-th roots of an external number. Results in this chapter are necessary to study the next chapters.

In Chapter 3 we will study vectors, matrices and determinants with external numbers. Adapted versions of notions and results in classical linear algebra are developed here, and deal with, for instance, properties of a determinant, the notions of linear independence and dependence of vectors with external numbers, the notion of rank of a matrix, and the relationship between the rank of a set of vectors and of a matrix. Unfortunately, the equality between the maximal number of independent row vector and the rank of a matrix determined via minors in this context is not clear. Different notions of rank of a matrix with external numbers are given. One is based on minors, one is based on the maximal number of independent row vectors and the other is based on both minor and the rank of a representative matrix which is a matrix over $\mathbb{R}$. Some conditions are considered such that at least one of them is equal to other.

Chapter 4 is dedicated to present results on systems of linear equations with external numbers. In general, the usual methods like Cramer's rule or the Gauss-Jordan elimination must be adapted. In the thesis "Nonstandard linear algebra with error analysis " by Júlia Maria da Rocha Vilaverde Justino [19], conditions were given to guarantee that the Cramer's rule can be applied to the non-singular non-homogeneous flexible systems of linear equations. We will extend these results to non-singular flexible systems. We will also present conditions such that the Gauss-Jordan elimination works well on non-singular flexible systems. We will apply the results in Chapter 3 to singular flexible systems. To be more precise, we will transform a non-singular system to an equivalent system whose rank is equal to the number of equations. Then some variables will be seen as parameters and we express the in terms of these parameters. We can apply Cramer's rule or the Gauss-Jordan elimination to the latter flexible system to find solutions. A solution formula is given. In the last section, a parameter method will be used to solve a flexible system. The neutrix parts of the constant terms of a flexible system will be seen as sets of parameters. Under certain conditions, exact solution formulas of flexible systems are expressed with these parameters.

Chapter 5 concern sequences with external numbers. We call them *flexible sequence*. An adapted version of the notion of convergence is developed for this kind of sequences. We will present several properties of convergence, which deal with operations, boundedness, convergence of subsequences and the Cauchy criterion. In particular, we will show that if a flexible sequence converges to an external number then all elements of this sequence, except at most finite terms of the sequence go inside the limit. We call this property *strong convergence*.

Similar properties of analysis are studied in Chapter 6 in the context of flexible functions. We will also consider the notions of continuity, derivative, higher derivatives as well as their properties they include an inverse function theorem and an implicit function theorem.

In Chapter 7 we will study linear programming with flexible objective and constraints. We consider two cases. In the first case, we investigate the problem in which the objective function is flexible, but the domain is precise. We provide a condition such that the problem has optimal solutions. Then we deal with the general case in which coefficients in both objective function and constraints are external numbers. We will use the results of the first case taking representatives of the coefficients in constraints. We will present conditions such that we can find an optimal solution of the original problem from an optimal solution of the problem with the precise domain.

In Chapter 8 we will investigate non-linear optimization problems with flexible objective and precise domain. We will build necessary and sufficient conditions for the existence of optimal or approximate optimal solutions. We will use different approaches to construct optimality conditions.

First of all, we generalize a well-known classical result which says that the derivative of a differentiable function $f\colon X \to \mathbb{R}$ vanishes at an extreme point. In this context, the notion of N-derivative of a flexible function given in Chapter 6 is used instead of ordinary derivative should be equal to a neutrix and does not need to be zero.

Next representatives of the objective function are used to construct optimality conditions. Conditions are given to guarantee that we can find optimal/approximate optimal solutions of a given optimization problem from an optimal solution of the problem whose objective function is a representative of the original function. This corresponds to a conventional problem, so we can apply results in classical optimization theory. We also use the relationship between the notions of the external infimum or supremum and the optimal value of a representative function of the objective function to give optimality conditions. If an optimization problem whose objective function is a representative has optimal solutions, and the relationships between external supremum or infimum and the value of the flexible function at an optimal solution of this problem satisfy some conditions then we can know that the original problem has optimal solutions or not. In some case, we also can find these optimal or approximate optimal solutions.

Also, by treating each external number as a collection of parameters we present conditions such that optimal solutions of the original problem can be found from the set of optimal solutions of optimization problems with respect to parameters.

Then each flexible function will be seen as a set-value mapping. We will modify the notion of derivative in the theory of set-valued mapping and then use this notion to build necessary and sufficient conditions for the existence of optimal or approximate optimal solutions.

Finally we extend the Lagrange multiplier method for non-linear optimization to the setting of external numbers.

2

Neutrices and external numbers

The purpose of this chapter is to present common background necessary to the rest of the thesis. First of all, this thesis will use nonstandard analysis introduced by Robinson [15, 28] in the setting of model theory. Our setting is the axiomatic approach to nonstandard analysis Internal Set Theory (IST) which was introduced by Nelson [25], see also [14, 23], for background.

IST is an extension of Zermelo-Fraenkel set theory ZFC. The language of IST adds to the primitive symbol $\in$ a new unary predicate st, which stands for standard. Formulas that involve the predicate st are called *external*, the others are called *internal* and correspond to formulas of conventional set theory. We can freely use the terms such as *external function, external sequence* corresponding to external formulas or *internal function, internal sequence* corresponding to internal formulas.

The set of nonstandard numbers $\mathbb{R}$ is defined in IST by the same formula as in conventional set theory. However, the nonstandard real line $\mathbb{R}$ has not only standard numbers but also nonstandard numbers. A real number ϵ is called *infinitesimal* if for all standard numbers $n \in \mathbb{N}$, one has $|\epsilon| < \dfrac{1}{n}$. A real number ω is called *unlimited* or *infinitely large* if $|\omega| > n$, for all standard $n \in \mathbb{N}$. Non-zero infinitesimals and unlimited numbers are nonstandard.

An *internal set* is a collection of mathematical entities defined by an internal formula. For example, let $\epsilon > 0$ be an infinitesimal, the collection of real numbers $S = \{x \in \mathbb{R} : -\epsilon \leq x \leq \epsilon\}$ is an internal set but not a standard set. IST only deals with internal sets. Now consider the collection of all infinitesimals $\oslash = \left\{\epsilon \in \mathbb{R} \middle| \forall st(n) \in \mathbb{N},\ |\epsilon| < \dfrac{1}{n}\right\}$. This is not an internal set. Also, obviously, some properties of classical mathematics is not true for this collection. For example, $\oslash$ is bounded but the infimum of this set does not exist. In [21] extended the bounded part of IST it extended to a theory of external sets, which we adopt in this thesis; external sets are collections of mathematical entities which satisfy external formulas. More examples of external sets are the set of all positive unlimited numbers, denoted by $\not\infty$, the set of all limited numbers , denoted by £, which are real numbers but not unlimited numbers, or the set of all *appreciable numbers*, denoted by @, which are positive limited numbers that are not infinitesimal.

Neutrices and External numbers were introduced in [22, 23], typically are external subsets of $\mathbb{R}$. Many notions and properties are common with ordinary real numbers, for instance, one may define algebra operations, order relations, supremum, infimum, the opposite entry of a given external number, its absolute value and related properties, norm, and n-th roots. Notions and results are contained in [14, 11, 22, 23, 25, 26]. Many of them are recalled but we also prove some new properties. The properties presented here are necessary for the next chapters.

2.1 Neutrices

Definition 2.1.1. A *neutrix* is an additive convex subgroup of the set of nonstandard real numbers $\mathbb{R}$.

So a neutrix is an external subset of $\mathbb{R}$.

Note that a non-empty convex set N of $\mathbb{R}$ is a neutrix if and only if

(i) N is symmetric with respect to 0,

(ii) $n \cdot N = N$ for all standard $n \in \mathbb{N}$.

In contrast to the conventional real line which has only two trivial neutrices $\{0\}$ and $\mathbb{R}$, the nonstandard real line has an infinity of neutrices. All neutrices are external sets, except the two neutrices above. The most common neutrices are $\oslash$ and £. Some other neutrices, with ϵ a positive infinitesimal, are $\epsilon\oslash$, ϵ£, £$\epsilon^{\not\infty} = \bigcap_{st(n)\in\mathbb{N}} [-\epsilon^n, \epsilon^n]$,

$$£e^{-\frac{@}{\epsilon}} = \bigcup_{st(n)\in\mathbb{N}} [-e^{-\frac{1}{n\epsilon}}, e^{-\frac{1}{n\epsilon}}].$$

Let $\epsilon > 0$ be infinitesimal and

$$S = \{f\colon \mathbb{R} \longrightarrow \mathbb{R}_+ \setminus \{0\}\}.$$

Then $\bigcap_{\mathrm{st}(f)\in S}[-f(\epsilon),f(\epsilon)]$ is a neutrix. We see that the cardinal of S is larger than the cardinal of $\mathbb{N}$. In fact, we can define a neutrix with similar constructions using arbitrary cardinalities.

We denote by $\mathcal{N}$ the collection of all neutrices. It is an "extended class", but in [12, 13] Dinis and Van den Berg show that a model can be found where it acts as a set. In this work we use the term "set of neutrices" with abuse of language. Note that $\mathcal{N}$ is totally ordered by inclusion. For more details on neutrices, we refer to [11, 22, 23, 3].

Operations on $\mathcal{N}$ are defined as the Minkowski operations.

Definition 2.1.2. The operations on $\mathcal{N}$ are defined as follows: for $A, B \in \mathcal{N}$, and $t \in \mathbb{R}$,

$$A + B = \{a + b \mid (a, b) \in (A \times B)\},$$

$$A \cdot B = \{a \cdot b \mid (a, b) \in (A \times B)\},$$

$$tA = \{t \cdot x \big| x \in A\},$$

and

$$A : B = \{c \in \mathbb{R} \mid c.B \subseteq A\}.$$

We will present some useful properties of operations on $\mathcal{N}$ which will be frequently used in this work. For more details and their proofs, we refer readers to [22].

Proposition 2.1.3 ([22]). *Let $A, B \in \mathcal{N}$. One has*

(i) $A + B = \max\{A, B\}$.

(ii) $tA = A$ *for all* $|t| \in @$ *and* $£A = A$.

Proof. (i). Assume that $A \subseteq B$. Then one has

$$B \leq A + B \leq B + B = B.$$

(ii). Without loss of generality, we assume that $t > 0$. Then there exists standard $n \in \mathbb{N}$ such that $\frac{1}{n} \leq t \leq n$. Also, one proves by external induction [IST] shows that $nA = A$, this implies that $A = \frac{nA}{n} = \frac{1}{n}A \subseteq tA \subseteq nA = A$ and hence $tA = A$.

Moreover, obviously, $A \subseteq £A$. Now let $x \in £A$. Then there exists $u \in £, v \in \mathcal{A}$ such that $x = uv$. Because $u \in £$, there exists $t \in @$ such that $u \in [-t, t]$. This implies that $uA \subseteq tA = A$. So $x \in A$ and hence $£.A \subseteq A$. We conclude that $A = £A$. $\square$

So the neutrix £ acts as the identity element for multiplication on $\mathcal{N}$ and neutrices are invariant under multiplication by appreciable numbers.

Example 2.1.4. One has

(i) $\oslash + \pounds = \pounds$; $\oslash + \epsilon\pounds = \oslash$.

(ii) $\pounds \cdot \pounds = \pounds$; $\pounds \cdot \oslash = \oslash$; $\epsilon \oslash \cdot \epsilon\pounds = \epsilon^2 \oslash$.

(iii) $\oslash : \oslash = \pounds$; $\pounds : \oslash = \pounds$.

Definition 2.1.5. ([22, p. 53]). A neutrix N is said to be *idempotent* if $N \cdot N = N$.

For example, $\oslash$ and $\pounds$ are idempotent neutrices since $\oslash \cdot \oslash = \oslash$ and $\pounds \cdot \pounds = \pounds$. Let $\epsilon > 0$ be infinitesimal. Using the Minkowski operations it is easy to verify that $@ + @ = @$ and $\not\propto + \not\propto = \not\propto$. As a consequence $\pounds\epsilon^{\not\propto}$, $\pounds e^{\frac{@}{\epsilon}}$ are idempotent neutrices. However, $\epsilon\pounds$ is not idempotent since $\epsilon\pounds \cdot \epsilon\pounds = \epsilon^2\pounds \subset \epsilon\pounds$.

If N is an idempotent neutrix and $n \in \mathbb{N}$ is standard, by External induction we have $N^n = N \cdot N \cdots N = N$.

The next result states that every neutrix is represented by the product of a positive real number and an idempotent neutrix.

Proposition 2.1.6 ([23, 22])**.** *Let A is a neutrix. Then there exists a real positive number t and a unique idempotent neutrix I such that $A = t \cdot I$.*

2.2 External numbers

An *external number* is the sum of a real number and a neutrix. So each external number has the form $\alpha = a + A$, where $a \in \mathbb{R}$ is called a *representative* of α and $A \in \mathcal{N}$ is called the *neutrix part* of α, denoted by $N(\alpha)$. We also call a the *real part* of α. If $0 \notin \alpha = a + N(\alpha)$, we call α *zeroless*

For example, $\alpha = 1 + \epsilon\oslash$, $\beta = \oslash$ and $\gamma = \epsilon$ are external numbers, here ϵ is a positive infinitesimal. In particular, neutrices and real numbers are external numbers.

Convention 2.2.1. From now on, we write an external number α in the form $\alpha = a + A$, we always using the lower-case as a representative of α and the upper-case as the neutrix part of α.

Note that for each external number α, the neutrix part $N(\alpha)$ is unique but its representative is not. In fact, $\alpha = b + N(\alpha)$ for all $b \in \alpha$. For example, $1 + \pounds = 0 + \pounds = \pounds$.

Once again, the collection of all external numbers is a class, denoted by $\mathbb{E}$. Similarly to neutrices, we also use the term "subset" of $\mathbb{E}$ with an abuse of language.

Two external numbers are either distinct or one is contained in the other.

Lemma 2.2.2 ([22])**.** *Let $\alpha = a + A, \beta = b + B$ be two external numbers. Then*

$$\alpha \cap \beta = \emptyset \vee \alpha \subseteq \beta \vee \beta \subseteq \alpha.$$

Proof. Assume that $\alpha \cap \beta \neq \emptyset$. Let $x \in \alpha \cap \beta$. Then $\alpha = x + A$ and $\beta = x + B$. So, if $A \subseteq B$ then $\alpha \subseteq \beta$, if $B \subseteq A$ then $\beta \subseteq \alpha$. $\square$

Notation 2.2.3. Let α, β be two external numbers. We write $\alpha \# \beta$ if $\alpha \cap \beta = \emptyset$.

Next we define the opposite element of a given external number. Then we will present some of its properties.

Definition 2.2.4. Let $\alpha = a + A$ be an external number. We call $-\alpha \equiv -a + A$ the *opposite number* of α.

Proposition 2.2.5. *Let $\alpha \in \mathbb{E}$ be an external number. Then $-\alpha = \{-x | x \in \alpha\}$.*

Proof. Let $\alpha \in \mathbb{E}$ be an external number. One has $-\alpha = -a + A = -a - A = -(a + A) = \{-x | x \in \alpha\}$. ☐

Corollary 2.2.6. *Definition 2.2.4 does not depend on the choice of the representative of α.*

Proof. Assume that $\alpha = b + A$. One has $-\alpha = \{-x | x \in \alpha\} = -a + A$. ☐

Definition 2.2.7. Let $\alpha = a + A$ be an external number. We call α *positive* if it contains only positive numbers, *negative* if it contains only negative numbers, and *neutricial* if α is a neutrix. A external number is called *non-negative* if it is either positive or neutricial and *non-positive* if it is either negative or neutricial.

Proposition 2.2.8. *Let $\alpha = a + A$ be a zeroless external number. Then*

(i) α is positive if and only if $a > A$,

(ii) α is negative if and only if $a < A$.

Proof. We will prove the first part, the second can be done similarly. Assume that α is positive. If $a < A$ then $u = a + x < 0$ for all $x \in A$, which is a contradiction. If $a \in A$ then $0 \in \alpha$, a contradiction. Hence $a > A$. Conversely, assume that $a > A$. Then for all $u \in \alpha$ one has $u = a + x > 0$. So $\alpha = a + A$ is positive. ☐

Example 2.2.9. Let $\epsilon > 0$ be infinitesimal. Consider the external numbers $\alpha = \epsilon £, \beta = 2 + \oslash, \gamma = -1 + \epsilon\oslash$. Then α is neutricial, β is positive and γ is negative.

Definition 2.2.10. Let $\alpha = a + A$ be an external number. The *absolute value* of α is defined by

$$|\alpha| = |a| + A.$$

Example 2.2.11. Let $\alpha_1 = \oslash, \alpha_2 = -1 + \oslash, \alpha_3 = 3 + \epsilon £$ with $\epsilon > 0$ an infinitesimal. Then $|\alpha_1| = \oslash, |\alpha_2| = 1 + \oslash$ and $|\alpha_3| = 3 + \epsilon £$.

Proposition 2.2.12. *Let $\alpha = a + A$ be a zeroless external number. Then $|\alpha| = \big\{|x| \big| x \in \alpha\big\}$.*

Proof. Let $\xi = \big\{|x| \big| x \in \alpha\big\}$. We show that $\xi = |\alpha|$. We consider two cases: $a > A$ and $a < A$.

For the first case, let $x \in |\alpha|$. Then $x = |a| + u$ with $u \in A$. Since $a > A$, one has $x = |a| + u = a + u > 0$. Then $x = |a + u| \in \xi$. Hence $|\alpha| \subseteq \xi$. On the other hand, let $y \in \xi$. Then $y = |a + v| = a + v = |a| + v$ since $a > A$. So $y \in |\alpha|$. Hence $\xi \subseteq |\alpha|$. One concludes that $\xi = |\alpha|$.

For the second case, let $x \in |\alpha|$. Then $x = |a| + u$ with $u \in A$. Because $a < A$ one has $x = -a + u = -(a + (-u))$. Also, $(a + (-u)) < 0$, so $x = |a + (-u)| \in \xi$. Hence $|\alpha| \subseteq \xi$. Conversely, let $y \in \xi$. Then $y = |a + u|$ for some $u \in A$. Because $a < A$, one has $a + u < 0$. So $y = -(a + u) = -a + (-u) = |a| + (-u) \in |\alpha|$ since $(-u) \in A$. Hence $\xi \subseteq |\alpha|$. We obtain $|\alpha| = \xi$. □

Note that the conclusion above is not true in the case α is neutricial.

Corollary 2.2.13. *Let $\alpha = a + A$ be a zeroless external number. The definition of absolute value of α does not depend on the choice of the representative of α.*

Proof. Assume that $\alpha = b + A$. Then $|\alpha| = |b| + A = \big\{|x| \big| x \in \alpha\big\} = |a| + A$. □

2.2.1 Operations on external numbers

In this section we recall operations and some of their properties on $\mathbb{E}$. For more details, we refer to [22, 11].

Operations on external numbers such as: subtraction, addition, multiplication, division are defined by the Minkowski law. However, in practice, implementing these operations are much easier as shown below. The formulas in Proposition 2.2.14 were introduced in [22, 23] without proof. We rewrite them here with full proof.

Proposition 2.2.14. ([23, p.151], [22, p.89]). *Let $\alpha = a + A, \beta = b + B \in \mathbb{E}$ be external numbers. Then*

(i) $\alpha \pm \beta = a \pm b + \max\{A, B\} = a \pm b + A + B$,

(ii) $\alpha\beta = ab + \max\{aB, bA, AB\} = ab + Ab + Ba + AB$.

Proof. (i) We will prove only the addition, the subtraction can be done similarly. One has $\alpha + \beta = \{x + y \big| x \in \alpha, y \in \beta\} = \{(a + u) + (b + v) \big| u \in A, v \in B\} = \{(a + b) + (u + v) \big| u \in A, v \in B\} = (a + b) + (A + B) = a + b + \max\{A, B\}$.

(ii) One has

$$\begin{aligned}\alpha\beta &= \big\{x \cdot y \big| x \in \alpha, y \in \beta\big\} = \big\{(a + u)(b + v) \big| u \in A, v \in B\big\}\\ &= \big\{ab + au + bv + uv \big| u \in A, v \in B\big\} \subseteq ab + Ab + aB + AB.\end{aligned}$$

Conversely, let $x \in ab + Ab + aB + AB$. We will show that $x \in \alpha\beta$ and hence $ab + Ab + aB + AB \subseteq \alpha\beta$. We consider three cases. Firstly, we assume that α is a neutrix. Then we can take $a = 0$. It follows that $ab + Ab + aB + AB = Ab + AB$. If β is zeroless then $AB \subseteq bA$. So $ab + Ab + aB + AB = bA$. Hence $x \in bA = b\alpha \subseteq \alpha\beta$. If β is a neutrix then $ab + Ab + aB + AB = AB$. It implies that $x \in AB = \alpha\beta$. Secondly, we assume that β is a neutrix, and α is zeroless. With analogous arguments we have $x \in \alpha\beta$. Finally we assume that both α, β are zeroless. Then $ab + Ab + aB + AB = ab + bA + aB$. If $aB \subseteq bA$ then $ab + Ab + aB + AB = ab + bA = \{ab + bu \big| u \in A\} \subseteq \alpha\beta$. If $bA \subseteq aB$ then $ab + Ab + aB + AB = ab + aB = \{ab + av \big| v \in B\} \subseteq \alpha\beta$. Thus one always has $ab + Ab + Ba + AB \subseteq \alpha\beta$.

We conclude that $\alpha\beta = ab + bA + Ba + AB = ab + \max\{aB, bA, AB\}$. $\square$

Definition 2.2.15. Let $\alpha = a + A \in \mathbb{E}$ be an external number and $\operatorname{st}(n) \in \mathbb{N}$. We define with External induction the power α^n of α by

$$\alpha^n = \alpha \cdot \alpha \cdots \alpha.$$

For example, $\oslash^2 = \oslash \cdot \oslash = \oslash$ and $(1 + \epsilon£)^2 = (1 + \epsilon£)(1 + \epsilon£) = 1 + \epsilon£$.

Note that for $n \in \mathbb{N}$ standard, in general, $\alpha^n \neq \{x^n | x \in \alpha\}$. For example, $\oslash^2 = \oslash \cdot \oslash = \oslash$ but $\{\epsilon^2 | \epsilon \in \oslash\} = \oslash^+$ is the set of non-negative infinitesimals, which is strictly included in $\oslash$. Also $\{x^2 | x \in \alpha\} \subseteq \alpha^2$.

Remark 2.2.16. (i) If α, β are zeroless, we have $\alpha\beta = ab + \max\{aB, bA\}$.

(ii) In general, $(\alpha + \beta)C \neq \alpha C + \beta C$ if C is a neutrix. For example, $\big(1 + (-1)\big)\oslash = 0 \cdot \oslash = 0$ whereas $1 \cdot \oslash + (-1)\oslash = \oslash$. See [11] for conditions of the equality.

We below list some properties of operations on external numbers.

Lemma 2.2.17 ([22, 20])**.** *Let $\alpha = a + A, \beta = b + B, \gamma = c + C$ be external numbers and N be a neutrix. Then*

(i) If β is zeroless, one has $N\beta = bN$, and $\dfrac{N}{\beta} = \dfrac{N}{b}$.

(ii) $(a + A) \cdot N = aN + AN$.

(iii) $(a + A)\beta = a\beta + A\beta$.

(iv) $x(\alpha + \beta) = x\alpha + x\beta$ for all $x \in \mathbb{R}$.

(v) Subdistributivity: $(\alpha + \beta)\gamma \subseteq \alpha\gamma + \beta\gamma$.

(vi) If $|a| > A$, it holds that $N\big((a + A)^n\big) = a^{n-1}A$, for $n \in \mathbb{N}$ standard.

Proof. (i) Since β is zeroless, one has $|b| > B$. So $BA \subset bA$. Hence $bA + BA = \max\{bA, BA\} = bA$. Moreover, by Lemma 2.2.20 one has $\dfrac{A}{\beta} = A\dfrac{\beta}{b^2} = Ab/b^2 = A/b$.

(ii) If $\alpha = a + A$ is a neutrix, the conclusion is trivial. We assume that α is zeroless. For each real number $x \in N$, one has $x(a + A) = xa + xA \subseteq Na + NA$. Then $(a + A)N \subseteq aN + AN$. Conversely, Since α is zeroless, one has $AN \subset aN$. So $aN + AN = aN$. Obviously, $aN + AN = aN \subseteq (a + A)N$. Hence $(a + A)N = aN + AN$.

(iii) One has $(a + A)\beta = ab + aB + bA + AB$. By (ii), $Ab + AB = A(b + B)$. Also, $ab + aB = a(b + B)$. So $(a + A)\beta = a\beta + A\beta$.

(iv) By (iii), for all $x \in \mathbb{R}$ one has $x(\alpha + \beta) = x\big((a + b) + (A + B)\big) = x(a + b) + x(A + B) = xa + xb + xA + xB = x(a + A) + x(b + B) = x\alpha + x\beta$.

(v) Let $x \in (\alpha+\beta)\gamma$. There exist real numbers $u \in \alpha, v \in \beta, t \in \gamma$ such that $x = (u+v)t = ut+vt \in \alpha\gamma+\beta\gamma$. Hence $(\alpha+\beta)\gamma \subseteq \alpha\gamma+\beta\gamma$.

(vi) One has $(a+A)^2 = (a+A)(a+A) = a^2+2aA+A^2$. Since $|a| > A$, it holds that $A^2 \subseteq |a|A$. So $(a+A)^2 = a^2+2aA = a^2+aA$. Assume that the claim is true for $n \in \mathbb{N}$ with n standard. That is $(a+A)^n = a^n+a^{n-1}A$. Then $(a+A)^{n+1} = (a+A)^n(a+A) = (a^n+a^{n-1}A)(a+A) = a^{n+1}+2a^nA+a^{n-1}A^2$. Once again, since $|a| > A$, one has $a^{n-1}A \subseteq a^nA$. So $a^{n+1}+2a^nA+a^{n-1}A^2 = a^{n+1}+a^nA$. Hence, $(a+A)^{n+1} = a^{n+1}+a^nA$. By external induction, we conclude that $(a+A)^n = a^n+a^{n-1}A$, for all $n \in \mathbb{N}$ standard. □

Lemma 2.2.18 ([22, Prop. 2.7.15, p. 73])**.** *, Let $\alpha = 1+A$, where $A \subseteq \oslash$. Then $\frac{1}{\alpha} = \alpha = 1+A$.*

Proof. Let $y \in \mathbb{R}$ be an infinitesimal. If $y \geq 0$, we have

$$1-y \leq \frac{1}{1+y} \leq 1-\frac{y}{2} \tag{2.1}$$

or

$$\frac{1}{1-y/2} \leq 1+y \leq \frac{1}{1-y}. \tag{2.2}$$

If $y < 0$, we have

$$1-\frac{y}{2} \leq \frac{1}{1+y} \leq 1-y \tag{2.3}$$

or

$$\frac{1}{1-y} \leq 1+y \leq \frac{1}{1-y/2}. \tag{2.4}$$

Using these inequalities we first verify that $\frac{1}{1+A} \subseteq 1+A$. Let $\epsilon \in A$. If $\epsilon \geq 0$, by inequality (2.1) one has $1-\epsilon \leq \frac{1}{1+\epsilon} \leq 1-\epsilon/2$. Since $1-\epsilon \in \alpha, 1-\epsilon/2 \in \alpha$ and α is convex, this implies that $\frac{1}{1+\epsilon} \in [1-\epsilon, 1-\epsilon/2] \subseteq \alpha = 1+A$. If $\epsilon < 0$, then by inequality (2.3), one has $1-\frac{\epsilon}{2} \leq \frac{1}{1+\epsilon} \leq 1-\epsilon$. Similarly, it implies that $\frac{1}{1+\epsilon} \in \alpha = 1+A$. Hence $\frac{1}{1+A} \subseteq 1+A$.

Next we check $1+A \subseteq \frac{1}{1+A}$. Let $\epsilon \in A$. If $\epsilon > 0$ then, by inequality (2.2) one has $\frac{1}{1-\epsilon/2} \leq 1+\epsilon \leq \frac{1}{1-\epsilon}$. Since $\frac{1}{1-\epsilon/2}, \frac{1}{1-\epsilon} \in \frac{1}{1+A}$ and α is convex which implies that $\frac{1}{\alpha}$ is convex, we have $1+\epsilon \in [\frac{1}{1-\epsilon/2}, \frac{1}{1-\epsilon}] \subseteq \frac{1}{\alpha}$. If $\epsilon < 0$ then, by inequality (2.4) it holds that $\frac{1}{1-\epsilon} \leq 1+\epsilon \leq \frac{1}{1-\epsilon/2}$. For the same arguments, we have $1+\epsilon \in [\frac{1}{1-\epsilon}, \frac{1}{1-\epsilon/2}] \subseteq \frac{1}{\alpha}$. Hence $1+A \subseteq \frac{1}{1+A}$. We conclude that $1+A = \frac{1}{1+A}$. □

Lemma 2.2.19 ([22, Prop. 3.2.10, p. 83])**.** *Let $\alpha = a+A$ be zeroless. Then $A/a \subseteq \oslash$.*

Proof. Because α is zeroless, $|a| > A$. Then for all $x \in \frac{A}{a} = \frac{A}{|a|}$ we have $|x| < 1$. On the other hand $\frac{A}{a}$ is a neutrix, so $\frac{A}{a} \subset$ £. Also, there is no neutrix between $\oslash$ and £, consequently $\frac{A}{a} \subseteq \oslash$. □

Proposition 2.2.20 ([23, p.151], [22, p. 89]). *Let $\alpha = a + A, \beta = b + B \in \mathbb{E}$ be zeroless. Then*

(i) $\frac{1}{\alpha} = \frac{\alpha}{a^2} = \frac{1}{a} + \frac{A}{a^2}$.

(ii) $\frac{\alpha}{\beta} = \frac{\alpha\beta}{b^2} = \frac{a}{b} + \frac{1}{b^2}\max\{aB, bA\}$.

Proof. (i) By Lemma 2.2.18, one has

$$\frac{1}{\alpha} = \frac{1}{a+A} = \frac{1}{a(1+A/a)} = \frac{1}{a}\frac{1}{1+A/a} = \frac{1}{a}(1+A/a) = \frac{1}{a} + \frac{N(\alpha)}{a^2} = \frac{\alpha}{a^2}.$$

(ii) Because $\frac{\alpha}{\beta} = \alpha\frac{1}{\beta}$, the conclusion follows by Part (i). □

Definition 2.2.21. Let $\alpha \in \mathbb{E}$ be zeroless. We define $N(\alpha)/a$ the *relative uncertainty* of α, denoted by $R(\alpha)$.

The relative uncertainty of an external number α is independent of the choice a by Lemma 2.2.17(i). Moreover $R(\alpha) \subseteq \oslash$.

Lemma 2.2.22 ([20]). *Let $\alpha = a + A$ be a zeroless external number. Then $\alpha \cap \oslash\alpha = \emptyset$.*

Proof. Suppose that $\alpha \cap \oslash\alpha \neq \emptyset$. Then there exists $x \in \alpha \cap \oslash\alpha$. So $x = a\epsilon \in a + A$ for some $a \in \alpha, a \neq 0$ and $\epsilon \in \oslash$. It follows that $\epsilon \in 1 + A/a \subseteq 1 + \oslash$, which is a contradiction. Hence $\alpha \cap \oslash\alpha = \emptyset$. □

Definition 2.2.23. ([20, Def. 2.3, p10]). Let A be a neutrix and $\alpha \in \mathbb{E}$ be an external number. The number α is called an *absorber* of A if $\alpha A \subset A$, and an *exploder* of A if $A \subset \alpha A$. We denote by $\oslash_N$ the set of all real numbers which are absorbers of N and by $\not\oslash_N$ the set of all real numbers which are exploders of N.

Example 2.2.24. Let $\epsilon > 0$ be infinitesimal. Then ϵ is an absorber of $\oslash$ since $\epsilon\oslash \subset \oslash$. It is also an absorber of £ because $\epsilon£ \subset \oslash \subset £$. Moreover $\frac{1}{\epsilon}$ is an exploder of $\oslash$ since $\oslash \subset £ \subset \frac{\oslash}{\epsilon}$.

Remark 2.2.25. By Proposition 2.1.3(i), no appreciable number is an absorber of a neutrix. Also, any unlimited number is not an absorber. Hence, if a number is an absorber of a neutrix, it must be an infinitesimal number. However, it is not true that every infinitesimal is an absorber of a given neutrix. For instance, let $\epsilon > 0$ be infinitesimal. Then $\epsilon \cdot £\epsilon^{\not\oslash} = \epsilon \cdot \bigcap_{st(n)\in\mathbb{N}} [-\epsilon^n, \epsilon^n] = \bigcap_{st(n)\in\mathbb{N}} [-\epsilon^n, \epsilon^n] = £\epsilon^{\not\oslash}$.

Proposition 2.2.26. *Let $c \in \mathbb{R} \setminus \{0\}$ be a limited number and B be a neutrix. If c is not an absorber of B then $cB = \frac{B}{c} = B$.*

Proof. Because c is limited, we have $B \subseteq \frac{B}{c}$. On the other hand, c is not an absorber of B, it holds that $\frac{B}{c} \subseteq B$. So $\frac{B}{c} = B$ and hence $B = cB$. □

Remark 2.2.27. Let $\alpha \in \mathbb{E}$ be *limited*, i.e., there exists a limited number $t \in \mathbb{R}$ such that $|\alpha| \leq t$ and B be a neutrix. Then $\alpha \cdot B \subseteq B$.

The multiplication on external numbers is not distributive but it is subdistributive as shown in Property (v) of Proposition 2.2.17. As a consequence, the multiplication on matrices over $\mathbb{E}$ are not associative or distributive [20, p.35]. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} \oslash \\ \oslash \end{bmatrix}$. One has

$$(AB)C = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} \oslash \\ \oslash \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \oslash \\ \oslash \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \oslash \\ \oslash \end{bmatrix} = \begin{bmatrix} \oslash \\ 0 \end{bmatrix}.$$

So $(AB)C \neq A(BC)$.

However, in some special cases the associative law is true.

Lemma 2.2.28. *Let $A = [a_{ij}]$ be an $m \times n$ real matrix, $B = [B_{ij}]$ be an $n \times p$ neutrix matrix, i.e, all of its entries are neutrices, and $C = [C_i]_{p\times 1}$ be a matrix whose entries are either real numbers or neutrices. Then $A(BC) = (AB)C$.*

Proof. Put $AB \equiv D_1 = [\beta_{ij}]_{m\times p}$. One has

$$\beta_{ij} = a_{i1}B_{1j} + \cdots + a_{in}B_{nj}$$

for all $1 \leq i \leq m, 1 \leq j \leq p$. Because B_{ij} is a neutrix for $1 \leq i \leq n, q \leq j \leq p$, also β_{ij} is a neutrix for all $1 \leq i \leq m, 1 \leq j \leq p$.

So $(AB)C = D_1C \equiv [\alpha_i]_{m\times 1}$, where

$$\begin{aligned} \alpha_i =& \beta_{i1}C_1 + \cdots + \beta_{ip}C_p \\ =& (a_{i1}B_{11} + \cdots + a_{in}B_{n1})C_1 + \cdots + (a_{i1}B_{1p} + \cdots + a_{in}B_{np})C_p. \end{aligned} \tag{2.5}$$

On the other hand, let $BC \equiv D_2 \equiv [\eta_i]_{n\times 1}$. Then $\eta_i = B_{i1}C_1 + \cdots + B_{ip}C_p$ for all $1 \leq i \leq n$. Put $A(BC) = AD_2 \equiv [\zeta_i]_{m\times 1}$. By Lemma 2.2.17(iv) we have

$$\begin{aligned} \zeta_i =& a_{i1}\eta_1 + \cdots + a_{in}\eta_n \\ =& a_{i1}(B_{11}C_1 + \cdots + B_{1p}C_p) + \cdots + a_{in}(B_{n1}C_1 + \cdots + B_{np}C_p) \\ =& a_{i1}B_{11}C_1 + \cdots + a_{i1}B_{1p}C_p + \cdots + a_{in}B_{n1}C_1 + \cdots + a_{in}B_{np}C_p \\ =& (a_{i1}B_{11} + \cdots + a_{in}B_{n1})C_1 + \cdots + (a_{i1}B_{1p} + \cdots + a_{in}B_{np})C_p. \end{aligned} \tag{2.6}$$

Hence $(AB)C = A(BC)$. $\square$

2.2.2 Order relations on external numbers

In this work we use the order relation on external numbers were studied by Koudjeti and Van den Berg [22]. In [12, 13] another order relation on external numbers was constructed, but it does not contemplate the inequalities " $\geq$ " or " $>$ ", so it lacks flexibility in some situations like optimization. The relation presented here overcomes this restriction, however, care is needed in interchanging $\leq$ and $\geq$ as shown below. We choose this order relation because it is easy to transform a minimization problem to a maximization problem and vice versa. In particular, it enables to transform constraints in linear programming (see Chapter 7) into the canonical form. Also, it is suitable to define optimal solutions in chapters 7 and 8.

Definition 2.2.29. Let $\alpha = a + A, \beta = b + B$ be two external numbers. We define the order relations on external numbers as follows:

(i) $\alpha \geq \beta$ if and only if $\forall x \in \alpha\, \exists y \in \beta (x \geq y)$.

(ii) $\alpha > \beta$ if and only if $\forall x \in \alpha\, \forall y \in \beta (x > y)$.

(iii) $\alpha \leq \beta$ if and only if $\forall x \in \alpha\, \exists y \in \beta (x \leq y)$.

(iv) $\alpha < \beta$ if and only if $\forall x \in \alpha\, \forall y \in \beta (x < y)$.

Example 2.2.30. Let ϵ be a positive infinitesimal. Then $1 + \epsilon £ > \oslash$, $\epsilon \leq \oslash$ and $\epsilon \geq \oslash$.

Remark 2.2.31. (i) Intuitively, we have $\alpha \leq \beta$ if and only if $(-\infty, \alpha] \subseteq (-\infty, \beta]$ and $\alpha \geq \beta$ if and only if $[\alpha, +\infty) \subseteq [\beta, +\infty)$.

(ii) If A, B are two neutrices, then $A \subseteq B$ if and only if $A \leq B$ or $A \geq B$. Note that the larger neutrix is always on the right side.

(iii) Clearly $\alpha < \beta$ if and only if $\beta > \alpha$. However $\alpha \geq \beta$ is not equivalent to $\beta \leq \alpha$. For example, we have $\oslash \geq £$ and $\oslash \leq £$, yet $£ \not\leq \oslash$. As a consequence, it can occur simultaneously that $\alpha \leq \beta \wedge \alpha \geq \beta$ without α and β being equal. In fact, it happens if $\alpha \subseteq \beta$.

(iv) Let α be an external number and N be a neutrix. One has $\alpha \subseteq N$ if and only if $|\alpha| \leq N$.

The next proposition present characterizations of the order relations on external numbers.

Lemma 2.2.32 ([22])**.** *Let α, β be two external numbers. Then*

(i) $\alpha < \beta$ *if and only if* $\alpha \leq \beta \wedge \alpha \cap \beta = \emptyset$.

(ii) $\alpha > \beta$ *if and only if* $\alpha \geq \beta \wedge \alpha \cap \beta = \emptyset$.

(iii) $\alpha \leq \beta$ *if and only if* $\alpha < \beta \vee \alpha \subseteq \beta$.

(iv) $\alpha \geq \beta$ *if and only if* $\alpha > \beta \vee \alpha \subseteq \beta$.

Proof. (i) Assume that $\alpha < \beta$. It follows directly by the definition that $\alpha \leq \beta$. Suppose that $\alpha \cap \beta \neq \emptyset$. Then there exists $x \in \alpha, x \in \beta$. It contradicts the definition of $\alpha < \beta$.

Conversely, assume that $\alpha \leq \beta \vee \alpha \cap \beta = \emptyset$. We suppose that $\alpha \not< \beta$. There exists $x \in \alpha, y \in \beta$ such that $x \geq y$. Since $\alpha \cap \beta = \emptyset$, it holds that $x > y$. On the other hand $\alpha \leq \beta$, by the definition, there exists $z \in \beta$ such that $x \leq z$. Again, since $\alpha \cap \beta = \emptyset$, it holds that $y < x < z$. Because β is convex, it follows that $x \in [y, z] \subset \beta$, which is a contradiction to $\alpha \cap \beta = \emptyset$. Hence $\alpha < \beta$.

(ii) The argument is similar.

(iii) If $\alpha < \beta$ then $\alpha \leq \beta$ by the definitions. If $\alpha \subseteq \beta$ then for all $x \in \alpha$, there exists $y \in \beta$ such that $x \leq y$. So $\alpha \leq \beta$.

Conversely, assume that $\alpha \leq \beta$. If $\alpha \cap \beta = \emptyset$ then by Part (i) we have $\alpha < \beta$. Otherwise, suppose that $\beta \subset \alpha$. Then there exists $x_0 \in \alpha$ such that $x_0 > y$ for all $y \in \beta$, a contradiction to $\alpha \leq \beta$. Hence $\alpha \subseteq \beta$.

(iv) The argument is similar. □

We below present some properties of order relations on external numbers which will be used in Chapters 4 and 5.

Let α, β be two external numbers. We first remark that $\alpha \not\leq \beta$ does not imply $\alpha > \beta$ (or $\beta < \alpha$). For example, $\alpha = £, \beta = \oslash$. Then $\alpha \not\leq \beta$ and $\alpha \not> \beta$. However, we have the following results.

Lemma 2.2.33. *Let N be a neutrix and α, β be two external numbers such that $N(\alpha) \subseteq N, N(\beta) \subseteq N$. Then $\alpha + N \not\leq \beta + N$ implies $\alpha + N > \beta + N$.*

Proof. Observe that $N(\alpha+N) = N(\beta+N) = N$. If $(\alpha+N) \cap (\beta+N) \neq \emptyset$, there exists $x \in (\alpha+N) \cap (\beta+N)$. It follows that $\alpha + N = \beta + N = x + N$, a contradiction to the hypothesis. So $(\alpha + N) \cap (\beta + N) = \emptyset$. If $\alpha + N < \beta + N$ we have a contradiction. Hence $\alpha + N > \beta + N$. □

Corollary 2.2.34. *Let α, β be two external numbers such that $N(\alpha) = N(\beta)$. Then $\alpha \not\leq \beta$ implies $\alpha > \beta$.*

Proof. It follows from Lemma 2.2.33 with $N = N(\alpha) = N(\beta)$. □

Lemma 2.2.35. *Let α, β be two external numbers. One has $\alpha \leq \beta$ if and only if $-\alpha \geq -\beta$.*

Proof. One has $\alpha \leq \beta$ if and only $\forall x \in \alpha, \exists y \in \beta (x \leq y)$. This is equivalent to $\forall\, (-x) \in (-\alpha), \exists\, (-y) \in (-\beta)\, (-x \geq -y)$. Once again it holds if and only if $-\alpha \geq -\beta$. □

Proposition 2.2.36. *Let α, β, γ be external numbers. If $\alpha - \beta < \gamma$ then $\alpha + N(\beta) < \beta + \gamma$.*

Proof. Write $\alpha = a + A, \beta = b + B, \gamma = c + C$. If $N(\beta) = B \subseteq C = N(\gamma)$, then $\beta + \gamma = c + b + C$. On the other hand, $\alpha - \beta < \gamma$ implies that $\alpha + N(\beta) < \gamma + b = \gamma + \beta$.

If $N(\gamma) \subset N(\beta)$ then $N(\gamma+\beta) = B \subseteq A + B = N(\alpha - \beta)$. We first prove that $\alpha + N(\beta) \cap \gamma + \beta = \emptyset$. Suppose that there exists $x \in \alpha + N(\beta) \cap \gamma + \beta$. Then $\alpha + N(\beta) = x + A + B$ and $\beta + \gamma = x + B$. It follows that $\beta + \gamma \subseteq \alpha + N(\beta)$. In particular, $b + c \in \alpha + N(\beta)$ implies that $c \in \alpha + \beta$, a contradiction. Also, one has $a \in \alpha + N(\beta), b + c \in \beta + \gamma$ and $a < b + c$ by the assumption. Hence $\alpha + N(\beta) < \beta + \gamma$. □

Corollary 2.2.37. *Let α, β, γ be external numbers. If $\alpha - \beta < \gamma$ then $\alpha < \beta + \gamma$.*

Proof. By Proposition 2.2.36 one has $\alpha + N(\beta) < \beta + \gamma$. Also, $0 \in N(\beta)$ so $\alpha < \beta + \gamma$. □

With an analogous argument to the proof of Proposition 2.2.36, one has

Proposition 2.2.38. *Let α, β, γ be external numbers. If $\eta < \alpha - \beta$ then $\eta + \beta < \alpha + N(\beta)$. In particular, $\eta + \beta < \alpha$.*

Proposition 2.2.39. *Let α, β, γ be external numbers. If $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$.*

Proof. Let $x \in \alpha + \gamma$. Then $x = u + v$ with $u \in \alpha$ and $v \in \gamma$. Since $\alpha \leq \beta$, there exists $t \in \beta$ such that $u \leq t$. It follows that $x = u + v \leq t + v = y$ with $y \in \beta + \gamma$. Hence $\alpha + \gamma \leq \beta + \gamma$. □

Proposition 2.2.40. *Let α be an external number and β be a zeroless, positive external number. Then $|\alpha| < \beta$ if and only if $-\beta < \alpha < \beta$.*

Proof. Assume that $|\alpha| < \beta$. We consider two cases:

Case 1: α is neutricial. We need to show that $-\beta < \alpha$. Suppose that there exists $u \in \alpha$ and $v \in -\beta$ such that $u \leq v$. Observe that $-\beta$ is negative, so $v \leq 0$. Since α is a neutrix and $u, 0 \in \alpha$, one derives that $v \in [u, 0] \subseteq \alpha$. Consequently $-v \in \alpha$ and by Proposition 2.2.5 one has $-v \in \beta$ and hence $\alpha \cap \beta = v$, a contradiction. So $-\beta < \alpha < \beta$.

Case 2: α is zeroless. Let $u \in \alpha$ and $v \in \beta$. Then $|u| \in |\alpha|$ by Proposition 2.2.12. So $|u| < v$, that is $-v < u < v$ for all $u \in \alpha$. Also, for $v \in -\beta$ one has $v < u$. Hence $-\beta < \alpha$. Moreover, $\alpha \leq |\alpha| < \beta$. So $-\beta < \alpha < \beta$.

Conversely, assume that $-\beta < \alpha < \beta$. If α is neutricial then $|\alpha| = \alpha < \beta$. If α is zeroless, let $y \in |\alpha|$ and $v \in \beta$. Then $y = |u|$ with some $u \in \alpha$ by Proposition 2.2.12. Since $-\beta < \alpha < \beta$, it holds that $-v < u < v$. So $y = |u| < v$. Since $y \in |\alpha|, v \in \beta$ are arbitrary, one concludes that $|\alpha| < \beta$. □

Proposition 2.2.41. *Let $\epsilon > 0$ be a positive real number and α, β be two external numbers. If $|\alpha - \beta| < \epsilon$ then $\beta - \epsilon < \alpha < \epsilon + \beta$.*

Proof. By Proposition 2.2.40, one has $-\epsilon < \alpha - \beta < \epsilon$. It follows that $-\epsilon + \beta < \alpha < \epsilon + \beta$ by Corollary 2.2.37 and Proposition 2.2.38. □

Proposition 2.2.42. *Let $\alpha, \beta \in \mathbb{E}$ be two external numbers. Then* *(i)* $|\alpha| \leq \beta$ *if and only if* $\alpha \leq \beta$ *and* $-\alpha \leq \beta$.

(ii) $\beta \leq |\alpha|$ *if and only if* $\beta \leq \alpha$ *or* $\beta \leq -\alpha$.

Proof. (i) Assume that $|\alpha| \leq \beta$. If $N(\alpha) < \alpha$, then $-\alpha < N(\alpha)$. Hence $-\alpha < N(\alpha) < \alpha = |\alpha| \leq \beta$. Hence $\alpha \leq \beta$ and $-\alpha \leq \beta$. If $\alpha < N(\alpha)$, then $N(\alpha) < -\alpha$. So $\alpha < N(\alpha) < -\alpha = |\alpha| \leq \beta$. Again $\alpha \leq \beta$ and $-\alpha \leq \beta$. If $\alpha = N(\alpha)$, then $|\alpha| = \alpha = \alpha \leq \beta$.

Conversely, assume that $\alpha \leq \beta$ and $-\alpha \leq \beta$. If $N(\alpha) \leq \alpha$, then $|\alpha| = \alpha \leq \beta$ and if $\alpha < N(\alpha)$, then $|\alpha| = -\alpha \leq \beta$.

(ii) Assume firstly that $\beta \leq |\alpha|$. If $N(\alpha) \leq \alpha$, then $\beta \leq |\alpha| = \alpha$ and if $\alpha < N(\alpha)$, then $\beta \leq |\alpha| = -\alpha$. Suppose secondly that $\beta \leq \alpha$ or $\beta \leq -\alpha$. In the case $\beta \leq \alpha$, if $N(\alpha) \leq \alpha$, then $\beta \leq \alpha = |\alpha|$ and if $\alpha < N(\alpha)$, then $\beta \leq \alpha < N(\alpha) \leq -\alpha = |\alpha|$. In the case $\beta \leq -\alpha$, if $N(\alpha) \leq \alpha$, one has $\beta \leq -\alpha \leq N(\alpha) \leq \alpha = |\alpha|$, and if $\alpha < N(\alpha)$, one has $\beta \leq -\alpha = |\alpha|$. Hence always $\beta \leq |\alpha|$. □

Proposition 2.2.43 (Triangular inequality)**.** *Let* $\alpha, \beta \in \mathbb{E}$. *Then* $|\alpha + \beta| \leq |\alpha| + |\beta|$.

Proof. Clearly $\alpha + \beta \leq |\alpha| + |\beta|$. Also $-(\alpha + \beta) = -\alpha - \beta \leq |\alpha| + |\beta|$. By Proposition 2.2.42(i) one concludes that $|\alpha + \beta| \leq |\alpha| + |\beta|$. □

Lemma 2.2.44. *Let* $\alpha, \beta \in \mathbb{E}$. *Then*

$$|\alpha - \beta| + N(|\beta|) = |\alpha - \beta| + N(|\alpha|) = |\alpha - \beta| . \tag{2.7}$$

Proof. One has $N(|\alpha - \beta|) = N(\alpha) + N(\beta)$. Hence $|\alpha - \beta| = |\alpha - \beta| + N(\alpha) + N(\beta)$. Also $N(\alpha) = N(|\alpha|)$ and $N(\beta) = N(|\beta|)$. This implies (2.7). □

Proposition 2.2.45. *Let* $\alpha, \beta \in \mathbb{E}$. *Then* $|\alpha| - |\beta| \leq |\alpha - \beta|$.

Proof. Using Proposition 2.2.43 one has

$$|\alpha| \leq |\alpha + N(\beta)| = |\alpha + \beta - \beta| \leq |\alpha - \beta| + |\beta| . \tag{2.8}$$

By compatibility with addition $|\alpha| - |\beta| \leq |\alpha + \beta| + N(|\beta|)$. Then $|\alpha| - |\beta| \leq |\alpha - \beta|$ by Lemma 2.2.44. □

Proposition 2.2.46. *Let* $\alpha, \beta \in \mathbb{E}$ *be two external numbers. Then* $||\alpha| - |\beta|| \leq |\alpha - \beta|$.

Proof. By Proposition 2.2.45 one has $|\alpha| - |\beta| \leq |\alpha - \beta|$ and $|\beta| - |\alpha| \leq |\beta - \alpha| = |\alpha - \beta|$. Then $||\alpha| - |\beta|| \leq |\alpha - \beta|$ by Proposition 2.2.42.(i). □

2.3 n-th roots of an external number

The n-th roots of an external number appear naturally when we use the neutrix-derivative to find approximate optimal solutions in Section 8.2. Let $\alpha = a + A$ be an external number and $n \in \mathbb{N}$ be standard. An *n-th root* of α is an external number β whose n-th power is α. For $n = 2$ we call it a *square root* and for $n = 3$ we call it a *cube root*.

For example, a square root of $\oslash$ is $\oslash$, a square root of £ is £ because $\oslash^2 = \oslash$ and $£^2 = £$.

In general, if $\alpha = I$ is an idempotent neutrix, the n-th root of I is I. We denote by $\sqrt[n]{I}$ the n-th root of I. If $\alpha = A$ is an arbitrary neutrix, by Proposition 2.1.6, there exists a real number $t > 0$ and the unique idempotent I such that $A = t.I$. Then the n-th root of A is $\sqrt[n]{A} = \sqrt[n]{t}I$. Clearly, this definition does not depends on t.

Definition 2.3.1. Let $\alpha = a + A$ be a positive external number. We define the positive n-th root of α and denoted $\sqrt[n]{\alpha}$ as the *principal n-th root* of α .

We have

$$\sqrt[n]{\alpha} = \sqrt[n]{a} + \frac{A}{\sqrt[n]{a^{n-1}}}. \tag{2.9}$$

Indeed, by Lemma 2.2.17(vi) we have $(\sqrt[n]{\alpha})^n = (\sqrt[n]{a})^n + (\sqrt[n]{a})^{n-1}\left(\frac{A}{\sqrt[n]{a^{n-1}}}\right) = a + A = \alpha.$

We will prove that formula (2.9) does not depend on the choice of the representative of α. Indeed, let $\beta = \sqrt[n]{\alpha} = \sqrt[n]{a} + \frac{A}{\sqrt[n]{a^{n-1}}} \equiv b + B$. Note that $\beta^n = \alpha$. Assume that $\alpha = a' + A$ and $\sqrt[n]{\alpha} = b' + B'$, where $b' = \sqrt[n]{a'}$ and $B' = \frac{A}{\sqrt[n]{(a')^{n-1}}}$. We will show that

$$\beta = b' + B'. \tag{2.10}$$

Because $a' = a + x$ with $x \in A$, it holds that $(b')^n = a' = a + x \in a + A = \alpha = \beta^n$. This implies that $b' \in \beta$. To complete the proof, we show that

$$B = B'.$$

Since $b' \in \beta$, one has $\beta = b' + B$. On the other hand, $(b' + B')^n = \alpha$ and $\beta^n = (b' + B)^n = \alpha = a + A$. By Lemma 2.2.17(vi), we obtain $(b')^{n-1}B' = (b')^{n-1}B = A$. Hence $B = B'$. Thus formula (2.10) holds and so formula (2.9) does not depend on the choice of a representative a of α.

Remark 2.3.2. If $\alpha = a + A$ is negative and if $n \in \mathbb{N}$ is even, there is no n-th root of α. However, if $n \in \mathbb{N}$ is odd, the n-th root of α is defined by formula (2.9).

Example 2.3.3. Let $\alpha = 4+\oslash, \beta = -27+\epsilon\oslash, \gamma = \epsilon+\epsilon\oslash, \delta = \omega£$, where $\epsilon > 0$ is an infinitesimal and $\omega > 0$ is an unlimited number. Then $\sqrt{\alpha} = 2 + \oslash$, $\sqrt[3]{\beta} = -3 + \epsilon\oslash$, $\sqrt{\gamma} = \sqrt{\epsilon} + \frac{\epsilon\oslash}{\sqrt{\epsilon}} = \sqrt{\epsilon} + \sqrt{\epsilon}\oslash$ and $\sqrt[4]{\delta} = \sqrt[4]{\omega}£$.

2.4 External supremum and infimum of an external set

In classical mathematics, it is well-known that every bounded set in $\mathbb{R}$ has an infimum and a supremum . However, it is no longer true in nonstandard analysis. For example, $\oslash$ is bounded but it has neither infimum nor supremum. In this section we present notions of external supremum and external infimum such that every external subset of $\mathbb{R}$ has both an infimum and a supremum.

We will present some notions such as *cut, external cut, upper boundary and lower boundary* of an external cut, *infimum, supremum* and some of their properties. They will be used to prove results related to optimization problem with a flexible objective function in Chapter 8.

Definition 2.4.1. A cut of $\mathbb{R}$ is a pair $(\mathcal{A}, \mathcal{B})$ of subsets of $\mathbb{R}$ satisfying

(i) $\mathcal{A} \cup \mathcal{B} = \mathbb{R}$

(ii) $\mathcal{A} \cap \mathcal{B} = \emptyset$,

(iii) For all $x \in \mathcal{A}, y \in \mathcal{B}$ one has $x < y$.

The set $\mathcal{A}$ is called a *lower halfline* of $\mathbb{R}$ and $\mathcal{B}$ is called an *upper halfline* of $\mathbb{R}$.

The cut is said to be external if either $\mathcal{A}$ or $\mathcal{B}$ is external set. Otherwise, it is said to be internal .

For example $\mathcal{A} = (-\infty, 1 + \oslash)$ and $\mathcal{B} = [1 + \oslash, +\infty)$ is an external cut.

Remark 2.4.2. Note that for each halfline of $\mathbb{R}$ there always exists one boundary which is an external number. In fact, let $(\mathcal{A}, \mathcal{B})$ be a cut of $\mathbb{R}$. As a consequence of results in [4], there exists an external number α such that $\mathcal{A} = (-\infty, \alpha]$ or $\mathcal{A} = (-\infty, \alpha)$. We say α is the *upper boundary* of the lower halfline $\mathcal{A}$. It is also said to be the *lower boundary* of the upper halfline.

Convention 2.4.3. Let $\gamma \in \mathbb{E}$. To be unambiguous, we denote by $(-\infty, \gamma) = \{x \in \mathbb{R} \big| x < \gamma\}$, $(-\infty, \gamma] = \{x \in \mathbb{R} \big| x \leq \gamma\}$, $(\gamma, +\infty) = \{x \in \mathbb{R} \big| \gamma < x\}$, $[\gamma, +\infty) = \{x \in \mathbb{R} \big| x \geq \gamma\}$.

Definition 2.4.4. Let S be a set of external numbers. We define $P(S)$ as the set of all real numbers which belong to at least one external number in S. That is,

$$P(S) = \big\{x \in \mathbb{R} \big| \exists \alpha \in S, x \in \alpha\big\}.$$

We call it the *projection* of S on $\mathbb{R}$.

Example 2.4.5. Let $S_1 = \{\xi \in \mathbb{E} \big| \xi \leq \oslash\}$ and $S_2 = \{x \in \mathbb{R} \big| \oslash < x\}$. Then $P(S_1) = (-\infty, \oslash]$ and $P(S_2) = (\oslash, +\infty)$.

Remark 2.4.6. Considering a set of external numbers, we will always determine the external infimum or the external supremum in the context of the projection of that set on $\mathbb{R}$.

Definition 2.4.7. [22] Let S be a set of external numbers.

The *lower convexification* of S is the set

$$\underline{\operatorname{conv}}(S) = \{\alpha \in \mathbb{E} \mid \exists \delta \in S \wedge \alpha \leq \delta\}. \tag{2.11}$$

The *upper convexification* of S is the set

$$\overline{\operatorname{conv}}(S) = \{\alpha \in \mathbb{E} \mid \exists \delta \in S \wedge \alpha \geq \delta\}. \tag{2.12}$$

The *convexification* of S, denoted by $\operatorname{conv}(S)$, is the intersection in $\mathbb{E}$ of $\underline{\operatorname{conv}}(S)$ and $\overline{\operatorname{conv}}(S)$.

Example 2.4.8. Let $S(\oslash, £]$. Then $\underline{\operatorname{conv}}(S) = (-\infty, £]$ and $\overline{\operatorname{conv}}(S) = (\oslash, +\infty)$. So $\operatorname{conv}(S) = S$.

Let S be any non-empty set of external numbers not containing the external number $\mathbb{R}$. Then

(i) $P(\underline{\operatorname{conv}}(S))$ is a lower halfline of $\mathbb{R}$.

(ii) $P(\overline{\operatorname{conv}}(S))$ is a upper halfline of $\mathbb{R}$.

(iii) $P(\operatorname{conv}(S))$ is an interval whose upper boundary coincides to the one of $P(\underline{\operatorname{conv}}(S))$ and whose lower boundary coincides to one of $P(\overline{\operatorname{conv}}(S))$.

Definition 2.4.9 ([22])**.** Let S be a non-empty set external numbers. We define

(i) The *least upper bound* of S is the upper boundary of $P(\operatorname{conv}(S))$, denoted by $\sup(S)$.

(ii) The *greatest lower bound* of S is the lower boundary of $P(\operatorname{conv}(S))$, denoted by $\inf(S)$.

Example 2.4.10. (a) Let $S_1 = \{\xi \in \mathbb{E} \big| \xi \leq \oslash\}, S_2 = \{x \in \mathbb{R} \big| \oslash < x\}$. Then $P(S_1) = (-\infty, \oslash]$ and $P(S_2) = (\oslash, +\infty)$. Hence $\sup(S_1) = \oslash$ and $\inf(S_2) = \oslash$.

(b) Let $@ = (\oslash, £]$. Then $P(@) = @$. So $\inf(@) = \oslash, \sup(@) = £$. This example is somewhat surprising because $@ \subset £$ and the infimum of $@$ is strictly included in the supremum of $@$.

The following proposition shows some properties of infimum and supremum of external subsets. We will refer to them in optimization problems of the next chapters.

Proposition 2.4.11 ([22])**.** *Let S be a non-empty set of external numbers being bounded from above and let γ be a given external number. Then γ is the least upper bound of S if and only if one of the following statements holds:*

(i) $(\gamma \cap \underline{\operatorname{conv}}(S) = \emptyset) \wedge (\forall \alpha < \gamma,\ \exists \delta \in S \bigm| \delta > \alpha)$.

(ii) $(\gamma \subset \underline{\operatorname{conv}}(S)) \wedge (\forall \delta \in S,\ \delta \leq \gamma)$.

Proof. We assume that γ is the least upper bound of S. Then by the definition one has $P(\underline{\text{conv}}(S)) = (-\infty, \gamma)$ or $P(\underline{\text{conv}}(S)) = (-\infty, \gamma]$. We will prove that γ satisfies condition (i) or (ii). We always have $\gamma \cap \underline{\text{conv}}(S) = \emptyset$ or $\gamma \cap \underline{\text{conv}}(S) \neq \emptyset$.

If $\gamma \cap \underline{\text{conv}}(S) = \emptyset$, we have $P(\underline{\text{conv}}(S)) = (-\infty, \gamma)$. One also has $P\left(\underline{\text{conv}}(S)\right) = \bigcup_{\delta \in S} (-\infty, \delta] \subset \mathbb{R}$.

Let $\alpha < \gamma$. Then $\alpha \subset \bigcup_{\delta \in S} (-\infty, \delta] = (-\infty, \gamma)$. It follows that there is $\delta_0 \in S$ such that $\alpha \subseteq (-\infty, \delta_0]$. We denote $V = \{\delta \in S | \alpha \subseteq \delta\}$. If $V = \emptyset$, then $\alpha < \delta_0$. If $V \neq \emptyset$ then $\beta = \bigcup_{\delta \in V} \delta$. Then $\beta \in \mathbb{E}$. Clearly $\beta < \gamma$ and $P(\beta, \gamma) \equiv \{x \in \mathbb{R} | \beta < x < \gamma\} \subset \bigcup_{\delta \in S} (-\infty, \delta] = (-\infty, \gamma)$. Let $\eta \in (\beta, \gamma)$. Then there exists $\delta_1 \in S$ such that $\eta \leq \delta_1$. Because of $\alpha \leq \beta < \eta$, this implies $\alpha < \delta_1$.

If $\gamma \cap \underline{\text{conv}}(S) \neq \emptyset$ then $P(\underline{\text{conv}}(S)) = (-\infty, \gamma]$ and hence $\gamma \subset \underline{\text{conv}}(S)$. For $\delta \in S$, clearly, $\delta \in \underline{\text{conv}}(S)$. So $\delta \subset (-\infty, \gamma]$ and thus $\delta \leq \gamma$.

Conversely, we assume that $\gamma \in \mathbb{E}$ satisfies condition (i) or (ii). We need to prove that γ is the least upper bound of S.

Assume that γ satisfies (i). We will show that

$$P\left(\underline{\text{conv}}(S)\right) = \bigcup_{\delta \in S} (-\infty, \delta] = (-\infty, \gamma). \tag{2.13}$$

Since $\gamma \cap \underline{\text{conv}}(S) = \emptyset$, for all $\delta \in S$ one has $\delta < \gamma$. This implies that

$$\bigcup_{\delta \in S} (-\infty, \delta] \subseteq (-\infty, \gamma). \tag{2.14}$$

Conversely, let $x \in (-\infty, \gamma)$. By (i), there exists $\delta \in S$ such that $x < \delta$. It follows $x \in \bigcup_{\delta \in S} (-\infty, \delta]$ and hence

$$(-\infty, \gamma) \subseteq \bigcup_{\delta \in S} (-\infty, \delta]. \tag{2.15}$$

Formulas (2.14) and (2.15) imply formula (2.13) and hence γ is the least upper bound of S.

Assume that γ satisfies (ii). We will show that $P(\underline{\text{conv}}(S)) = (-\infty, \gamma]$. Indeed, the assumption $\gamma \subset \underline{\text{conv}}(S)$ implies that $(-\infty, \gamma] \subseteq P\left(\underline{\text{conv}}(S)\right)$. Also for $x \in P\left(\underline{\text{conv}}(S)\right)$, there exists $\xi \in \underline{\text{conv}}(S)$ such that $x \in \xi \subset P(\underline{\text{conv}}(S))$. By formula (2.11) there is $\delta \in S$ such that $\xi \leq \delta$. Again by (ii), we have $\delta \leq \gamma$ and hence $x \in (-\infty, \gamma]$. So $P(\underline{\text{conv}}(S)) = (-\infty, \gamma]$. This implies that γ is the least upper bound of S. □

Similarly, we have the following property for the greatest lower bound of a subset of $\mathbb{E}$.

Proposition 2.4.12 ([22]). *Let S be a non-empty set of external numbers being bounded from below. An external number η is the greatest lower bound of the set S if and only if one of the following statements holds*

(i) $\eta \cap \overline{\text{conv}}(S) = \emptyset \wedge (\forall \alpha > \eta,\ \exists \delta \in S | \delta < \alpha)$.

(ii) $(\eta \subset \overline{\text{conv}}(S)) \wedge (\forall \delta \in S,\ \delta \geq \eta)$.

Remark 2.4.13. Let $\gamma = \inf(S)$ and $\eta = \sup(S)$. Then $\gamma \cap \underline{\text{conv}}(S) \neq \emptyset$ is equivalent to $\gamma \subset \underline{\text{conv}}(S)$ and $\eta \cap \overline{\text{conv}}(S) \neq \emptyset$ is equivalent to $\eta \subset \overline{\text{conv}}(S)$.

2.5 Norm on $\mathbb{E}^n$

In this section we define a norm of vectors whose components are external numbers. We will use it to study Lagrange multipliers and also to investigate the N-derivative of vector functions presented in chapters 6 and 8.

Definition 2.5.1. Let $n \in \mathbb{N}$ be standard. A mapping $\|\cdot\| :\ \mathbb{E}^n \to \mathbb{E}$ is said to be a *norm* on $\mathbb{E}^n$ if it satisfies conditions:

(i) $0 \leq \|\alpha\|$, for all $\alpha \in \mathbb{E}^n$ and $\|\alpha\| = 0 \Longleftrightarrow \alpha = 0$.

(ii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ for all $\alpha, \beta \in \mathbb{E}$.

(iii) $\|r\alpha\| = |r|\|\alpha\|$ for all $r \in \mathbb{R}, \alpha \in \mathbb{E}^n$.

Example 2.5.2. Let $\|\cdot\| :\ \mathbb{E}^n \to \mathbb{E}$ be given by

$$\|\alpha\| = \max_{i\in\{1,\ldots,n\}} |\alpha_i| \quad \text{for all} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{E}^n.$$

Then one has

(i) Obviously $0 \leq \|\alpha\|$ for all $\alpha \in \mathbb{E}^n$ and $\|\alpha\| = 0 \Longleftrightarrow \alpha = 0$.

(ii) The triangular inequality $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ holds for all $\alpha, \beta \in \mathbb{E}^n$. Indeed, by Proposition 2.2.43 we have

$$\|\alpha+\beta\| = \max_{i\in\{1,\ldots,n\}} |\alpha_i+\beta_i| = |\alpha_{i_0}+\beta_{i_0}| \leq |\alpha_{i_0}|+|\beta_{i_0}| \leq \max_{i\in\{1,\ldots,n\}} |\alpha_i| + \max_{i\in\{1,\ldots,n\}} |\beta_i| = \|\alpha\|+\|\beta\|.$$

(iii) Clearly, $\|r\alpha\| = |r|\|\alpha\|$ for all $r \in \mathbb{R}, \alpha \in \mathbb{E}^n$.

So $\|\cdot\|$ is a norm on $\mathbb{E}^n$.

Example 2.5.3. Let $\|\cdot\| :\ \mathbb{E}^n \to \mathbb{E}$ be given by

$$\|\alpha\| = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2} \quad \text{for all} \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{E}^n.$$

Then

(i) Obviously $0 \leq \|\alpha\|$, for all $\alpha \in \mathbb{E}^n$ and $\|\alpha\| = 0 \Longleftrightarrow \alpha = 0$.

(ii) Let $\alpha = (\alpha_1, \dots, \alpha_i), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{E}^n$. To verify the triangle inequality we first prove that

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\| \cdot \|\beta\| \quad \text{for all} \quad \alpha, \beta \in \mathbb{E}^n, \tag{2.16}$$

where $\langle \alpha, \beta \rangle = \sum\limits_{i=1}^{n} \alpha_i \beta_i$. Note that

$$\{\|u\| \mid u \in \alpha\} \subseteq \|\alpha\| \quad \text{for all} \quad \alpha \in \mathbb{E}^n. \tag{2.17}$$

and if $\alpha, \beta \in \mathbb{E}$ are non-negative, then

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta. \tag{2.18}$$

Let $x \in |\langle \alpha, \beta \rangle|$. We need to prove that there is $y \in \|\alpha\| \cdot \|\beta\|$ such that $x \leq y$. Because $x \in |\langle \alpha, \beta \rangle|$, we have $x = |\langle u, v \rangle|$ with $u \in \alpha, v \in \beta$. It follows from the Cauchy-Schwarz inequality that $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$. Put $y = \|u\| \cdot \|v\| \in \|\alpha\| \cdot \|\beta\|$ by (2.17). Then $x \leq y$. So $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \cdot \|\beta\|$.

On the other hand, by subdistributivity and formulas (2.17), (2.18) we have

$$\begin{aligned} \|\alpha + \beta\|^2 =& \sum_{i=1}^{n} (\alpha_i + \beta_i)^2 \leq \sum_{i=1}^{n} \left(\alpha_i^2 + \beta_i^2 + 2\alpha_i \cdot \beta_i\right) = \sum_{i=1}^{n} \alpha_i^2 + \sum_{i=1}^{n} \beta_i^2 + 2 \sum_{i=1}^{n} \alpha_i \beta_i \\ =& \|\alpha\|^2 + \|\beta\|^2 + 2\langle \alpha, \beta \rangle \leq \|\alpha\|^2 + \|\beta\|^2 + 2\|\alpha\| \cdot \|\beta\| = (\|\alpha\| + \|\beta\|)^2. \end{aligned}$$

It follows that $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

(iii) Clearly $\|r\alpha\| = |r| \|\alpha\|$ for all $r \in \mathbb{R}, \alpha \in \mathbb{E}^n$.

We conclude that $\| \cdot \|$ is a norm on $\mathbb{E}^n$.

3

Matrices and vectors with external numbers

In this chapter we study matrices and vectors with uncertainties in terms of external numbers. In Section 3.1 we will start by introducing some special matrices and vectors which will be useful in the sequel. In Section 3.2 we present some properties of the determinant and its minors of a matrix with external numbers which are necessary for the study of flexible systems in the next chapter. Some of these properties are not identical with classical results. In Section 3.3 we will generalize some notions of traditional linear algebra such as linearly independent and dependent vectors. Some properties will be presented. The relationships between linear dependence and linear independence of a set of vectors with external numbers as well between vectors with external numbers and their representatives are investigated. In section 3.4 we study the rank of a matrix with external numbers. In classical linear algebra, it is well-known that the rank of a matrix determined via determinants is equal to the maximum number of independent row vectors, but in our context this relation is less evident. So different notions of rank of a matrix with external numbers are given. The first, based on the determinant, is called *minor-rank*, the second, based on the maximum number of independent row vectors, is called *row-rank*, and the last one, based on both the determinant and the rank of a representative matrix, is called *strict rank*. It is shown that under some conditions we have equalities.

3.1 Some notions and notation of matrices and vectors over $\mathbb{E}$

We introduce some special matrices and vectors with external numbers which will be used frequently in this work. Let

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$$

be an $m \times n$ matrix with $\alpha_{ij} \in \mathbb{E}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Let $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$ or $\beta = (\beta_1, \ldots, \beta_m)^T$ with $\beta_i \in \mathbb{E}$ be a column vector over $\mathbb{E}$. We denote by $\mathcal{M}_{m,n}(F)$ the set of all $m \times n$ matrices in F, where F is either $\mathbb{R}$ or $\mathbb{E}$. When $m = n$ we simply write $\mathcal{M}_n(F)$.

We will use the following notations: $|\overline{\alpha}| = \max\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} |\alpha_{ij}|$, $\quad \overline{A} = \max\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} A_{ij}$, $\quad \underline{A} = \min\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} A_{ij}$, $|\overline{\beta}| = \max\limits_{1 \leq i \leq m} |\beta_i|$, $\overline{B} = \max\limits_{1 \leq i \leq m} B_i$, $\quad \underline{B} = \min\limits_{1 \leq i \leq m} B_i$. We will always assume that $\overline{\alpha}$ is zeroless. We define below some special matrices.

Definition 3.1.1. A matrix of the form

$$I_A = \begin{bmatrix} 1 + A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & 1 + A_{nn} \end{bmatrix},$$

where neutrices $A_{ij} \subseteq \oslash$ for $1 \leq i, j \leq n$, is called a *near identity matrix* .

Example 3.1.2. The matrix $\begin{bmatrix} 1 + \oslash & \epsilon £ \\ \epsilon \oslash & 1 + \epsilon £ \end{bmatrix}$ is a near identity matrix.

Matrices with $|\overline{\alpha}| = 1 + A$, $A \subseteq \oslash$ play an important role in our work.

Definition 3.1.3. A matrix $\mathcal{A} = [\alpha_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{E})$, with $|\overline{\alpha}| = 1 + A$ and $A \subseteq \oslash$, is called a *reduced matrix* . A matrix $P = [a_{ij}] \in \mathcal{M}_{m,n}(\mathbb{R})$ is called *reduced* if $|a_{ij}| \leq 1$ and $a_{11} = 1$.

Example 3.1.4. The following matrix is a reduced matrix

$$\mathcal{A} = \begin{bmatrix} 1 + \oslash & 0.5 + \epsilon \oslash & -1 + \epsilon £ \\ -0.2 + \epsilon^2 \oslash & 0.3 + \epsilon \oslash & -0.4 + \epsilon £ \\ 0.1 + \oslash & 0.2 + \epsilon £ & 0.7 + \epsilon £ \end{bmatrix}.$$

Definition 3.1.5. A matrix $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ is called *non-singular* if $m = n$ and det($\mathcal{A}$) is zeroless. Otherwise we call it *singular*.

Definition 3.1.6. Let $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$. A matrix $R = [a_{ij}] \in \mathcal{M}_{m,n}(\mathbb{R})$, with $a_{ij} \in \alpha_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$, is called a *representative matrix* of $\mathcal{A}$. In particular, if $\mathcal{A}$ is a reduced matrix and R is a representative matrix of $\mathcal{A}$ which is reduced then we call R a *reduced representative matrix* of $\mathcal{A}$.

Definition 3.1.7. Let $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{E}^n$. If $\overline{\beta}$ is a neutrix, β is called an *upper neutrix vector*. Moreover, a vector $b = (b_1, \dots, b_n)$, where $b_i \in \beta_i$ for $1 \leq i \leq n$, is said to be a *representative* of β.

Example 3.1.8. The vector $\beta = \big(\epsilon + \epsilon^2 \oslash, \oslash, \epsilon + \epsilon^2 £\big)$ is an upper neutrix vector since $\overline{\beta} = \oslash$ is a neutrix. The vector $\beta_1 = \big(1 + \epsilon^2 \oslash, \oslash, 2 + \epsilon £\big)$ is not an upper neutrix vector since $\overline{\beta} = 2 + \epsilon £$ is zeroless.

Definition 3.1.9. For each $1 \leq k \leq n$, a vector in the form $e_A^{(k)} = (A_1, \dots, A_{k-1}, 1+A_k, A_{k+1}, \dots, A_n) \in \mathbb{E}^n$, where neutrices $A_i \subseteq \oslash$ for $1 \leq i \leq n$, is called a *near unit vector* .

3.2 Properties of determinants with external numbers

We start this section by showing that the Laplace expansion of a determinant along a column or a row is not an equality, but an inclusion.

We denote by $\Delta_{i,j}$ the (i, j) minor of $\mathcal{A}$, that is the determinant of $(n-1) \times (n-1)$ submatrix of $\mathcal{A}$ that results from removing the i-th row and the j-th column of $\mathcal{A}$.

Proposition 3.2.1 ([20]). *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_n(\mathbb{E})$ and $\Delta = \det(\mathcal{A})$. Then for all $j \in \{1, ..., n\}$,*

$$(-1)^{j+1}\alpha_{1j}\Delta_{1,j} + \cdots + (-1)^{j+n}\alpha_{nj}\Delta_{n,j} \subseteq \Delta.$$

Proof. We only prove it for $j = 1$, the other cases are similar. Let S_n be the set of all permutations of $\{1, \dots, n\}$ and $\sigma \in S_n$. The Laplace expansion along the first column and subdistributivity yields

$$\begin{aligned}
&\alpha_{11}\Delta_{1,1} - \alpha_{21}\Delta_{2,1} + \cdots + \alpha_{n1}(-1)^{1+n}\Delta_{n,1} \\
=&\alpha_{11} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \big(\operatorname{sgn}(\sigma)\alpha_{\sigma(2)2} \cdots \alpha_{\sigma(n)n}\big) + \cdots + \alpha_{n1} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=n}} \big(\operatorname{sgn}(\sigma)\alpha_{\sigma(1)1} \cdots \alpha_{\sigma(n-1)(n-1)}\big) \\
\subseteq& \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \alpha_{11}\big(\operatorname{sgn}(\sigma)\alpha_{\sigma(2)2} \cdots \alpha_{\sigma(n)n}\big) + \cdots + \sum_{\substack{\sigma \in S_n \\ \sigma(1)=n}} \alpha_{n1}\big(\operatorname{sgn}(\sigma)\alpha_{\sigma(1)1} \cdots \alpha_{\sigma(n-1)(n-1)}\big) \\
=& \sum_{\sigma \in S_n} \big(\operatorname{sgn}(\sigma)\alpha_{\sigma(1)1} \cdots \alpha_{\sigma(n-1)(n-1)}\alpha_{\sigma(n)n}\big) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} = \Delta.
\end{aligned}$$

□

The next result shows that, for reduced matrices corresponding to each column (row), there is a minor of $(n-1)^{th}$-order such that the minor is the same order of magnitude as the determinant. It also gives a lower bound for absolute values of minors of $(n-1)^{th}$-order.

Proposition 3.2.2 ([20]). *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_n(\mathbb{E})$ be a reduced square matrix of order n. Suppose that $\Delta = \det\mathcal{A}$ is zeroless. Then for each $j \in \{1, \ldots, n\}$, there exists $i \in \{1, \ldots, n\}$ such that*

$$|\Delta_{i,j}| > \oslash\Delta.$$

Proof. For simplicity we prove only the case $j = 1$, the other cases are proved analogously. By Proposition 3.2.1 one has

$$\alpha_{11}\Delta_{1,1} - \alpha_{21}\Delta_{2,1} + \cdots + \alpha_{n1}(-1)^{n+1}\Delta_{n,1} \subseteq \Delta.$$

Suppose that $\Delta_{i,1} \subseteq \oslash\Delta$ for all $i = 1, \ldots, n$. Because the matrix is reduced, it holds that $|\alpha_{ij}| \leq 1 + \oslash$, for all $1 \leq i, j \leq n$. So $\alpha_{i1}\Delta_{i,1} \subseteq (1 + \oslash) \oslash \Delta = \oslash\Delta$ for all $i = 1, \ldots, n$. Consequently,

$$\alpha_{11}\Delta_{1,1} - \alpha_{21}\Delta_{2,1} + \cdots + \alpha_{n1}(-1)^{n+1}\Delta_{n,1} \subseteq \oslash\Delta.$$

So $\alpha_{11}\Delta_{1,1} - \alpha_{21}\Delta_{2,1} + \cdots + \alpha_{n1}(-1)^{n+1}\Delta_{n,1} \subseteq \Delta \cap \oslash\Delta$, a contradiction to Lemma 2.2.22, for Δ is zeroless. □

The results below give an upper bound of the minors and their neutrix parts of a reduced matrix.

Let $A \in \mathcal{M}_{m,n}(\mathbb{E})$. We denote by $M_{i_1 \ldots i_k, j_1 \ldots j_k}$ the $k \times k$ minor of $\mathcal{A}$ by holding only rows $\{i_1 \ldots i_k\}$ and columns $\{j_1 \ldots j_k\}$ from $\mathcal{A}$.

Proposition 3.2.3. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_n(\mathbb{E})$ be a reduced matrix. Let $k \in \{1, \ldots, n\}$ and $1 \leq i_1 < \cdots < i_k \leq n,\ 1 \leq j_1 < \cdots < j_k \leq n$. Then*

$$|M_{i_1 \ldots i_k, j_1 \ldots j_k}| \leq \pounds.$$

Proof. Let $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$. Let S_k be the set of all bijections $\sigma \colon I \to J$. Because $\mathcal{A}$ is a reduced matrix, it follows that $|\alpha_{ij}| \leq 1 + \oslash$ for all $1 \leq i, j \leq n$. So

$$\begin{aligned}
|M_{i_1 \ldots i_k, j_1 \ldots j_k}| &= \left| \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha_{i_1\sigma(i_1)} \ldots \alpha_{i_k\sigma(i_k)} \right| \\
&\leq \sum_{\sigma \in S_k} \left|\alpha_{i_1\sigma(i_1)}\right| \ldots \left|\alpha_{i_k\sigma(i_k)}\right| \leq \sum_{\sigma \in S_k} (1 + \oslash)^k \\
&= k!(1 + \oslash).
\end{aligned}$$

Because $n \in \mathbb{N}$ is standard and $k \leq n$, it follows that $k! \leq \pounds$. Consequently, $k!(1 + \oslash) \leq \pounds$. Hence $|M_{i_1 \ldots i_k, j_1 \ldots j_k}| \leq \pounds$. □

Lemma 3.2.4. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_n(\mathbb{E})$ be a reduced non-singular matrix. Let $\Delta = \det\mathcal{A}$, $k \in \{1, \ldots, n\}$ and $1 \leq i_1 < \cdots < i_k \leq n,\ 1 \leq j_1 < \cdots < j_k \leq n$. Then for all $1 \leq k \leq n$, one has*

$$N\left(M_{i_1 \ldots i_k, j_1 \ldots j_k}\right) \subseteq \overline{A}.$$

In particular $N(\Delta) \subseteq \overline{A}$.

Proof. Let $I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\}$. Let S_k be the set of all bijections $\sigma\colon I \to J$. Because $\mathcal{A}$ is a reduced matrix, it follows that $|\alpha_{ij}| \leq 1 + \overline{A}$ for all $1 \leq i, j \leq n$. So

$$\begin{aligned} N\left(M_{i_1 \ldots i_k, j_1 \ldots j_k}\right) =& N\left(\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \alpha_{i_1\sigma(i_1)} \ldots \alpha_{i_k\sigma(i_k)}\right) \\ &= \sum_{\sigma \in S_k} N\left(\alpha_{i_1\sigma(i_1)} \cdots \alpha_{i_k\sigma(i_k)}\right) \subseteq \sum_{\sigma \in S_k} N\left((1+\overline{A})^k\right) = \sum_{\sigma \in S_k} \overline{A} = k!\overline{A} = \overline{A}. \end{aligned}$$

When $k = n$ we obtain that $N(\Delta) \subseteq \overline{A}$. $\square$

3.3 Linear dependence and independence of vectors

A neutrix can be seen as a generalization of zero, so a vector such that all of its components are neutrices plays a role as zero vector in classical linear algebra. In the definition of linear dependence of vectors with external numbers below, a neutrix vector is used instead of the zero vector.

Definition 3.3.1. A vector $A = (A_1, \ldots, A_n)$, where A_i is a neutrix for all $1 \leq i \leq n$, is called a *neutrix vector*.

Definition 3.3.2. A set of vectors in $\mathbb{E}^n$

$$V = \{\alpha_1, \ldots, \alpha_m\}$$

where $\alpha_i \in \mathbb{E}^n$ for $1 \leq i \leq m$ is called *linearly dependent* if there exist real numbers $t_1, t_2, ..., t_m \in \mathbb{R}$, at least one of them being non-zero, and a neutrix vector $A = (A_1, A_2, ..., A_n)$ such that

$$t_1\alpha_1 + t_2\alpha_2 + \cdots + t_m\alpha_m = A.$$

Otherwise, the set of vectors $V = \{\alpha_1, \ldots, \alpha_m\}$ is called *linearly independent*.

Remark 3.3.3. A set $V = \{\alpha_1, \cdots, \alpha_m\}$ of vectors in $\mathbb{E}^n$ is linearly independent if and only if that $t_1\alpha_1 + t_2\alpha_2 + \cdots + t_m\alpha_m = A$, where A is a neutrix vector, implies $t_1 = \cdots = t_m = 0$.

Example 3.3.4. Let $\epsilon > 0$ be infinitesimal. Then the vectors $\alpha_1 = (1+\oslash, \epsilon\oslash, -2+\epsilon£),\ \alpha_2 = (-2+\oslash, \epsilon£, 4+\epsilon£)$ in $\mathbb{E}^3$ are linearly dependent, since $2\alpha_1 + \alpha_2 = (\oslash, \epsilon£, \epsilon£)$ is a neutrix vector.

Example 3.3.5. The vectors $\alpha_1 = (1 + \oslash, \epsilon\oslash)$, $\alpha_2 = (\oslash, 1 + \epsilon£)$ with $\epsilon > 0$ in $\mathbb{E}^2$ are linearly independent.

Indeed, let $t_1, t_2 \in \mathbb{R}$ and $A = (A_1, A_2)$ is a neutrix vector such that $t_1\alpha_1 + t_2\alpha_2 = A$. Then there are vectors $x_1 = (1 + \eta, \epsilon\zeta) \in \alpha_1$ and $x_2 = (\vartheta, 1 + \epsilon\upsilon) \in \alpha_2$, where η, ζ, ϑ are infinitesimal, such that $t_1x_1 + t_2x_2 = 0$. It is equivalent to the following

$$\begin{cases} t_1(1+\eta) + t_2\vartheta = 0 \\ t_1\zeta + t_2(1+\epsilon\upsilon) = 0 \end{cases}$$

This implies that $t_1 = t_2 = 0$. Hence the vectors α_1, α_2 are linearly independent.

Example 3.3.6. Let $\epsilon > 0$ be infinitesimal. The set of vectors $\{\alpha_1 = (1+\oslash, \epsilon\oslash, -2+\epsilon£), \alpha_2 = (\oslash, \epsilon£, \epsilon£)\} \subset \mathbb{E}^3$ is linearly dependent, since $0\alpha_1 + \alpha_2 = (\oslash, \epsilon£, \epsilon£)$ is a neutrix vector.

A generalization of this example above is the following result, which has an obvious proof.

Proposition 3.3.7. *Any set of vectors with external numbers including a neutrix vector is linearly dependent.*

We next present some useful properties of vectors in $\mathbb{E}^n$. We start by characterizing linearly independence and dependence of vectors with external numbers via representatives.

Theorem 3.3.8. *Let*

$$V = \{\xi_1 = (\xi_{11}, \dots, \xi_{1n}), \xi_2 = (\xi_{21}, \dots, \xi_{2n}), \dots, \xi_m = (\xi_{m1}, \dots, \xi_{mn})\} \subset \mathbb{E}^n$$

be a set of vectors, with $\xi_{ij} = a_{ij} + A_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$. Then

(i) The set V of vectors in $\mathbb{E}^n$ is linearly dependent if and only if for all $1 \le i \le m$, there exist representatives $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ of ξ_i such that $x_1, \dots, x_m$ are linearly dependent.

(ii) The set V of vectors in $\mathbb{E}^n$ is linearly independent if and only if every set $\{x_1, \dots, x_m\}$ of vectors in $\mathbb{R}^n$, where $x_i \in \xi_i$ for $1 \le i \le m$, is linearly independent.

Proof. (i) Suppose that the vectors $\xi_1, \dots, \xi_m$ are linearly dependent. By the definition, there exist real numbers $t_1, \dots, t_m$, at least one of them being non-zero, and a neutrix vector $A = (A_1, \dots, A_n)$ such that

$$t_1\xi_1 + t_2\xi_2 + \cdots + t_m\xi_m = A.$$

Consequently, the vector $\theta = (0, ..., 0) \in t_1\xi_1+t_2\xi_2+\cdots+t_m\xi_m$. Hence there exist vectors $x_i \in \xi_i, i = 1, ..., m$ such that $t_1x_1 + t_2x_2 + \cdots + t_mx_m = 0$. That is, the set of vectors $\{x_1, ..., x_m\}$ is linearly dependent.

Conversely, suppose that there exists a set of vectors $V' = \{x_1, ..., x_m\} \subset \mathbb{R}^n$, with $x_i \in \xi_i$ for all $i = 1, ..., m$ such that $x_1, ..., x_n$ are linearly dependent. Then there exist real numbers $t_1, ..., t_m$, at least one of them being non-zero, such that $t_1x_1 + t_2x_2 + \cdots + t_mx_m = 0$. Let $x_i = (x_{i1}, ..., x_{in})$ for all $i = 1, ..., m$. Then

$$t_1x_{1j} + \cdots + t_mx_{mj} = 0 \text{ for all } j = 1, ..., n. \tag{3.1}$$

Because $x_i \in \xi_i, i = 1, ..., m$, one has $\xi_{ij} = x_{ij} + A_{ij}$ for all $i = 1, ..., m; j = 1, ..., n$. From (3.1) one obtains that

$$\begin{aligned} t_1\xi_{1j} + \cdots + t_m\xi_{mj} =& t_1(x_{1j} + A_{1j}) + \cdots + t_m(x_{mj} + A_{mj}) \\ =& t_1x_{1j} + \cdots + t_mx_{mj} + t_1A_{1j} + \cdots + t_mA_{mj} \\ =& t_1A_{1j} + \cdots + t_mA_{mj} = A_j \end{aligned}$$

for all $j = 1, ..., n$. Hence the vectors $\xi_1, ..., \xi_m$ are linearly dependent.

(ii) Assume that the set of vectors $\{\xi_1, \ldots, \xi_m\}$ in $\mathbb{E}^n$ are linearly independent. Suppose on contrary that there exist representatives x_i of ξ_i for $1 \leq i \leq m$ such that $\{x_1, \ldots, x_m\}$ are linearly dependent. By Part (i) one concludes that $\xi_1, \ldots, \xi_m$ are linearly dependent, a contradiction. Hence there exists no representative x_i of ξ_i for $1 \leq i \leq m$ such that $x_1, \ldots, x_m$ are linearly dependent.

Conversely, suppose that for all $1 \leq i \leq m$ and for all representatives x_i of ξ_i one has $\{x_1, \ldots, x_m\}$ are linearly independent. Suppose that $\{\xi_1, \ldots, \xi_n\}$ are linearly dependent. By Part (i), there exists representatives x_i of ξ_i for all $1 \leq i \leq m$ such that $\{x_1, \ldots, x_m\}$ are linearly dependent, a contradiction. Hence $\{\xi_1, \ldots, \xi_m\}$ are linearly independent. $\square$

Observe that a set of linearly dependent vectors may have a set of linearly independent representative vectors.

Example 3.3.9. Let $\epsilon > 0$ be infinitesimal. Consider the set of vectors $\{\xi_1 = (\oslash, \oslash), \xi_2 = (0, \epsilon)\}$. Then $\{\xi_1, \xi_2\}$ is linearly dependent, since $\xi_1 + \xi_2 = (\oslash, \oslash)$. Now we take $x_1 = (\epsilon, 0) \in \xi_1$ and $x_2 = \xi_2$. Then $\{\xi_1, \xi_2\}$ is linearly independent.

Proposition 3.3.10. *Every set of vectors $\{\xi_1, \ldots, \xi_m\} \subset \mathbb{E}^n$, where $m > n$, is linearly dependent.*

Proof. Let $a_i = (a_{i1}, \ldots, a_{in}) \in \xi_i$ be a representative of $\xi_i, 1 \leq i \leq m$. Because $m > n$ then the set of vectors

$$V = \{a_1 = (a_{11}, \ldots, a_{1n}), \ldots, a_m = (a_{m1}, \ldots, a_{mn})\}$$

in $\mathbb{R}^n$ is linearly dependent. Hence, by Theorem 3.3.8, the set of vectors $\{\xi_1, \ldots, \xi_m\}$ is linearly dependent. $\square$

Proposition 3.3.11. *Let $S = \{\xi_1, \cdots, \xi_m\}$ be a set of vectors in $\mathbb{E}^n$ and $k \in \mathbb{N}$ be standard.*

(i) If the set S is linearly dependent, any set of k vectors including S is linearly dependent.

(ii) If the set S is linearly independent, any set of vectors included in S is linearly independent.

Proof. (i) Let

$$V = \{\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_k\} \subset \mathbb{E}^n.$$

Because the set S of vectors is linearly dependent, there exists real numbers $t_1, \ldots, t_m$, all of them are not equal to zero simultaneously, and a neutrix vector $A = (A_1, \ldots, A_n)$ such that $t_1\xi_1 + \cdots + t_m\xi_m = A$. Let $t' = (t_1, \ldots, t_m, 0, \ldots, 0)$. Then $t' \neq \theta = (0, \ldots, 0)$ and $t_1\xi_1 + \cdots + t_m\xi_m + 0\xi_{m+1} + \cdots + 0\xi_k = (A_1, \ldots, A_n)$, which is a neutrix vector in $\mathbb{E}^n$. Hence V is linearly dependent.

(ii) Let V' be a set of vectors included in S. Suppose that V' is linearly dependent. Because $V' \subseteq S$, by Part (i) the set S of vectors is linearly dependent, a contradiction. $\square$

Definition 3.3.12. Let $V = \{\xi_1, \ldots, \xi_m\}$ be a set of vectors in $\mathbb{E}^n$. The maximum number of linearly independent vectors of V is called the *rank* of the given set of vectors.

Example 3.3.13. Let $\xi_1 = (1 + \oslash, \oslash, -1 + \epsilon\oslash), \xi_2 = (-1 + \epsilon£, \epsilon\oslash, 1 + \oslash)$ with $\epsilon > 0$ is infinitesimal. Then the set of vectors $\{\xi_1, \xi_2\}$ is linearly dependent, since $\xi_1 + \xi_2 = (\oslash, \oslash, \oslash)$. The rank of given set of vectors is 1.

Definition 3.3.14. Let $\xi_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{E}^n, 1 \leq i \leq m$. The matrix

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix}$$

is called the *coordinate matrix* of the given vectors and is denoted by $[\xi_1, \ldots, \xi_m]^T$.

For a set of n vectors in $\mathbb{E}^n$ linear independence and dependence is determined by the determinant of its coordinate matrix.

Theorem 3.3.15. *A set $V = \{\alpha_1, \cdots, \alpha_n\}$ of n vectors in $\mathbb{E}^n$, where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$ for $1 \leq i \leq n$, is linearly independent if and only if*

$$\det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$

is zeroless.

Proof. Put $\alpha_{ij} = a_{ij} + A_{ij}$ for all $1 \leq i, j \leq n$ and

$$\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}.$$

Assume that $\det(\mathcal{A})$ is zeroless. Suppose that the set V of vectors is linearly dependent. By Theorem 3.3.8, there exists a set of vectors $a_i = (a_{i1}, ..., a_{in}) \in \mathbb{R}^n$, where $a_i \in \alpha_i$ is a representative of α_i for all $i = 1, .., n$, which is linearly dependent. It follows that

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = 0.$$

So

$$\begin{aligned} \det(\mathcal{A}) =& \det \begin{bmatrix} a_{11} + A_{11} & a_{12} + A_{12} & \cdots & a_{1n} + A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + A_{n1} & a_{n2} + A_{n2} & \cdots & a_{nn} + A_{nn} \end{bmatrix} \\ =& \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + N\big(\det(\mathcal{A})\big) = N\big(\det(\mathcal{A})\big), \end{aligned}$$

which is a contradiction.

Conversely, assume that the set of vectors $\{\alpha_1, ..., \alpha_n\}$ is linearly independent. Suppose that $\det(\mathcal{A})$ is a neutrix. Then there exist vectors $\{x_1, \ldots, x_n\}$, where $x_i = (a_{i1}, \ldots, a_{in}) \in \alpha_i$ for all $i \in \{1, ...n\}$, such that

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = 0.$$

Hence the set of vectors $\{x_1, x_2, ..., x_n\}$ is linearly dependent. By Theorem 3.3.8, the set of vectors $\{\alpha_1, ..., \alpha_n\}$ is linearly dependent, a contradiction. □

This proposition enables us to verify whether a set of vectors with external numbers is linearly independent or not.

Example 3.3.16. (a) The set of vectors

$$\{\alpha_1 = (1 + \oslash, 2 + \epsilon\oslash, \epsilon£), \alpha_2 = (-1 + \epsilon\oslash, \epsilon\oslash, -1 + \epsilon£), \alpha_3 = (\epsilon\oslash, 2 + \epsilon\oslash, -1 + \epsilon^2£)\}$$

is linearly dependent, since

$$\begin{vmatrix} 1 + \oslash & 2 + \epsilon\oslash & \epsilon£ \\ -1 + \epsilon\oslash & \epsilon\oslash & -1 + \epsilon£ \\ \epsilon\oslash & 2 + \epsilon\oslash & -1 + \epsilon^2£ \end{vmatrix} = \oslash.$$

(b) The set of vectors

$$\{\eta_1 = (1 + \oslash, 2 + \epsilon\oslash), \eta_2 = (-1 + \epsilon\oslash, \epsilon\oslash)\} \subset \mathbb{E}^2$$

is linearly independent, since

$$\begin{vmatrix} 1 + \oslash & 2 + \epsilon\oslash \\ -1 + \epsilon\oslash & \epsilon\oslash \end{vmatrix} = 2 + \epsilon \oslash .$$

Corollary 3.3.17. *If a matrix $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ over $\mathbb{E}$ has two identical representatives of rows then* $\det(\mathcal{A})$ *is neutricial.*

Proof. Put $\alpha_i = (\alpha_{i1}, ..., \alpha_{in})$ for all $i = 1, ..., n$ and $S = \{\alpha_1, ..., \alpha_n\}$. Because the matrix $\mathcal{A}$ has two identical representatives rows, the set of vectors S is linearly dependent. By Theorem 3.3.15, $\det(\mathcal{A})$ is neutricial. □

The corollary is also true if we use columns instead of rows.

3.4 On the ranks of a matrix over $\mathbb{E}$

In this section three notions of rank of a matrix over $\mathbb{E}$ are given, respectively based on minors, based on the maximum number of independent row vectors, and based on the minors and the rank of a representative matrix. In general, these three notions do not match. Conditions for the equality of these notions of ranks are presented.

Firstly we define the rank of a matrix over $\mathbb{E}$ via minors of the matrix.

Definition 3.4.1. Let $\mathcal{A} = [\alpha_{ij}]$ be an $m \times n$ matrix over $\mathbb{E}$. We say that the *minor-rank* of $\mathcal{A}$ is r, denoted by $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$, if there exists a zeroless minor of order r of $\mathcal{A}$ and every minor of order k of $\mathcal{A}$, with $k > r$, is neutricial.

Obviously $\mathrm{mr}(\mathcal{A}) \leq \min\{\mathrm{m}, \mathrm{n}\}$.

Example 3.4.2. Let $\mathcal{A} = \begin{bmatrix} 1+\oslash & 2+\oslash & -1+\epsilon£ \\ -2 & -4+\epsilon & 2+\epsilon\oslash \end{bmatrix}$. Then

$M_{12,12} = \begin{vmatrix} 1+\oslash & 2+\oslash \\ -2 & -4+\epsilon \end{vmatrix} = \oslash$, $M_{12,13} = \begin{vmatrix} 1+\oslash & -1+\epsilon£ \\ -2 & 2+\epsilon£ \end{vmatrix} = \oslash$,
$M_{12,23} = \begin{vmatrix} 2+\oslash & -1+\epsilon£ \\ -4+\epsilon & 2+\epsilon\oslash \end{vmatrix} = \oslash$, and $M_{1,1} = 1+\oslash$ zeroless. Hence $\mathrm{mr}(\mathcal{A}) = 1$.

Proposition 3.4.3. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $m \times n$ matrix over $\mathbb{E}$. Then* $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}(\mathrm{A}^{\mathrm{T}})$.

Proof. It is a consequence of the fact that $\det(\mathcal{A}) = \det(\mathcal{A}^T)$. $\square$

Next we define the rank of a matrix over $\mathbb{E}$ through the maximum number of linearly independent row vectors of the matrix.

Definition 3.4.4. Let $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$. The maximal number of linearly independent row vectors is called the *row-rank* of A, denoted by $r(\mathcal{A})$.

So the row-rank of a matrix is equal to the rank of the set of row vectors of the matrix presented in Definition 3.3.12.

Theorem 3.4.5. *Let $\mathcal{A} = [\alpha_{ij}]_{m\times n}$ be a matrix over $\mathbb{E}$. such that* $\mathrm{mr}(\mathcal{A}) = \mathrm{r} \leq \min\{\mathrm{m}, \mathrm{n}\}$. *Then there exist r row vectors of $\mathcal{A}$, which are linearly independent. As a consequence $r(\mathcal{A}) \geq$* $\mathrm{mr}(\mathcal{A})$.

Proof. Because $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$, we may suppose without loss of generality that the minor

$$\det(M) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix}$$

is zeroless. Let $\xi_i = (\alpha_{i1}, \ldots, \alpha_{in})$, for $1 \leq i \leq m$ be row vectors of $\mathcal{A}$ and $\xi_i' = (\alpha_{i1}, \ldots, \alpha_{ir})$, for $1 \leq i \leq m$ be vectors in $\mathbb{E}^r$. By Theorem 3.3.15 and the fact that $\det(M)$ is zeroless, the set of vectors $\{\xi_1', ..., \xi_r'\}$ is linearly independent.

We will prove that $\xi_1, \ldots, \xi_r$ are linearly independent. Assume that $t_1\xi_1 + \cdots + t_r\xi_r = (A_1, \ldots, A_n)$. That is, for all $1 \leq j \leq n$, one has $t_1\alpha_{1j} + t_2\alpha_{2j} + \cdots + t_r\alpha_{rj} = A_j$. It follows that $t_1\xi_1' + \cdots + t_r\xi_r' = (A_1, \ldots, A_r)$. Because $\{\xi_1', \ldots, \xi_r'\}$ is linearly independent, one has $t_1 = \cdots = t_r = 0$. Hence the set of vectors $\{\xi_1, ..., \xi_r\}$ is linearly independent by Remark 3.3.3. $\square$

For a square matrix over $\mathbb{E}$, linear independence and dependence of row vectors in a matrix are completely characterized by the determinant as shown in Theorem 3.3.15. However, unlike classical linear algebra, for rectangle matrices $\mathcal{A}$ of order $m \times n$ with $m \neq n$, it is difficult, in practice, to verify whether the maximum number of independent row vectors is equal to the minor-rank of $\mathcal{A}$ or not, or equivalently to verify that if all minors of order r of $\mathcal{A}$ are neutricial, the respective r row vectors are linearly dependent or not. To overcome this difficulty we introduce below another notion of the rank for matrices over $\mathbb{E}$ charactered via both minors and the rank of a representative. We call it *strict rank*.

Definition 3.4.6. Let $\mathcal{A} = [\alpha_{ij}]_{m\times n}$ be a matrix over $\mathbb{E}$. A number $r \in \mathbb{N}$ is called the *strict rank* of the matrix $\mathcal{A}$, denoted by $\text{sr}(\mathcal{A}) = r$, if the following holds:

(i) There is a zeroless minor of order r of $\mathcal{A}$.

(ii) There is a representative matrix $\hat{\mathcal{A}} = [a_{ij}] \in \mathcal{M}_{m,n}(\mathbb{R})$ of $\mathcal{A}$ such that the rank of $(\hat{\mathcal{A}})$ is r.

Clearly, in the case of non-singular matrices condition (i) implies the condition ii. So the three notions of rank coincide. However for non-singular matrices, in particular for non-square matrices, the equalities are not easy to verify. Below we show that if we know the strict rank, we know the minor-rank and the row-rank. Then conditions are given for the other relationships to hold.

Theorem 3.4.7. *Let $\mathcal{A}$ be an $m \times n$ matrix over $\mathbb{E}$. If* $\text{sr}(\mathcal{A}) = r$ *then* $\text{mr}(\mathcal{A}) = \text{r}$.

Proof. Because $\text{sr}(\mathcal{A}) = r$, there exists a zeroless minor of order r of $\mathcal{A}$. By the definition of minor-rank of a matrix one has $\text{mr}(\mathcal{A}) \geq \text{r}$. Let $\mathcal{A}_k = \mathcal{A}_{i_1\ldots i_k, i_1\ldots i_k}$ be a minor of order k of $\mathcal{A}$ with $k > r$. Because there exists a representative matrix $\hat{\mathcal{A}} = [a_{ij}]$ of $\mathcal{A}$ such that $\text{rank}(\hat{\mathcal{A}}) = r$, we have $\det\big(\hat{\mathcal{A}}_k\big) = \det\big(\hat{\mathcal{A}}_{i_1\ldots i_k, i_1\ldots i_k}\big) = 0$. So $\det\big(A_{i_1\ldots i_k, i_1\ldots i_k}\big)$ is a neutrix. One concludes that $\text{mr}(\mathcal{A}) = \text{r}$. □

Corollary 3.4.8. *Let $\mathcal{A}$ be an $m \times n$ matrix over $\mathbb{E}$. Then* $\text{sr}(\mathcal{A}) \leq \text{mr}(\mathcal{A})$.

Theorem 3.4.9. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $m \times n$ matrix over $\mathbb{E}$. If* $\text{sr}(\mathcal{A}) = r$ *then $r(\mathcal{A}) = r$.*

Proof. Assume that $\text{sr}(\mathcal{A}) = r$. Then there exists a representative matrix $\hat{\mathcal{A}}$ of $\mathcal{A}$ such that $\text{rank}(\hat{\mathcal{A}}) = r$. Without loss of generality, we may assume that $\det(\hat{\mathcal{A}}_r) = \det \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \cdots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \neq 0$. Let $i \in \{r+1, n\}$. Then the set of vectors

$$\{a_1 = (a_{11}, \ldots, a_{1n}), \ldots, a_r = (a_{r1}, \ldots, a_{rn}), a_i = (a_{i1}, \ldots, a_{in})\}$$

is linearly dependent. This implies that the set of vectors

$$\{\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1n}), \ldots, \alpha_r = (\alpha_{r1}, \ldots, \alpha_{rn}), \alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})\}$$

is linearly dependent. So the number of linearly independent vectors is at most r.

Moreover sr$(\mathcal{A}) = r$, by Theorem 3.4.7 we have mr$(\mathcal{A}) = \mathrm{r}$. By Theorem 3.4.5 there are exactly r linearly independent row vectors in $\mathcal{A}$. $\square$

Corollary 3.4.10. *Let $\mathcal{A}$ be an $m \times n$ matrix over $\mathbb{E}$. Then* sr$(\mathcal{A}) \leq r(\mathcal{A})$.

We end this section by studying several conditions such that the minor-rank, the row-rank, and the strict rank are equal.

Theorem 3.4.11. *Let $\mathcal{A} = [\alpha_{ij}]_{m\times n}$ be a matrix over $\mathbb{E}$. Assume that $r(\mathcal{A}) = r$ and there is a zeroless minor of order r of $\mathcal{A}$. Then* sr$(\mathcal{A}) = r$.

Proof. Because there are exactly r linearly independent row vectors in $\mathcal{A}$, by Theorem 3.3.8 there is a set of real vectors $V = \{a_1, \dots, a_m\}$, where $a_i \in \alpha_i = (\alpha_{i1}, \dots, \alpha_{in})$ for all $1 \leq i \leq m$, such that the maximum number of linearly independent vectors in V is r. It follows that the rank of the matrix $\hat{A} = [a_{ij}]$ is r. Also there is a zeroless minor of order r of $\mathcal{A}$, we conclude that sr$(\mathcal{A}) = r$. $\square$

The proposition below shows that if the minor-rank of a matrix is equal to the number of columns minus one, it is equal to the row-rank.

Proposition 3.4.12. *Let $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ such that* mr$(\mathcal{A}) = \mathrm{n} - 1$. *Then*

$$\mathrm{mr}(\mathcal{A}) = \mathrm{sr}(\mathcal{A}) = \mathrm{n} - 1 = \mathrm{r}(\mathcal{A}).$$

Proof. If $m = n - 1$, the conclusion follows by Theorem 3.4.5 and Theorem 3.4.11. Assume that $m > n - 1$. Let $\xi_i = (\alpha_{i1}, \dots, \alpha_{in})$, for $1 \leq i \leq m$. Because mr$(\mathcal{A}) = \mathrm{n} - 1$, by Theorem 3.4.5, there are $(n-1)$ linearly independent row vectors in $\mathcal{A}$. Suppose on contrary that $r(\mathcal{A}) > n - 1$. Then there are n linearly independent vectors in $\mathcal{A}$. Without loss of generality, we suppose that $\xi_1, \dots, \xi_n$ are linearly independent. By Theorem 3.3.15 it holds that det $\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$ is zeroless. Hence mr$(\mathcal{A}) \geq \mathrm{n}$, a contradiction. So $r(\mathcal{A}) = n - 1$. By Theorem 3.4.11 we have sr$(\mathcal{A}) = n - 1$. $\square$

Example 3.4.13. Let $\mathcal{A} = \begin{bmatrix} 1 + \oslash & 1 \\ 1 + \epsilon + \epsilon^2\oslash & 1 \\ 1 + \oslash & 1 \end{bmatrix}$. Then mr$(\mathcal{A}) = 1$. By Proposition 3.4.12, we have $r(\mathcal{A}) =$ sr$(\mathcal{A}) = 1$.

Next, we will show that if a matrix has a submatrix such that the relative uncertainty is included in all the neutrix parts of the remaining entries, the minor-rank is equal to the row-rank.

For convenience, we use the following notations.

Notation 3.4.14. Let $\mathcal{A} = [\alpha_{ij}] \equiv [a_{ij} + A_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ such that $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$. Let $I = \{i_1, \ldots, i_r\}$ and $J = \{i_1, \ldots, i_r\}$ be such that $\det(\mathcal{A}_{IJ}) = \det(\mathcal{A}_{i_1 \ldots i_r, j_1 \ldots j_r})$ is zeroless. We denote $\overline{A}_J = \max\limits_{\substack{j \in J \\ 1 \le i \le m}} A_{ij}$, and $\underline{A}_{J^C} = \min\limits_{\substack{j \notin J \\ 1 \le i \le m}} A_{ij}$.

Theorem 3.4.15. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $m \times n$ reduced matrix over $\mathbb{E}$, with $\alpha_{ij} = a_{ij} + A_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$. Suppose that $\dfrac{\overline{A}_J}{\det(\mathcal{A}_{IJ})} \subseteq \underline{A}_{J^C}$ for $1 \le i \le m$, and* $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$. *Then $r(\mathcal{A}) = r$. In particular* $\mathrm{st}(\mathcal{A}) = r$.

Proof. If $r = m$, the conclusion follows by Theorem 3.4.5. Assume that $r < m$. Because $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$, without loss of generality, assume that

$$\det(\mathcal{A}_{IJ}) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix} \tag{3.2}$$

is zeroless. We will demonstrate that every set $\{\xi_1, \ldots, \xi_r, \xi_i\}$ is linearly dependent for all $i \in \{r+1, \ldots, m\}$. To do this, we prove that there is a set of vectors

$$\{a_1 = (a_{11}, \ldots, a_{1n}), \ldots, a_r = (a_{r1}, \ldots, a_{rn}), a_i = (a_{i1}, \ldots, a_{in})\},$$

with $a_{pq} \in \alpha_{pj}, p \in \{1, \ldots, r, i\}, q \in \{1, \ldots, n\}$ satisfying

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rj} \\ a_{i1} & \cdots & a_{ir} & a_{ij} \end{bmatrix} = 0, \tag{3.3}$$

for all $j \in \{r+1, \ldots, n\}$.

For $j = r+1$, because $\mathrm{mr}(\mathcal{A}) = \mathrm{r}$ one has

$$\det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} & \alpha_{1(r+1)} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} & \alpha_{r(r+1)} \\ \alpha_{i1} & \cdots & \alpha_{ir} & \alpha_{i(r+1)} \end{bmatrix}$$

is a neutrix. Consequently, there exist $a_{ps} \in \alpha_{ps}$ for all $p \in \{1, \ldots, r, i\}, s \in \{1, \ldots, r+1\}$ such that

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a_{1(r+1)} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{r(r+1)} \\ a_{i1} & \cdots & a_{ir} & a_{i(r+1)} \end{bmatrix} = 0. \tag{3.4}$$

Hence formula (3.3) is true for $j = r+1$. Let $k \in \mathbb{N}, r+1 < k \le n$ be arbitrary. We need to prove that there is a column $a_k = (a_{1k}, \dots, a_{rk}, a_{ik})^T$ such that $a_{pk} \in \alpha_{pk}$ for all $p \in \{1, \dots, r, i\}$ and

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rk} \\ a_{i1} & \cdots & a_{ir} & a_{ik} \end{bmatrix} = 0, \tag{3.5}$$

where a_{ps} is defined by formula (3.4) $p \in \{1, \dots, r, i\}$ and $s \in \{1, \dots, r\}$. Because mr$(\mathcal{A}) =$ r, one has

$$\det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} & \alpha_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} & \alpha_{rk} \\ \alpha_{i1} & \cdots & \alpha_{ir} & \alpha_{ik} \end{bmatrix}$$

is a neutrix. As a result, there are row vectors

$$a'_1 = (a'_{11}, \dots, a'_{1r}, a'_{1k}), \dots, a'_r = (a'_{r1}, \dots, a'_{rr}, a'_{rk}), a'_i = (a'_{i1}, \dots, a'_{ir}, a'_{ik})$$

such that $a'_{ij} \in \alpha_{ij}$ and det$(T) = 0$ with

$$T \equiv \begin{bmatrix} a'_{11} & \cdots & a'_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a'_{r1} & \cdots & a'_{rr} & a'_{rk} \\ a'_{i1} & \cdots & a'_{ir} & a'_{ik} \end{bmatrix}. \tag{3.6}$$

To complete the proof, one shows that there exists $\epsilon_{ik} \in A_{ik}$ such that the column vector

$$\epsilon_{.k} = (0, \dots, 0, \epsilon_{ik})^T$$

satisfies

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{rk} \\ a_{i1} & \cdots & a_{ir} & a'_{ik} + \epsilon_{ik} \end{bmatrix} = 0.$$

That is, we need to find ϵ_{ik} such that

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{rk} \\ a_{i1} & \cdots & a_{ir} & a'_{ik} \end{bmatrix} + \det \begin{bmatrix} a_{11} & \cdots & a_{1r} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & 0 \\ a_{i1} & \cdots & a_{ir} & \epsilon_{ik} \end{bmatrix} = 0.$$

Using the Laplace expansion along the $(r+1)$-th column for the second determinant, the above condition becomes

$$d.\epsilon_{ik} + \det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{rk} \\ a_{i1} & \cdots & a_{ir} & a'_{ik} \end{bmatrix} = 0, \tag{3.7}$$

where

$$d = \det \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}.$$

Because $a_{qs}, a'_{qs} \in \alpha_{qs}$ for all $q, s \in \{1, \ldots, r\}$, we have

$$a_{qs} = a'_{qs} + \epsilon_{qs}, \text{ where } \epsilon_{qs} \in A_{qs}.$$

So

$$\eta \equiv \det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{rk} \\ a_{i1} & \cdots & a_{ir} & a'_{ik} \end{bmatrix} = \det \begin{bmatrix} a'_{11} + \epsilon_{11} & \cdots & a'_{1r} + \epsilon_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a'_{r1} + \epsilon_{r1} & \cdots & a'_{rr} + \epsilon_{rr} & a'_{rk} \\ a'_{i1} + \epsilon_{i1} & \cdots & a'_{ir} + \epsilon_{ir} & a'_{ik} \end{bmatrix}.$$

Let $S_{(r+1)}$ be the set of all bijections σ: $\{1, \ldots, r, i\} \to \{1, \ldots, r, k+1\}$. Then

$$\begin{aligned} \eta = & \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma)(a'_{1\sigma(1)} + \epsilon_{1\sigma(1)}) \cdots (a'_{r\sigma(r)} + \epsilon_{r\sigma(r)}).a'_{i\sigma(i)} \\ \equiv & \det T + \nu. \end{aligned}$$

Because $\det(T) = 0$, one has $\eta = \nu$. We will show that $\eta = \nu \in \overline{A}_J$. Observe that ν is the sum of terms which contains at least one ϵ_{ps} with $p \in \{1, \ldots, r, i\}$ and $s \in \{1, \ldots, r\}$. Because $\epsilon_{ps} \in A_{ps} \subseteq \overline{A}_J$ for all $s \in \{1, \ldots, r\}$, $p \in \{1, \ldots, r, i\}$ and $|a'_{kl}| \leq |\alpha_{kl}| \leq 1 + \oslash$ for $k \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m\}$, each term of ν is included in $\overline{A}_J(1+\oslash)^r = \overline{A}_J$. Hence, $\eta = \nu \in \overline{A}_J$.

Also, by formula (3.2) we have $d \in \det(\mathcal{A}_{IJ})$. It follows that $\dfrac{\overline{A}_J}{d} \subseteq \underline{A}_{JC} \subseteq A_{ik}$ by the assumption. So

$$\epsilon_{ik} = -\frac{\eta}{d} \in \frac{\overline{A}_J}{d} \subseteq \underline{A}_{JC} \subseteq A_{ik}. \tag{3.8}$$

Hence the vector

$$\epsilon_{.k} = [0, \ldots, 0, -\eta/d]^T$$

satisfies (3.7). That is, formula (3.5) is satisfied with $a_{.k} = (a'_{1k}, \ldots, a'_{rk}, a'_{ik} - \eta/d)^T$.

Because $k \in \{r+1, \ldots, n\}$ is arbitrary, one concludes that (3.3) holds for all $k \in \{r+1, \ldots, n\}$. So the set of vectors

$$\{a_1 = (a_{11}, \cdots, a_{1n}), \cdots, a_r = (a_{r1}, \cdots, a_{rn}), a_i = (a_{i1}, \cdots, a_{in})\}$$

is linearly dependent for all $i \in \{r+1, \ldots, m\}$. By Theorem 3.3.8, the set of vectors

$$V = \{\alpha_1 = (\alpha_{11}, \cdots, \alpha_{1n}), \cdots, \alpha_r = (\alpha_{r1}, \cdots, \alpha_{rn}), \alpha_i = (\alpha_{i1}, \cdots, \alpha_{in})\}$$

is linearly dependent for each $i \in \{r+1, \cdots, m\}$. Hence $r(\mathcal{A}) = r$.

The last conclusion follows by Theorem 3.4.11. □

Finally we show if all entries of a given matrix $\mathcal{A}$ have the same neutrix parts, the minor-rank is equal to the strict rank, and therefore to the row-rank.

Theorem 3.4.16. *Let $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ be a reduced matrix with $N(\alpha_{ij}) = A_{ij} \equiv A$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Assume that* mr$(\mathcal{A}) =$ r. *Then $r(\mathcal{A}) =$ sr$(\mathcal{A}) = r$.*

Proof. If $m = r$, the conclusion follows by Theorem 3.4.5 and Theorem 3.4.11. Assume that $r < m$. Because mr$(\mathcal{A}) =$ r, there exists a submatrix $\mathcal{A}_r$ of $\mathcal{A}$ such that det$\mathcal{A}_r$ is zeroless. We can also choose det$(\mathcal{A}_r)$ such that the absolute value of det$\mathcal{A}_r$ is the maximum minor comparing to all the absolute value of the minors of order r of $\mathcal{A}$. For simplicity, we may assume that

$$\det(\mathcal{A}_r) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix} \equiv d + D = \Delta \quad \text{is zeroless.}$$

Let $p \in \{r+1, \ldots, m\}$ be arbitrary. We will prove that, there exists a set of representative vectors

$$\{a_1 = (a_{11}, \ldots, a_{1n}), \ldots, a_r = (a_{r1}, \ldots, a_{rn}), a_p = (a_{p1}, \ldots, a_{pn})\}$$

of $\{\alpha_1, \ldots, \alpha_r, \alpha_p\}$, such that the set of vectors $\{a_1, \ldots, a_r, a_p\}$ is linearly dependent.

With an analogous argument as in the proof of Theorem 3.4.15 we show that

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rj} \\ a_{p1} & \cdots & a_{pr} & a_{pj} \end{bmatrix} = 0 \text{ for all } r+1 \leq j \leq n, \tag{3.9}$$

where $a_{ij} \in \alpha_{ij}$ for $i \in \{1, \ldots, r, p\}, j \in \{1, \ldots, r, r\}$ are fixed. Put $u_s = \big(a_{1s}, \ldots, a_{rs}, a_{ps}\big),\ s \in \{1, \ldots, n\}$.

For $j = 1 + r$, formula (3.9) is true. Let $k \in \{r+2, \ldots, n\}$ be arbitrary. Because mr$(\mathcal{A}) =$ r, one has $\det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} & \alpha_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} & \alpha_{rk} \\ \alpha_{p1} & \cdots & \alpha_{pr} & \alpha_{pk} \end{bmatrix}$ is a neutrix. Consequently, there exist $a'_{ij} \in \alpha_{ij}$ for all $i \in \{1, \ldots, r, k\}$ and

$j \in \{1, \ldots, r, p\}$ such that

$$\det \begin{bmatrix} a'_{11} & \cdots & a'_{1r} & a'_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a'_{r1} & \cdots & a'_{rr} & a'_{rk} \\ a'_{p1} & \cdots & a'_{pr} & a'_{pk} \end{bmatrix} = 0.$$

This means that the set of column vectors

$$\{u'_1 = (a'_{11}, \ldots, a'_{r1}, a'_{p1}), \ldots, u'_r = (a'_{1r}, \ldots, a'_{rr}, a'_{pr}), u'_{k+1} = (a'_{1k}, \ldots, a'_{rk}, a'_{pk})\}$$

is linearly dependent. As a consequence, there exist real numbers $t_1, \ldots, t_r$ such that

$$\begin{aligned} u'_k =& t_1 u'_1 + \cdots + t_r u'_r \\ =& t_1(u_1 + \epsilon_1) + \cdots + t_r(u_r + \epsilon_r) \\ =& (t_1 u_1 + \cdots + t_r u_r) + (t_1\epsilon_1 + \cdots + t_r\epsilon_r), \end{aligned} \tag{3.10}$$

where $\epsilon_q = (\epsilon_{1q}, \ldots, \epsilon_{rq}, \epsilon_{pq}) \in A^{r+1} = A \times \cdots \times A$. We also have $t_q = \dfrac{d_{pq}}{d}, 1 \leq q \leq r$, where

$$d_{pq} = \det \begin{bmatrix} a'_{11} & \cdots & a'_{1q-1} & a'_{1k} & a'_{1q+1} & \cdots & a'_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a'_{r1} & \cdots & a'_{rq-1} & a'_{rk} & a'_{rq+1} & \cdots & a'_{rr} \end{bmatrix}.$$

Moreover $d \neq 0, d_{pq} \in \Delta_{pq}$, and $\left|\dfrac{d_{pq}}{d}\right| \leq 1 + \oslash$ since $\det(\mathcal{A}_r)$ is maximum, one derives

$$t_1\epsilon_1 + \cdots + t_r\epsilon_r \in A^{r+1}.$$

Put

$$u_k = u'_k - (t_1\epsilon_1 + \cdots + t_r\epsilon_r) \equiv (a_{1k}, \ldots, a_{rk}, a_{pk}).$$

Then $a_{qk} \in \alpha_{qk}, q \in \{1, \ldots, r, p\}$. By formula (3.10) one has

$$u_k = t_1 u_1 + \cdots + t_r u_r.$$

So the set of vectors $\{u_1, \ldots, u_r, u_k\}$ is linearly dependent. Hence

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1r} & a_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rk} \\ a_{p1} & \cdots & a_{pr} & a_{pk} \end{bmatrix} = 0.$$

Hence formula (3.9) holds for $j = r + 1, \ldots, n$. Consequently, the set of vectors $\{a_1, \ldots, a_r, a_p\}$ is linearly dependent. It follows that $\{\alpha_1, \ldots, \alpha_r, \alpha_p\}$ is linearly dependent for all $p \in \{r + 1, \ldots, m\}$. So $r(\mathcal{A}) = r$. By

Theorem 3.4.11 we obtain $\mathrm{sr}(\mathcal{A}) = r$. □

4

Flexible systems of linear equations

4.1 Introduction

In this chapter we will study a system of linear equations in which coefficients are external numbers. It is well-known that many problems of engineering and economics are modelled in terms of systems of linear equations. Data which form the system often involve imprecisions. As a result, the coefficients of the system contain uncertainties. Also in practice solving a linear system implies many successive computer operations, so next to problems of propagation of errors appear problems of rounding off. We will use external numbers to model these imprecisions. A system of linear equations with external numbers is called a *flexible system*.

Part of this chapter is motivated by Chapter 7. We will apply results in this chapter to construct conditions such that a linear optimization problem with flexible objective function and constraints has optimal solutions.

In section 4.2 we define flexible systems, consider several types of solutions and also distinguish some special systems.

In general common methods like Cramer's rule, the Gauss-Jordan elimination do not work on flexible systems. In Section 4.3 we will present conditions such that Cramer's rule can be applied to non-singular flexible systems.

In Section 4.4 we will present an explicit formula for Gauss-Jordan elimination method in the case of linear systems. We will express Gauss operations in terms of multiplications of matrices. Then we apply this formula to study conditions such that we can use Gauss-Jordan elimination method to solve non-singular flexible systems of linear equations.

Using results developed in chapter 3, in Section 4.5 we give necessary and sufficient conditions so that singular flexible systems have solutions. Also, solution formulas are given.

In the final section we use a parameter method to deal with flexible systems. We will treat the neutrix parts of constant terms of a flexible system as sets of parameters. Taking advantages of the group properties of neutrices, under certain conditions formulas of solutions of a flexible system depending on parameters are given.

Convention 4.1.1. In this chapter we always assume that $m, n \in \mathbb{N}$ are standard.

4.2 Some basic notions

We start this section by defining some notions related to flexible systems, that is a system in which coefficients are external numbers. Then we will classify flexible systems into several different categories. Notions of solutions of a flexible system are also given.

Because an external number is an external set of real numbers, in flexible systems inclusions are used instead of equalities.

Definition 4.2.1 ([19])**.** Let $n, m \in \mathbb{N}$ be two standard natural numbers and $\alpha_{ij} = a_{ij} + A_{ij}, \beta_i = b_i + B_i$ for $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$ be external numbers. A system of the form

$$\begin{cases} \alpha_{11}\xi_1 + & \alpha_{12}\xi_2 + & \cdots & +\alpha_{1n}\xi_n & \subseteq b_1 + B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}\xi_1 + & \alpha_{m2}\xi_2 + & \cdots & +\alpha_{mn}\xi_n & \subseteq b_m + B_m \end{cases} \tag{4.1}$$

is called a *flexible system of linear equations*, or a *flexible system* (for short).

We call $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ the *coefficient matrix*, $\mathcal{B} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} b_1 + B_1 \\ \vdots \\ b_m + B_m \end{bmatrix}$ the *constant term vector* each β_i a *constant term* , $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$ the *variable vector*, and

$$[\mathcal{A}|\mathcal{B}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \beta_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} & \beta_m \end{bmatrix}$$

the *augmented matrix* of the system.

Then the system (4.1) can be represented in the matrix form $\mathcal{A}\xi \subseteq \mathcal{B}$.

Definition 4.2.2. The flexible system (4.1) is said to be

(i) *non-singular* if its coefficient matrix $\mathcal{A}$ is non-singular. Otherwise, we call it *singular*.

(ii) *homogeneous* if all the constant terms β_i for $i \in \{1, \dots, m\}$ are neutrices.

(iii) *upper homogeneous* if $\mathcal{B}$ is an upper neutrix vector.

(iv) *non-homogeneous* if it is not upper homogeneous.

Example 4.2.3. Let $\epsilon > 0$ be infinitesimal. Consider the flexible systems

(a) $\begin{cases} (1+\oslash)\xi_1 & + & \epsilon\pounds\xi_2 & \subseteq & \oslash \\ \epsilon\oslash\xi_1 & + & (1+\oslash)\xi_2 & \subseteq & \epsilon+\epsilon^2\oslash\,. \end{cases}$

(b) $\begin{cases} (1+\epsilon\oslash)\xi_1 & + & (2+\epsilon\pounds)\xi_2 & - & (3+\oslash)\xi_3 & \subseteq \oslash \\ (\epsilon\pounds)\xi_1 & + & \oslash\xi_2 & + & (\epsilon^2\oslash)\xi_3 & \subseteq \epsilon\pounds. \end{cases}$

(a) One has $\Delta = \det \begin{bmatrix} 1+\oslash & \epsilon\pounds \\ \epsilon\oslash & 1+\oslash \end{bmatrix} = 1+\oslash$ is zeroless and $|\overline{\beta}| = \max\{\oslash, \epsilon+\epsilon^2\oslash\} = \oslash$ is a neutrix. So the given flexible system is non-singular and upper homogeneous, although the constant term $\epsilon+\epsilon^2\oslash$ is zeroless. Hence the system is not homogeneous.

(b) Because $|\overline{\beta}| = \max\{\oslash, \epsilon\pounds\} = \oslash$ and the number of rows is $m = 2$ which differs from the number of columns $n = 3$, the given flexible system is singular. Also, all the constant terms are neutrices, hence the given flexible system is homogeneous.

From the definition and the examples above, it is clear that a homogeneous system is upper homogeneous, however, in general, the converse is not true, as shown by Example 4.2.3(a).

Remark 4.2.4. For an upper homogeneous flexible system $\mathcal{A}\xi \subseteq \mathcal{B}$ all the constant terms of the system are included in the largest neutrix. In fact, $\beta_i \subseteq \overline{B} = |\overline{\beta}|$ for all $i = 1, \dots, n$. Indeed, one has $|\beta_i| \le |\overline{\beta}| = \overline{B}$. Assume that $|\beta_i| \cap \overline{B} = \emptyset$, then for all $y \in |\beta_i|$, and for all $z \in \overline{B}$, it follows that $y < z$. Now we take $z \in \overline{B}, z < 0$, and $y \ge 0$, a contradiction. Hence $\beta_i \subseteq \overline{B}$.

An arbitrary flexible system can be transformed to an equivalent system such that the absolute value of every coefficient is less than or equal to $1 + \oslash$. For this kind of systems, easier to treat with, we call it a *reduced flexible system*.

Definition 4.2.5. A flexible system is called *reduced* if its coefficient matrix is reduced.

Example 4.2.6. For $\epsilon > 0$ an infinitesimal, the following system is a reduced system

$$\begin{cases} (1+\oslash)\xi_1 + (1/2+\epsilon £)\xi_2 - \epsilon \oslash \xi_3 & \subseteq 1+\oslash \\ (-1+\epsilon £)\xi_1 + \oslash\xi_2 - (1/3+\epsilon £)\xi_3 & \subseteq -2+\oslash. \end{cases}$$

The kinds of solutions of a given flexible systems are defined as follows.

Definition 4.2.7 ([19])**.** A vector of external numbers $\xi = (\xi_1, \ldots, \xi_n)$ is called an *admissible solution* of the flexible system (4.1) if it satisfies the system. In particular, if $\xi \in \mathbb{R}^n$ then we call it a *real admissible solution*. A solution $\xi = (\xi_1, \ldots, \xi_n)$ of the system (4.1) is said to be *maximal* if there is no external (internal) vector $\eta \supset \xi$ satisfying the system. If $\xi = (\xi_1, \ldots, \xi_n)$ satisfies the system with strict equalities, the vector $\xi = (\xi_1, \ldots, \xi_n)$ is called an *exact solution* of the system.

4.3 Cramer's rule for non-singular flexible systems

Note first that, in general, Cramer's rule is not true for flexible systems as shown in the following example.

Example 4.3.1. For $\epsilon > 0$ be infinitesimal, consider the homogeneous flexible system

$$\begin{cases} (1+\epsilon\oslash)\xi_1 & + & (\epsilon+\epsilon^2\oslash)\xi_2 & \subseteq \epsilon^2\oslash \\ \epsilon\oslash\xi_1 & + & (1+\epsilon £)\xi_2 & \subseteq \epsilon £. \end{cases}$$

One has

$$\Delta = \det \begin{bmatrix} 1+\epsilon\oslash & \epsilon+\epsilon^2\oslash \\ \epsilon\oslash & 1+\epsilon £ \end{bmatrix} = 1+\epsilon £ \quad \text{is zeroless.}$$

Hence the system is non-singular. Let

$$\begin{cases} \det(M_1) & = \det \begin{bmatrix} \epsilon^2\oslash & \epsilon+\epsilon^2\oslash \\ \epsilon £ & 1+\epsilon £ \end{bmatrix} = \epsilon^2 £, \\ \det(M_2) & = \det \begin{bmatrix} 1+\epsilon\oslash & \epsilon^2\oslash \\ \epsilon\oslash & \epsilon £ \end{bmatrix} = \epsilon £. \end{cases}$$

Applying classical Cramer's rule to the system one has

$$\begin{cases} \xi_1 & = \dfrac{\det(M_1)}{\Delta} = \dfrac{\epsilon^2 £}{1+\epsilon £} = \epsilon^2 £, \\ \xi_2 & = \dfrac{\det(M_2)}{\Delta} = \dfrac{\epsilon £}{1+\epsilon £} = \epsilon £. \end{cases}$$

However, it is not a valid solution of the system. Indeed, substituting ξ_1, ξ_2 into the first equation of the system, one obtains

$$(1+\epsilon\oslash)\epsilon^2\pounds + (\epsilon+\epsilon^2\oslash)\epsilon\pounds = \epsilon^2\pounds \not\subseteq \epsilon^2\oslash\,.$$

Hence ξ_1, ξ_2 do not satisfy the first equation.

In [19], J. Justino and I.P. van den Berg have shown that under certain conditions upon the sizes of uncertainties Cramer′s rule holds for non-singular, non-homogeneous flexible systems of linear equations. We will extend this result to all non-singular flexible systems of linear equations. This means that we can also apply the result to upper homogeneous, in particular to homogeneous non-singular flexible systems.

4.3.1 Main results on Cramer's rule

Consider a flexible system of the form

$$\begin{cases} \alpha_{11}\xi_1+ & \alpha_{12}\xi_2+ & \cdots & +\alpha_{1n}\xi_n & \subseteq b_1+B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1}\xi_1+ & \alpha_{n2}\xi_2+ & \cdots & +\alpha_{nn}\xi_n & \subseteq b_n+B_n, \end{cases} \tag{4.2}$$

where $n \in \mathbb{N}$ is a standard number. Put

$$\Delta = \det(\mathcal{A}) \equiv d + D,$$

where $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ is the coefficient matrix of the system. We define

$$M_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1j-1} & \beta_1 & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nj-1} & \beta_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}.$$

$$M_j(b) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1j-1} & b_1 & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nj-1} & b_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}.$$

$$M_j(a,b) = \begin{bmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}.$$

To study flexible systems we need to control uncertainties of entries in matrices and vectors. To do this, we will use the following definition.

Definition 4.3.2. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix and $\beta = (\beta_1, \ldots, \beta_n)^T$ be a column vector over $\mathbb{E}$.

(i) The *relative uncertainty* of $\mathcal{A}$ is defined by $R(\mathcal{A}) = \dfrac{\overline{A}\overline{\alpha}^{n-1}}{\Delta}$.

(ii) The *relative precision* of $\mathcal{B}$ is defined by $P(\mathcal{B}) = \underline{B}/\overline{\beta}$ if $\overline{\beta}$ is zeroless, and by $P(\mathcal{B}) = \underline{B} : \overline{B}$ if $\overline{\beta} = \overline{B}$ is a neutrix.

This definition is an extended version of the definition in [20]. In fact, we include the case where the maximal term is a neutrix. For the division of two neutrices, we refer to Definition 2.1.2

Definition 4.3.3. Consider the system (4.2). The following conditions are called the *Cramer conditions* of system (4.2).

(i) $R(\mathcal{A}) \subseteq P(\mathcal{B})$

(ii) $\Delta/\overline{\alpha}^n$ is not an absorber of $\underline{B}$

(iii) $\overline{B} = \underline{B}$.

Remark 4.3.4. For reduced flexible systems, if a given system is upper homogeneous, the condition $R(A) \subseteq P(\mathcal{B})$ implies $\dfrac{\overline{A}}{\Delta} \cdot \overline{B} \subseteq \underline{B}$. As a consequence, in the case of reduced upper homogeneous non-singular flexible system the conditions that $\overline{B} = \underline{B} = B$ and that Δ is not an absorber of B imply that $R(\mathcal{A}) \subseteq P(\mathcal{B})$. Also, the relative uncertainty of $\mathcal{A}$ becomes $R(\mathcal{A}) = \dfrac{\overline{A}}{\Delta}$, since $\overline{\alpha} = 1 + A$ with $A \subseteq \oslash$.

It is easier to work with reduced flexible systems. The following theorem says that every flexible system is equivalent to a reduced system and every Cramer condition satisfied by the original system is also satisfied by the reduced system.

Theorem 4.3.5. *Let $n \in \mathbb{N}$ be standard, $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a non-singular matrix over $\mathbb{E}$, $\overline{a} \in \overline{\alpha}$ and $\mathcal{B} = (\beta_1 \dots \beta_n)^T \in \mathbb{E}^n$ be a column vector. Let $\mathcal{A}' = [\alpha'_{ij}]_{n\times n}$ where $\alpha'_{ij} = \dfrac{\alpha_{ij}}{\overline{a}}$ for all $1 \le i, j \le n$ and $\mathcal{B}' = (\beta_1, \dots, \beta'_m)^T$ where $\beta'_i = \dfrac{\beta_i}{\overline{a}}$ for $1 \le i \le n$. Consider the two following flexible systems*

$$\mathcal{A}\xi \subseteq \mathcal{B} \tag{4.3}$$

and

$$\mathcal{A}'\xi \subseteq \mathcal{B}'. \tag{4.4}$$

The following statements hold:

(i) The flexible system (4.4) *is reduced.*

(ii) The two flexible systems above are equivalent, that is the set of solutions of the two systems are the same.

(iii) If a Cramer condition is satisfied by system (4.3)*, it is also satisfied by system* (4.4).

To prove this theorem we need some lemmas.

Lemma 4.3.6. *Let $n \in \mathbb{N}$ be standard, $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a non-singular matrix over $\mathbb{E}$, $\overline{a} \in \overline{\alpha}$ and $\mathcal{B} = (\beta_1, \dots, \beta_n)^T$ be a column vector, where $\beta_i \in \mathbb{E}$ for all $1 \le i \le n$. Let $\mathcal{A}' = [\alpha'_{ij}]_{n\times n}$ where $\alpha'_{ij} = \dfrac{\alpha_{ij}}{\overline{a}}$ for all $1 \le i, j \le n$ and $\mathcal{B}' = (\beta_1, \dots, \beta'_m)^T$ where $\beta'_i = \dfrac{\beta_i}{\overline{a}}$ for $1 \le i \le n$. One has* *(i)* $R(\mathcal{A}) = R(\mathcal{A}')$,

(ii) $P(\mathcal{B}) = P(\mathcal{B}')$.

Proof. (i) Because $\overline{\alpha}$ is zeroless, $\overline{\alpha'} = \dfrac{\overline{\alpha}}{\overline{a}}$ is zeroless. For $\overline{a}' \in \overline{\alpha}'$, there exists $\overline{c} \in \overline{\alpha}$ such that $\overline{a}' = \dfrac{\overline{c}}{\overline{a}}$. By Lemma 2.2.17(i), one has

$$R(\mathcal{A}') = \frac{\overline{A'}\,\overline{a'}^{n-1}}{\Delta'} = \frac{\left(\overline{A}/\overline{a}\right)\cdot\left(\overline{c}^{n-1}/\overline{a}^{n-1}\right)}{\Delta/\overline{a}^{n}} = \frac{\overline{A}.\overline{c}^{n-1}}{\Delta} = \frac{\overline{A}.\overline{\alpha}^{n-1}}{\Delta} = R(\mathcal{A}).$$

(ii) We consider two cases.

Case 1: $\overline{\beta}$ is zeroless. Then

$$P(\mathcal{B}') = \underline{B'}/\overline{\beta'} = \left(\underline{B}/\overline{a}\right)/\left(\overline{\beta}/\overline{a}\right) = \underline{B}/\overline{\beta} = P(\mathcal{B}).$$

Case 2: $\overline{\beta}$ is neutricial. Then

$$P(\mathcal{B}') = \left(\underline{B}/\overline{a}\right) : \left(\overline{B}/\overline{a}\right) = P(\mathcal{B}).$$

□

Lemma 4.3.7. *Let $n \in \mathbb{N}$ be standard, $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a non-singular matrix over $\mathbb{E}$, $\overline{a} \in \overline{\alpha}$ and $\mathcal{B} = (\beta_1 \cdots \beta_n)^T \in \mathbb{E}^n$ be a column vector. Let $\mathcal{A}' = [\alpha'_{ij}]_{n\times n}$, where $\alpha'_{ij} = \dfrac{\alpha_{ij}}{\overline{a}}$ for all $1 \le i, j \le n$ and $\mathcal{B}' = (\beta_1, \cdots, \beta'_m)^T$, where $\beta'_i = \dfrac{\beta_i}{\overline{a}}$ for $1 \le i \le n$. We denote $\Delta = \det(\mathcal{A}), \Delta' = \det(\mathcal{A}')$.*

Then

(i) $\Delta/\overline{\alpha}^n$ *is not an absorber of* $\underline{B}$ *implies that* $\Delta' = \det(\mathcal{A}')$ *is not an absorber of* $\underline{B}'$.

(ii) $\underline{B} = \overline{B}$ *if and only if* $\underline{B'} = \overline{B'}$.

Proof. (i) Because $\Delta/\overline{\alpha}^n$ is not an absorber of $\underline{B}$, also $\Delta/\overline{a}^n$ is not an absorber of $\underline{B}$. This means that $B \subseteq (\Delta/\overline{a}^n)\cdot B$. On the other hand, $\underline{B'} = \underline{B}/\overline{a} \subseteq (\Delta/\overline{a}^n)\cdot(\underline{B}/\overline{a}) = \Delta' \cdot \underline{B'}$. Hence Δ' is not an absorber of $\underline{B}'$.

(ii) This follows from the facts that $\overline{B}' = \dfrac{\overline{B}}{\overline{a}}$ and $\underline{B}' = \dfrac{\underline{B}}{\overline{a}}$. □

Proof of Theorem 4.3.5. (i) We have $\overline{\alpha}' = \overline{\alpha}/\overline{a} = 1 + A \subseteq 1 + \oslash$. The fact $|\alpha_{ij}| \le |\overline{\alpha}|$ for all $1 \le i, j \le n$ implies $\left|\alpha'_{ij}\right| = \left|\dfrac{\alpha_{ij}}{\overline{a}}\right| \le \left|\dfrac{\overline{\alpha}}{\overline{a}}\right| \le 1 + \oslash$ for all $1 \le i, j \le n$. Hence the system (4.4) is reduced.

(ii) Note that $\overline{\alpha}$ is zeroless, so $\overline{a} \neq 0$. A vector $\xi = (\xi_1, \ldots, \xi_n)^T$ is a solution of the system (4.3) if and only if $\sum_{j=1}^{n} \alpha_{ij}\xi_j \subseteq \beta_i$ for all $1 \le i \le n$, hence also $\left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right)/\overline{a} \subseteq \beta_i/\overline{a}$, for all $1 \le i \le n$. The latter is equivalent to $\sum_{j=1}^{n} (\alpha_{ij}/\overline{a})\,\xi_j \subseteq \beta_i/\overline{a}$, for all $1 \leq i \leq n$. Once again, these inclusions hold if and only if ξ is a solution of the system (4.4).

(iii) This follows by Lemmas 4.3.6 and 4.3.7. □

We show below that the Cramer conditions are sufficient to guarantee that Cramer's rule can be applied to non-singular flexible systems. This result is a generalization of Theorem 4.4 in [20, p.19] on non-homogeneous, non-singular flexible systems.

Theorem 4.3.8. *Assume that the flexible system* (4.2) *is non-singular. The following holds.*

(i) If $R(\mathcal{A}) \subseteq P(\mathcal{B})$ then

$$\xi = \left(\frac{detM_1(b)}{d}, \ldots, \frac{detM_n(b)}{d}\right) \tag{4.5}$$

is an admissible solution of the flexible system (4.2)*.*

(ii) If $R(\mathcal{A}) \subseteq P(\mathcal{B})$ and $\Delta/\overline{\alpha}^n$ is not an absorber of $\underline{B}$ then

$$\xi = \left(\frac{detM_1(b)}{\Delta}, \ldots, \frac{detM_n(b)}{\Delta}\right) \tag{4.6}$$

is an admissible solution of the flexible system (4.2)*.*

(iii) If $R(\mathcal{A}) \subseteq P(\mathcal{B})$, $\Delta/\overline{\alpha}^n$ is not an absorber of $\underline{B}$ and $\overline{B} = \underline{B}$ then

$$\xi = \left(\frac{detM_1}{\Delta}, \ldots, \frac{detM_n}{\Delta}\right) \tag{4.7}$$

is the maximal solution of the flexible system (4.2)*.*

Note that

$$\left(\frac{\mathrm{det}M_1(b)}{d}, \ldots, \frac{\mathrm{det}M_n(b)}{d}\right) \in \left(\frac{\mathrm{det}M_1(b)}{\Delta}, \ldots, \frac{\mathrm{det}M_n(b)}{\Delta}\right) \subseteq \left(\frac{\mathrm{det}M_1}{\Delta}, \ldots, \frac{\mathrm{det}M_n}{\Delta}\right). \tag{4.8}$$

Due to this fact, these vectors have at least one common representative vector $x = (x_1, \ldots, x_n)$ which is a solution of a linear system

$$\begin{cases} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1 \\ \vdots & \ddots & \vdots & & \vdots \\ a_{n1}x_1 & +\cdots & +a_{nn}x_n & = & b_n, \end{cases}$$

with $a_{ij} \in \alpha_{ij}$ and $b_i \in \beta_i$ for all $1 \leq i, j \leq n$.

The condition $\Delta/\overline{\alpha}^n$ being not too small to become an absorber of $\underline{B}$ can be seen as a generalization of the condition in classical linear algebra of the determinant of a non-singular system of linear equations being different from zero.

Definition 4.3.9 ([20])**.** The column vector $\xi = \left(\frac{\det M_1}{\Delta}, \ldots, \frac{\det M_n}{\Delta}\right)^T$ is called the *Cramer-solution*.

To prove Theorem 4.3.8 we need some auxiliary results.

Lemma 4.3.10. *Consider the system* (4.2)*. Assume that it is reduced, non-singular and upper homogeneous. Then*

(i) $|\det(M_j)| \subseteq \overline{B}$*, in particular* $N\big(\det(M_j)\big) \subseteq \overline{B}$.

(ii) $N\Big(\det\big(M_j(b)\big)\Big) \subseteq \overline{A} \cdot \overline{B}$.

Proof. Let S_n be the set of all permutations of $\{1, \ldots, n\}$ and $\sigma \in S_n$. Put

$$\gamma_\sigma = \alpha_{\sigma(1)1} \cdots \alpha_{\sigma(j-1)(j-1)}\alpha_{\sigma(j+1)(j+1)} \cdots \alpha_{\sigma(n)n}.$$

Because the system is reduced, $|\alpha_{ij}| \leq |\overline{\alpha}| = 1 + A \subseteq 1 + \oslash$ and $\overline{A} \subseteq \oslash$. So

$$|\gamma_\sigma| \leq \overline{\alpha}^{n-1} \leq (1 + \oslash)^{n-1} = 1 + \oslash. \tag{4.9}$$

Moreover,

$$|\det(M_j)| = \left|\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\gamma_\sigma\beta_{\sigma(j)}\right| \leq \sum_{\sigma \in S_n} \left|\gamma_\sigma\beta_{\sigma(j)}\right|.$$

(i) The system is upper homogeneous, so $\beta_i \subseteq \overline{B}$ by Remark 4.2.4. Formula (4.9) implies

$$|\det(M_j)| \leq \sum_{\sigma \in S_n} \left|\gamma_\sigma\beta_{\sigma(j)}\right| \leq \sum_{\sigma \in S_n} \left|(1 + \oslash)\overline{B}\right| = n!(1 + \oslash)\overline{B} = \overline{B}.$$

Hence $|\det(M_j)| \subseteq \overline{B}$.

(ii) By Lemma 2.2.17(vi) and the definition of γ_σ, one has $N(\gamma_\sigma) \subseteq N(1 + \overline{A})^{n-1} = \overline{A}$. Furthermore the system is upper homogeneous, so $b_i \subseteq \overline{B}, i = 1, \ldots, n$. As a result, for $1 \leq i \leq n$,

$$\begin{aligned} N\Big(\det\big(M_j(b)\big)\Big) &= N\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma b_{\sigma(j)}\right) = \sum_{\sigma \in S_n} N\left(\gamma_\sigma b_{\sigma(j)}\right) \\ &= \sum_{\sigma \in S_n} b_{\sigma(j)} N(\gamma_\sigma) \subseteq n!\overline{B} \cdot \overline{A} \subseteq \overline{B} \cdot \overline{A}. \end{aligned}$$

□

Lemma 4.3.11. *Assume that the system* (4.2) *is reduced, non-singular and upper homogeneous, and satisfies the condition* $R(\mathcal{A}) \subseteq P(\mathcal{B})$*. Let* $\Delta = \det(\mathcal{A}) = d + D$ *and* $\xi = (\xi_1, \ldots, \xi_n)^T$ *be an admissible solution, where* $\xi_j = x_i + X_j \in \mathbb{E}$ *for all* $j \in \{1, \ldots, n\}$*. Let* $\overline{x} = \max_{1 \leq j \leq n} |x_j|$*. Then*

(i) $\overline{A} \cdot \overline{x} \subseteq \big(\overline{A}/\Delta\big) \cdot \overline{B}$*, hence* $\overline{A}\,\overline{x} \subseteq \underline{B}$.

(ii) If $N(\xi_j) \subseteq \underline{B}$ for $j \in \{1, \ldots, n\}$, then $N\left(\sum\limits_{j=1}^{n} \alpha_{ij}\xi_j\right) \subseteq N(\beta_i)$, *for all* $i \in \{1, \ldots, n\}$.

Proof. (i) Because $\mathcal{A}$ is non-singular, the determinant Δ is zeroless. In particular $d \neq 0$. By Cramer's rule, the column vector $x = \left(\frac{\det\big(M_1(a,b)\big)}{d}, \ldots, \frac{\det\big(M_n(a,b)\big)}{d}\right)^T$ is the unique solution of the classical linear system $\hat{\mathcal{A}}X = b$, where $\hat{\mathcal{A}} = [a_{ij}]_{n\times n}$ is a representative of $\mathcal{A}$, $X = [x_i]_{n\times 1}$ is the variable column and $b = [b_i]_{n\times 1}$ is the constant term vector. It follows that $\overline{x} = \left|\frac{\det\big(M_k(a,b)\big)}{d}\right|$ for some $k \in \{1, \ldots, n\}$. By Lemma 4.3.10i, $\det\big(M_k(a,b)\big) \in \det M_j \subseteq \overline{B}$. So, condition $R(\mathcal{A}) \subseteq P(\mathcal{B})$ implies

$$\overline{A} \cdot \overline{x} = \overline{A}\left|\frac{\det\big(M_k(a,b)\big)}{d}\right| \subseteq \frac{\overline{A}}{d} \cdot \overline{B} = \frac{\overline{A}}{\Delta} \cdot \overline{B} \subseteq \underline{B}.$$

(ii) One has

$$N\left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right) = \sum_{j=1}^{n} \left(N\left(\alpha_{ij}\right)\xi_j + \alpha_{ij}N(\xi_j)\right). \tag{4.10}$$

Let $\overline{\xi} = \max\limits_{1\leq j\leq n} |\xi_j|$. One considers two cases.

Case 1: $\overline{\xi} = \xi_k$ is a neutrix with some $k \in \{1, \ldots, n\}$. We have $\xi_k = N(\xi_k)$. If $N(\xi_j) \subseteq \underline{B}$ for all $j \in \{1, \ldots, n\}$ one derives $\overline{\xi} = \xi_k = N(\xi_k) \subseteq \underline{B}$. Because $|\xi_i| \leq \overline{\xi} \subseteq \underline{B}$, one has $\xi_j \subseteq \underline{B}$ for all $1 \leq j \leq n$. It follows from (4.10) and the fact $N(\alpha_{ij}) \subseteq \overline{A} \subseteq \oslash$ that

$$N\Big(\sum_{j=1}^{n} \alpha_{ij}\xi_j\Big) \subseteq \sum_{j=1}^{n} \Big(\underline{B}\overline{A} + \overline{a}\underline{B}\Big) = \underline{B} \subseteq N(\beta_i).$$

Case 2: $\overline{\xi} = \max\limits_{1\leq j\leq n} |\xi_j|$ is zeroless. Then by (4.10) and Part (i), we have

$$N\Big(\sum_{j=1}^{n} \alpha_{ij}\xi_j\Big) \subseteq \sum_{j=1}^{n} \Big(\overline{x}\overline{A} + \overline{a}\underline{B}\Big) = n(\underline{B} + \overline{x}\,\overline{A}) \subseteq \underline{B} + \underline{B} = \underline{B} \subseteq N(\beta_j).$$

□

Note that

(i) Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{E}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a representative of ξ. Then

$$\sum_{j=1}^{n} \alpha_{ij}\xi_j = \sum_{j=1}^{n} \alpha_{ij}x_j + \sum_{j=1}^{n} \alpha_{ij}N(\xi_i). \tag{4.11}$$

So the vector $\xi = (\xi_1, \ldots, \xi_n)^T$ is a solution of the system (4.2) if and only if

$$\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n \subseteq \beta_i \quad \text{for all} \quad 1 \leq i \leq n, \tag{4.12}$$

and

$$\sum_{j=1}^{n} \alpha_{ij} N(\xi_j) \subseteq N(\beta_i) \quad \text{for all} \quad 1 \leq i \leq n. \tag{4.13}$$

(ii) If $\mathcal{A}$ is a non-singular matrix, then $\Delta = \det(A) \equiv d + D$ with $d \neq 0$ and $\dfrac{1}{\Delta} = \dfrac{1}{d} + \dfrac{D}{d^2}$ by Lemma 2.2.20(i). Hence

$$N(1/\Delta) = D/d^2 = D/\Delta^2. \tag{4.14}$$

Proof of Theorem 4.3.8. For non-homogeneous non-singular flexible systems, we refer to the proof in [19]. Now we suppose that the system is upper homogeneous. We consider two cases.

Case 1: the system is reduced. By formula (4.8), the vector $x = (x_1, \ldots, x_n)^T$, where $x_i = \dfrac{\det\big(M_i(a,b)\big)}{d}$ for $1 \leq i \leq n$, is a representative of all the three vectors above. Note that $x = (x_1, \ldots, x_n)^T$ is a solution of the system $\sum\limits_{j=1}^{n} a_{ij}x_j = b_i$, where $a_{ij} \in \alpha_{ij}$ for $1 \leq i, j \leq n$, by Cramer's rule. By Lemma 4.3.11(i) one has

$$\begin{aligned}
\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n &= (a_{i1} + A_{i1})x_1 + \cdots + (a_{in} + A_{in})x_n \\
&= (a_{i1}x_1 + \cdots + a_{in}x_n) + (A_{i1}x_1 + \cdots + A_{in}x_n) \\
&\subseteq b_i + \overline{A}\overline{x} \subseteq b_i + \underline{B} \subseteq b_i + B_i = \beta_i.
\end{aligned}$$

So formula (4.12) is satisfied. To complete the proof of this case we will verify the condition (4.13).

(i) Assume that $R(\mathcal{A}) \subseteq P(\mathcal{B})$. Because the system is reduced and upper homogeneous, this condition becomes $\dfrac{\overline{A}}{\Delta}\overline{B} \subseteq \underline{B}$. By Lemma 4.3.10(ii),

$$N\left(\frac{\det\big(M_j(b)\big)}{d}\right) = \frac{1}{d} N\left(\det\big(M_j(b)\big)\right) \subseteq \frac{\overline{B} \cdot \overline{A}}{d} = \big(\overline{A}/\Delta\big) \cdot \overline{B} \subseteq \underline{B}. \tag{4.15}$$

As a consequence, $N(\xi_j) = N\left(\dfrac{\det\big(M_j(b)\big)}{d}\right) \subseteq \underline{B}$ for all $j = \{1, \ldots, n\}$. By Lemma 4.3.11(ii), $\sum\limits_{i=1}^{n} \alpha_{ij} N(\xi_i) \subseteq N\left(\sum\limits_{j=1}^{n} \alpha_{ij}\xi_j\right) \subseteq N(\beta_i)$ for all $1 \leq i \leq n$, so formula (4.13) is satisfied. Hence $\xi = \left(\dfrac{\det\big(M_1(b)\big)}{d}, \ldots, \dfrac{\det\big(M_n(b)\big)}{d}\right)^T$ is a solution of the non-singular and upper homogeneous system (4.2).

(ii) Because Δ is not an absorber of $\underline{B}$, one has $\underline{B} \subseteq \Delta\underline{B}$ and therefore

$$\underline{B}/\Delta \subseteq \underline{B}. \tag{4.16}$$

Also by formula (4.14), for all $j \in \{1, \ldots, n\}$ it holds that

$$N(\xi_j) = N\left(\frac{\det\big(M_j(b)\big)}{\Delta}\right) = \frac{1}{\Delta}N\left(\det M_j(b)\right) + \det\big(M_j(b)\big) \cdot N\left(\frac{1}{\Delta}\right)$$
$$= \frac{1}{d}N\left(\det\big(M_j(b)\big)\right) + \det\big(M_j(b)\big) \cdot \frac{D}{\Delta^2} = \frac{N\left(\det\big(M_j(b)\big)\right)}{d} + \frac{\det\big(M_j(b)\big)}{\Delta} \cdot \frac{D}{\Delta}. \tag{4.17}$$

On the other hand, $R(\mathcal{A}) \subseteq P(\mathcal{B})$, so $\dfrac{\overline{A}}{d}\overline{B} = \dfrac{\overline{A}}{\Delta}\overline{B} \subseteq \underline{B}$. From formula (4.17), Lemma 4.3.10 and Lemma 3.2.4 we obtain

$$N(\xi_j) \subseteq \frac{\overline{A} \cdot \overline{B}}{d} + \big(\overline{B}/\Delta\big) \cdot \big(\overline{A}/\Delta\big) \subseteq \underline{B} + \frac{1}{\Delta}\left(\frac{\overline{A}}{\Delta} \cdot \overline{B}\right) \subseteq \underline{B} + \underline{B}/\Delta \subseteq \underline{B} + \underline{B} = \underline{B}.$$

By Lemma 4.3.11(ii), one has $\sum\limits_{i=1}^{n} \alpha_{ij} N(\xi_i) \subseteq N\Big(\sum\limits_{i=1}^{n} \alpha_{ij}\xi_i\Big) \subseteq N(\beta_i), \quad$ for all $\quad 1 \leq i \leq n.$

Hence $\xi = \left(\dfrac{\det\big(M_1(b)\big)}{\Delta}, \ldots, \dfrac{\det\big(M_n(b)\big)}{\Delta}\right)^T$ is a solution of the non-singular and upper homogeneous system (4.2).

(iii) Furthermore, if $\overline{B} = \underline{B} = B$ then by Lemma 4.3.10(i) and Lemma 3.2.4,

$$N(\xi_j) = N\left(\frac{\det(M_j)}{\Delta}\right) = \frac{1}{\Delta}N\left(\det(M_j)\right) + \left(\det(M_j)\right) \cdot N\left(\frac{1}{\Delta}\right) \subseteq \frac{1}{\Delta}B + B\frac{D}{\Delta^2} \subseteq B.$$

Then Lemma 4.3.11(ii) yields $\sum\limits_{j=1}^{n} \alpha_{ij} N(\xi_j) \subseteq N\Big(\sum\limits_{i=1}^{n} \alpha_{ij}\xi_i\Big) \subseteq N(\beta_i), \quad$ for all $\quad 1 \leq i \leq n$. Hence the column vector $\xi = \left(\dfrac{\det(M_1)}{\Delta}, \ldots, \dfrac{\det(M_n)}{\Delta}\right)^T$ is a solution of the flexible system $A\xi \subseteq B$.

Finally we will show that ξ defined as above is the maximum solution. Let $\zeta = (\zeta_1, \ldots, \zeta_n)^T$ be any solution of the given system and choose $y_j \in \zeta_j$ for all $j = 1, \ldots, n$. Then for every choice of representatives $a_{ij} \in \alpha_{ij}, 1 \leq i, j \leq n$ there exist $b_1 \in \beta_1, \ldots, b_n \in \beta_n$ such that

$$\begin{cases} a_{11}y_1 + & a_{12}y_2 + & \cdots & + a_{1n}y_n & = b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}y_1 + & a_{n2}y_2 + & \cdots & + a_{nn}y_n & = b_n. \end{cases}$$

Put

$$d = \det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

Then $y_j = \dfrac{M_j(a,b)}{d} \in \dfrac{\det(M_j)}{\Delta}$ for $1 \leq j \leq n$ and so $\zeta_j \subseteq \dfrac{\det(M_j)}{\Delta}$. Hence $\xi = \left(\dfrac{\det(M_1)}{\Delta}, \ldots, \dfrac{\det(M_n)}{\Delta}\right)^T$ is the maximal solution.

Case 2: the system is not reduced. By Theorem 4.3.5, the given system is equivalent to the reduced system

$$\begin{cases} \alpha'_{11}\xi_1 + & \alpha'_{12}\xi_2 + & \cdots & +\alpha'_{1n}\xi_n & \subseteq b_1 + B'_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha'_{n1}\xi_1 + & \alpha'_{n2}\xi_2 + & \cdots & +\alpha'_{nn}\xi_n & \subseteq b_n + B'_n. \end{cases} \tag{4.18}$$

A short calculation shows that for all $1 \leq i \leq n$,

$$\frac{\det\big(M'_i(b)\big)}{d'} = \frac{\det\big(M_i(b)\big)}{d}, \frac{\det M'_i(b)}{\Delta'} = \frac{\det\big(M_i(b)\big)}{\Delta} \text{ and } \frac{\det\big(M'_i\big)}{\Delta'} = \frac{\det(M_i)}{d}, \tag{4.19}$$

where

$$M'_j = \begin{bmatrix} \alpha'_{11} & \cdots & \alpha'_{1j-1} & \beta'_1 & \alpha'_{1(j+1)} & \cdots & \alpha'_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha'_{n1} & \cdots & \alpha'_{nj-1} & \beta'_n & \alpha'_{n(j+1)} & \cdots & \alpha'_{nn}, \end{bmatrix}$$

$$M'_j(b) = \begin{bmatrix} \alpha'_{11} & \cdots & \alpha'_{1j-1} & b'_1 & \alpha'_{1(j+1)} & \cdots & \alpha'_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha'_{n1} & \cdots & \alpha'_{nj-1} & b'_n & \alpha'_{n(j+1)} & \cdots & \alpha'_{nn}, \end{bmatrix}$$

and d' is a representative of Δ'.

Also, if the condition in Part (i) is satisfied by the given system, by Theorem 4.3.5 it is also satisfied by the system (4.18). By formula (4.19) and the conclusion in Case 1, the vector $\xi = (\xi_1, \ldots, \xi_n)$ defined by (4.5) is an admissible solution of the system (4.18) and hence it is an admissible solution of the system (4.2).

With analogous arguments the second and the last part can be proved. □

The following result provides another condition to guarantee that there exist real admissible solutions by Cramer's rule for non-singular flexible systems.

Theorem 4.3.12. *Consider the following non-singular reduced flexible system*

$$\begin{cases} \alpha_{11}x_1 + & \cdots & +\alpha_{1n}x_n & \subseteq & b_1 + B_1 \\ \vdots & \ddots & \vdots & & \vdots \\ \alpha_{n1}x_1 & +\cdots & +\alpha_{nn}x_n & \subseteq & b_n + B_n. \end{cases} \tag{4.20}$$

Let $\Delta = \det(\mathcal{A}) = d + D$, *with* $\mathcal{A} = [\alpha_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{E})$. *If* $\overline{A}/\Delta \subseteq \underline{B}$ *then* $u = (u_1, \ldots, u_n)$, *where* $u_j = \dfrac{\det\big(M_j(a,b)\big)}{d}$ *for* $1 \leq j \leq n$, *is a real admissible solution of the system.*

For reduced systems, condition $R(\mathcal{A}) = \overline{A}/\Delta \subseteq \underline{B}$ is weaker than condition $R(\mathcal{A}) \subseteq P(B)$ if $|\overline{\beta}| \in \not\infty$. Also if $\dfrac{\overline{A}}{\Delta} \subseteq P(\mathcal{B}) = \underline{B}/\overline{\beta}$, the point x defined above is a solution of the system. Combining this fact with the result in Theorem 4.3.12 we obtain the following.

Corollary 4.3.13. *Consider the non-singular reduced flexible system* (4.20). *If* $\dfrac{\overline{A}}{\Delta} \subseteq \max\left\{\underline{B}, \underline{B}/\overline{\beta}\right\}$ *then* $u = (u_1, \dots, u_n)$, *where* $u_j = \dfrac{\det\bigl(M_j(a,b)\bigr)}{d}$ *for* $1 \leq j \leq n$, *is a real solution of the system* (4.20).

Proof of Theorem 4.3.12. It is clear that $u = (u_1, \dots, u_n)$, where $u_j = \dfrac{\det\bigl(M_j(a,b)\bigr)}{d}$ for $1 \leq j \leq n$, is a solution of the system

$$\begin{cases} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1 \\ \vdots & \ddots & \vdots & & \vdots \\ a_{n1}x_1 & + \cdots & +a_{nn}x_n & = & b_n \end{cases} \tag{4.21}$$

by Cramer's rule. So

$$\begin{cases} a_{11}u_1 + & \cdots & +a_{1n}u_n & \subseteq & b_1 + B_1 \\ \vdots & \ddots & \vdots & & \vdots \\ a_{n1}u_1 & + \cdots & +a_{nn}u_n & \subseteq & b_n + B_n. \end{cases} \tag{4.22}$$

Also, by Proposition 3.2.3, one obtains that $u_i \cdot A_{ij} \subseteq \dfrac{\det\bigl(M_j(a,b)\bigr)}{d} \cdot \overline{A} \subseteq \pounds\underline{B} \subseteq B_i$ for all $1 \leq i \leq n$. This implies that

$$\sum_{j=1}^{n} \alpha_{ij}x_j = \sum_{j=1} a_{ij}x_j + A_{ij}x_j \subseteq b_i + B_i + \sum_{j=1}^{n} B_i = b_i + B_i, \quad \text{for all} \quad 1 \leq i \leq n.$$

Thus x is a solution of the system (4.20). ☐

The result below shows that if a homogeneous flexible system has the constant term vector with identical components, the Cramer-solution is equal to the neutrix vector. This is a generalization of the result in classical algebra which says that the zero vector is the unique solution of a non-singular homogeneous linear system.

Theorem 4.3.14. *Consider a homogeneous non-singular and flexible system* (4.2). *Assume also that the system satisfies all the Cramer conditions. Then the vector* $\Bigl(B, \dots, B\Bigr)^T$ *is the Cramer-solution of the system* (4.2).

To prove this result, we need the following.

Lemma 4.3.15. *Suppose that the flexible system* (4.2) *is non-singular and satisfies all the Cramer conditions. Then for all* $j \in \{1, \dots, n\}$,

$$N\left(\frac{\det(M_j)}{\Delta}\right) = B.$$

In addition, if the system is homogeneous, for all $1 \leq j \leq n$,

$$B = \frac{\det(M_j)}{\Delta}. \tag{4.23}$$

Proof. For the case of non-homogeneous systems, we refer to the proof in [20, p.78]. We now suppose that the system is homogeneous. By Theorem 4.3.5, we can also assume that the system is reduced. Let $j \in \{1, \dots, n\}$

be arbitrary. Because $\overline{B} = \underline{B} = B$ and by Lemma 4.3.10,

$$\det(M_j) \subseteq B. \tag{4.24}$$

On the other hand $|M_{ij}| > \oslash\Delta$ for some $i \in \{1, \ldots, n\}$ by Proposition 3.2.2, so there exists $t > \oslash$ such that $|M_{ij}| = |t\Delta|$. Moreover, Δ is not an absorber of B, hence $B \subseteq B\Delta \subseteq tB\Delta = BM_{ij}$ for some $i \in \{1, \ldots, n\}$. By Proposition 3.2.1 we obtain

$$\begin{aligned} B &\subseteq (-1)^{j+1} BM_{1,j} + \cdots + (-1)^{j+n} BM_{n,j} \\ &\subseteq \det \begin{bmatrix} 1 + A_{11} & \cdots & \alpha_{1(j-1)} & B & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & B & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix} \\ &= \det(M_j). \end{aligned}$$

Thus $B = \det(M_j)$. By Proposition 2.2.26,

$$B = \frac{B}{\Delta} = \frac{\det(M_j)}{\Delta}.$$

Hence $B = N\left(\dfrac{\det(M_j)}{\Delta}\right)$ for $1 \leq j \leq n$. □

Proof of Theorem 4.3.14. Because the system satisfies all the Cramer conditions, by Theorem 4.3.8, the vector $\xi = (\xi_1, \ldots, \xi_n)^T$ with $\xi_i = \dfrac{\det(M_i)}{\Delta}, 1 \leq i \leq n$ is the Cramer-solution of the system. Moreover, the system is homogeneous, so by Lemma 4.3.15 we have $\dfrac{\det(M_i)}{\Delta} = B$ for all $1 \leq i \leq n$. Hence the vector $\xi = (B, \ldots, B)^T$ is the Cramer-solution of the system (4.2). □

Remark 4.3.16. Assume that the flexible system (4.2) satisfies the Cramer conditions. By Lemma 4.3.15, the Cramer-solution of the system is of the form $\xi_i = x_i + B$, for all $i \in \{1, \ldots, n\}$, where $x = (x_1, \ldots, x_n)$ is a solution of the linear system

$$\begin{cases} a_{11}y_1 + & a_{12}y_2 + & \cdots & + a_{1n}y_n & = b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}y_1 + & a_{n2}y_2 + & \cdots & + a_{nn}y_n & = b_n, \end{cases}$$

where $a_{ij} \in \alpha_{ij}, b_i \in \beta_i$ for all $i, j \in \{1, \ldots, n\}$.

4.3.2 Some examples

The following example illustrates the conditions of Part (ii) of Theorem 4.3.8 for an upper homogeneous flexible system. It also shows that the Cramer's rule does not fully holds.

Example 4.3.17. Let $\epsilon > 0$ be infinitesimal. Consider the following upper homogeneous flexible system

$$\begin{cases} (1+\epsilon^2\oslash)\xi_1 & + & \epsilon£\xi_2 & \subseteq & \oslash \\ (2+\epsilon\oslash)\xi_1 & + & (1+\epsilon\oslash)\xi_2 & \subseteq & \epsilon+\epsilon\oslash\,. \end{cases}$$

One has $\Delta = \det\begin{bmatrix} 1+\epsilon^2\oslash & \epsilon£ \\ 2+\epsilon\oslash & 1+\epsilon\oslash \end{bmatrix} = 1+\epsilon£$ is zeroless, so the system is non-singular. Moreover $\underline{B} = \epsilon\oslash$ and $\Delta = 1+\epsilon£ \in @$, hence Δ is not an absorber of $\underline{B}$. Also $\overline{B} = \oslash$ implies $P(\mathcal{B}) = \underline{B} : \overline{B} = \epsilon\oslash : \oslash = \epsilon£$. In addition $\overline{A} = \epsilon£$ and $\overline{\alpha} = 2+\epsilon\oslash$, $n = 2$, so $R(\mathcal{A}) = \dfrac{\overline{A}\cdot\overline{\alpha}^{n-1}}{\Delta} = \epsilon£$. It follows that $R(\mathcal{A}) = P(\mathcal{B}) = \epsilon£$.

So the conditions of Theorem 4.3.8(ii) are satisfied by the system. For $\epsilon_1, \epsilon_2 \in \oslash$, let

$$\begin{aligned} \det\bigl(M_1(b)\bigr) &= \det\begin{bmatrix} \epsilon_1 & \epsilon£ \\ \epsilon+\epsilon\epsilon_2 & 1+\epsilon\oslash \end{bmatrix} &&= \epsilon_1+\epsilon(\epsilon_1\oslash+\epsilon£) \\ \det\bigl(M_2(b)\bigr) &= \det\begin{bmatrix} 1+\epsilon^2\oslash & \epsilon_1 \\ 2+\epsilon\oslash & \epsilon+\epsilon\epsilon_2 \end{bmatrix} &&= \epsilon(1+\epsilon_2)-2\epsilon_1+\epsilon(\epsilon^2\oslash+\epsilon_1\oslash). \end{aligned}$$

By Theorem 4.3.8 we conclude that the vector $\xi = (\xi_1, \xi_2) \in \mathbb{E}^2$ given by

$$\begin{aligned} \xi_1 &= \frac{\det\bigl(M_1(b)\bigr)}{\Delta} = \frac{\epsilon_1+\epsilon(\epsilon_1\oslash+\epsilon£)}{1+\epsilon£} &&= \epsilon_1+\epsilon(£\epsilon_1+£\epsilon) \\ \xi_2 &= \frac{\det\bigl(M_2(b)\bigr)}{\Delta} = \frac{\epsilon(1+\epsilon_2)-2\epsilon_1+\epsilon(\epsilon\oslash+\epsilon_1\oslash)}{1+\epsilon£} &&= \epsilon(1+\epsilon_2)-2\epsilon_1+\epsilon(\epsilon\oslash+\epsilon_1\oslash)+\epsilon^2£+\epsilon_1\epsilon£. \end{aligned}$$

is an admissible solution of the given system.

However, the vector $\eta = (\eta_1, \eta_2)$ with $\begin{cases} \eta_1 = \dfrac{\det(M_1)}{\Delta} = \oslash \\ \eta_2 = \dfrac{\det(M_2)}{\Delta} = \oslash \end{cases}$ is not a solution of the system. Indeed, one has $(2+\epsilon\oslash)\oslash+(1+\epsilon\oslash)\oslash = \oslash \not\subseteq \epsilon+\epsilon\oslash$, so it does not satisfy the second equation of the system.

Te following example deal with an homogeneous flexible system which satisfies only the conditions of Theorem 4.3.8(ii).

Example 4.3.18. Let $\epsilon > 0$ be infinitesimal. Consider the homogeneous flexible system

$$\begin{cases} (1+\epsilon\oslash)\xi_1 & + & (\epsilon+\epsilon^2\oslash)\xi_2 & \subseteq \epsilon^2\oslash \\ \epsilon\oslash\xi_1 & + & (1+\epsilon^2£)\xi_2 & \subseteq \epsilon£. \end{cases}$$

Because $\Delta = \det\begin{bmatrix} 1+\epsilon\oslash & \epsilon+\epsilon^2\oslash \\ \epsilon\oslash & 1+\epsilon^2£ \end{bmatrix} = 1+\epsilon\oslash$ is zeroless, the system is non-singular. Moreover, $\Delta \in @$ and $\underline{B} = \epsilon^2\oslash$, so Δ is not an absorber of $\underline{B}$. Furthermore,

$$R(\mathcal{A}) = \frac{\overline{A}.\overline{\alpha}^{n-1}}{\Delta} = \frac{\epsilon\oslash\cdot(1+\epsilon\oslash)}{1+\epsilon\oslash} = \epsilon\oslash,$$

and

$$P(\mathcal{B}) = \underline{B} : \overline{B} = \epsilon^2 \oslash : \epsilon £ = \epsilon \oslash .$$

Hence

$$R(\mathcal{A}) = P(B).$$

So the system just satisfies conditions of Theorem 4.3.8(ii) and does not satisfy those of Theorem 4.3.8(iii) since $\underline{B} \neq \overline{B}$. For $b_1 \in \epsilon^2 \oslash, b_2 \in \epsilon £$, let

$$\begin{aligned} \det\big(M_1(b)\big) &= \det \begin{bmatrix} b_1 & \epsilon + \epsilon^2 \oslash \\ b_2 & 1 + \epsilon^2 £ \end{bmatrix} &= b_1 - b_2\epsilon + b_1\epsilon^2 £ + b_2\epsilon^2 \oslash, \\ \det\big(M_2(b)\big) &= \det \begin{bmatrix} 1 + \epsilon \oslash & b_1 \\ \epsilon \oslash & b_2 \end{bmatrix} &= b_2 + b_1\epsilon \oslash + b_2\epsilon \oslash . \end{aligned}$$

Applying Theorem 4.3.8(ii) one concludes that the vector $\xi = (\xi_1, \xi_2)^T$ given by

$$\begin{aligned} \xi_1 &= \frac{\det\big(M_1(b)\big)}{\Delta} &= \frac{b_1 - b_2\epsilon + \epsilon b_1 £ + b_2\epsilon^2 \oslash}{1 + \epsilon \oslash} &= b_1 - b_2\epsilon + \epsilon b_1 £ + b_2\epsilon^2 \oslash, \\ \xi_2 &= \frac{\det\big(M_2(b)\big)}{\Delta} &= \frac{b_2 + \epsilon b_1 \oslash + b_2\epsilon \oslash}{1 + \epsilon \oslash} &= b_2 + \epsilon b_1 \oslash + b_2\epsilon \oslash, \end{aligned}$$

is an admissible solution of the given system.

However, note that $(\xi_1, \xi_2) = (\dfrac{\det(M_1)}{\Delta}, \dfrac{\det(M_2)}{\Delta})$, with

$$\begin{aligned} \det(M_1) &= \det \begin{bmatrix} \epsilon^2 \oslash & \epsilon + \epsilon^2 \oslash \\ \epsilon £ & 1 + \epsilon^2 £ \end{bmatrix} &= \epsilon^2 £ \\ \det(M_2) &= \det \begin{bmatrix} 1 + \epsilon \oslash & \epsilon^2 \oslash \\ \epsilon \oslash & \epsilon^2 £ \end{bmatrix} &= \epsilon^2 £ \end{aligned}$$

is not a valid solution. Indeed, we have

$$\begin{aligned} \xi_1 &= \frac{\det(M_1)}{\Delta} = \frac{\epsilon^2 £}{1 + \epsilon \oslash} = \epsilon^2 £ \\ \xi_2 &= \frac{\det(M_2)}{\Delta} = \frac{\epsilon^2 £}{1 + \epsilon \oslash} = \epsilon^2 £. \end{aligned}$$

Substituting it into the first equation of the system, we have

$$(1 + \epsilon \oslash)\epsilon^2 £ + (\epsilon + \epsilon^2 \oslash)\epsilon^2 £ = \epsilon^2 £ \supset \epsilon^2 \oslash .$$

Hence this vector does not satisfy the first equation.

Next we have a homogeneous flexible system satisfying all Cramer's conditions.

Example 4.3.19. Let $\epsilon > 0$ be infinitesimal. Consider the homogeneous flexible system

$$\begin{cases} (1+\epsilon\oslash)\xi_1 & + & (\epsilon+\epsilon^2\oslash)\xi_2 & \subseteq & \epsilon£ \\ \epsilon\oslash\xi_1 & + & (1+\epsilon^2£)\xi_2 & \subseteq & \epsilon£. \end{cases}$$

Because $\Delta = \det \begin{bmatrix} 1+\epsilon\oslash & \epsilon+\epsilon£ \\ \epsilon\oslash & 1+\epsilon^2£ \end{bmatrix} = 1+\epsilon\oslash$ is zeroless, the system is non-singular. Moreover $\Delta \in @$, so Δ is not an absorber of $\underline{B}$. Furthermore, $\overline{A} = \epsilon\oslash$, $\underline{B} = \overline{B} = \epsilon£$, so $P(\mathcal{B}) = \underline{B} : \overline{B} = £$ and $R(\mathcal{A}) = \dfrac{\overline{A}\cdot\overline{\alpha}^{n-1}}{\Delta} = \epsilon\oslash$; hence $R(\mathcal{A}) \subseteq P(B)$. Thus the system satisfies all the Cramer conditions. Let

$$\det(M_1) = \det \begin{bmatrix} \epsilon£ & \epsilon+\epsilon^2\oslash \\ \epsilon£ & 1+\epsilon^2£ \end{bmatrix} = \epsilon£$$
$$\det(M_2) = \det \begin{bmatrix} 1+\epsilon\oslash & \epsilon£ \\ \epsilon\oslash & \epsilon£ \end{bmatrix} = \epsilon£.$$

Theorem 4.3.8 says that the vector $(\xi_1, \xi_2)^T$ given by

$$\xi_1 = \frac{\det M_1}{\Delta} = \frac{\epsilon£}{1+\epsilon£} = \epsilon£$$
$$\xi_2 = \frac{\det M_2}{\Delta} = \frac{\epsilon£}{1+\epsilon£} = \epsilon£$$

is the maximal solution of the system. Moreover, if we verify it by substituting ξ_1, ξ_2 into the system, we obtain that

$$\begin{cases} (1+\epsilon\oslash)\epsilon£ & + & (\epsilon+\epsilon^2\oslash)\epsilon£ & = \epsilon£ \\ (\epsilon\oslash)(\epsilon£) & + & (1+\epsilon^2£)\epsilon£ & = \epsilon£. \end{cases}$$

Hence it is a valid solution of the system.

The following example shows that although the determinant is infinitesimal, the Cramer conditions still are satisfied.

Example 4.3.20. Let $\epsilon > 0$ be infinitesimal. Consider the system

$$\begin{cases} x & + & y & \subseteq & 1+£\epsilon^{\not\infty} \\ (1+\epsilon)x & + & y & \subseteq & £\epsilon^{\not\infty}. \end{cases}$$

One has $\Delta = \det \begin{bmatrix} 1 & 1 \\ 1+\epsilon & 1 \end{bmatrix} = -\epsilon$ is zeroless and not an absorber of $\underline{B} = £\epsilon^{\not\infty}$. A short calculation shows that $R(\mathcal{A}) = 0$, $P(\mathcal{B}) = £\epsilon^{\not\infty}$ and hence $R(\mathcal{A}) \subset P(\mathcal{B})$. Applying Theorem 4.3.8 we conclude that the vector

$\xi = (\xi_1, \xi_2)$ given by

$$\begin{aligned} \xi_1 &= \frac{\det\begin{bmatrix} 1+£\epsilon^{-\not\infty} & 1 \\ £\epsilon^{-\not\infty} & 1 \end{bmatrix}}{-\epsilon} &= -\frac{1}{\epsilon} + £\epsilon^{\not\infty} \\ \xi_2 &= \frac{\det\begin{bmatrix} 1 & 1+£\epsilon^{\not\infty} \\ 1+\epsilon & £\epsilon^{-\not\infty} \end{bmatrix}}{-\epsilon} &= \frac{1+\epsilon}{\epsilon} + £\epsilon^{-\not\infty} \end{aligned}$$

is the maximal solution of the system.

Finally, we have a flexible system satisfying only the condition $R(\mathcal{A}) \subseteq P(\mathcal{B})$ of Theorem 4.3.8(i).

Example 4.3.21. Let $\epsilon > 0$ be infinitesimal. Consider the system

$$\begin{cases} (1+\epsilon+\epsilon\oslash)x & + & y & \subseteq & £\epsilon \\ (1+\epsilon\oslash)x & + & (1+\epsilon\oslash)y & \subseteq & 1+£\epsilon. \end{cases}$$

We have $\Delta = \det\begin{bmatrix} 1+\epsilon+\epsilon\oslash & 1 \\ 1+\epsilon\oslash & 1+\epsilon\oslash \end{bmatrix} = \epsilon + \epsilon\oslash$ is zeroless. Moreover, $\underline{B} = \overline{B} = \epsilon£$. So Δ is an absorber of $\underline{B}$. In addition, the system is reduced, so $R(\mathcal{A}) = \dfrac{\overline{A}}{\Delta} = \oslash$ and $P(\mathcal{B}) = \epsilon£ : £\epsilon = £$. Hence $R(\mathcal{A}) \subseteq P(\mathcal{B})$. Hence the conditions of Part (i) of Theorem 4.3.8 are satisfied. For $t_1 \in \epsilon£, t_2 \in 1+\epsilon£$ let

$$\begin{aligned} \det\big(M_1(b)\big) = & \det\begin{bmatrix} t_1 & 1 \\ t_2 & 1+\epsilon\oslash \end{bmatrix} & = & \ t_1 - t_2 + t_1\epsilon\oslash . \\ \det\big(M_2(b)\big) = & \det\begin{bmatrix} 1+\epsilon+\epsilon\oslash & t_1 \\ 1+\epsilon\oslash & t_2 \end{bmatrix} & = & \ t_2 + \epsilon t_2 - t_1 + t_1\epsilon\oslash + t_2\epsilon\oslash . \end{aligned}$$

Using Theorem 4.3.8(i) we conclude that the vector $\xi_0 = (\xi_1, \xi_2)$ given by

$$\begin{aligned} \xi_1 &= \frac{\det\big(M_1(b)\big)}{d} = \frac{t_1 - t_2 + t_1\epsilon\oslash}{\epsilon} \\ \xi_2 &= \frac{\det\big(M_2(b)\big)}{d} = \frac{t_2 + \epsilon t_2 - t_1 + t_1\epsilon\oslash + t_2\epsilon\oslash}{\epsilon} \end{aligned}$$

is an admissible solution of the system.

4.4 Gauss-Jordan elimination method for non-singular flexible systems

The Gauss-Jordan elimination is a well-known and widely used method for solving linear systems and computing inverses of matrices. The procedure is simple to state and implement. However, if we apply the Gauss-Jordan elimination to transform a matrix over $\mathbb{E}$ into a near identity matrix I_A we may change the orders of magnitudes of neutrix parts of elements of the matrix. To know how Gauss operations affect the neutrix parts we will explicit these operations and then we will apply it to deal with non-singular flexible systems.

Also, the Gauss-Jordan elimination method does not work on all flexible systems. For example, consider the flexible system

$$\begin{cases} 2x & - & y & \subseteq & 1+\oslash \\ -x & + & y & \subseteq & \epsilon £. \end{cases} \tag{4.25}$$

By adding the first one to the second row we have

$$\begin{cases} 2x & - & y & \subseteq & 1+\oslash \\ x & & & \subseteq & 1+\oslash. \end{cases} \tag{4.26}$$

Now $\xi_0 = (1+\oslash, 1+\oslash)^T$ is a solution of the system below. However, the vector ξ_0 is not a valid solution of the original system. This means that the two systems are not equivalent.

In this section we will provide conditions to guarantee that the Gauss-Jordan elimination can be applied to non-singular flexible systems. We also consider some special kinds of systems which satisfy these conditions.

From now on we use the following notations.

Notation 4.4.1. Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{n,n}(\mathbb{E})$. For each $k \in \{1, ..., n\}$, let $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$.

(a) We denote by $\mathcal{A}_{i_1 \dots i_k, j_1 \dots j_k}$ the $k \times k$ matrix formed by removing from $\mathcal{A}$ the rows whose indices do not belong to $\{i_1, \dots, i_k\}$ and columns whose indices do not belong to $\{j_1, \dots, j_k\}$ and by $M_{i_1 \dots i_k, j_1 \dots j_k} = \det(\mathcal{A}_{i_1 \dots i_k, j_1 \dots j_k})$ a $k \times k$ minor of $\mathcal{A}$.

(b) For $k \geq 1, k+1 \leq i, j \leq n$ we write by $M_{i,j}^{(k)} = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1k} & \alpha_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kk} & \alpha_{kj} \\ \alpha_{i1} & \cdots & \alpha_{ik} & \alpha_{ij} \end{bmatrix}$. Note that we added j-th column $(\alpha_{1j}, \dots, \alpha_{kj}, \alpha_{ij})^T$ and the i-th row $(\alpha_{i1}, \dots, \alpha_{ik}, \alpha_{ij})$ to the $k \times k$ principal submatrix $\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1k} \\ \vdots & \ddots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kk} \end{bmatrix}$ of $\mathcal{A}$. For $k = 0$ we write $M_{i,j}^{(0)} = \alpha_{ij}$ for all $1 \leq i, j \leq n$.

(c) We write $M_{k,k}^{(k-1)} = M^{(k)}$.

(d) We write $m_{i_1 \dots i_k, j_1 \dots j_k}$ as a representative of $M_{i_1 \dots i_k, j_1 \dots j_k}$, $m_{i,j}^{(k)}$ as a representative of $M_{i,j}^{(k)}$ and $m^{(k)}$ as a representative of $M^{(k)}$.

We start by showing that we can modify a given matrix such that the resulting matrix satisfies the condition $\left|m_{i,j}^{(k)}\right| \leq \left|m_{k,k}^{(k)}\right|$.

Proposition 4.4.2. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E})$ be a non-singular matrix. Let $\mathcal{P} = [a_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{R})$ be a representative of $\mathcal{A}$. We can change rows and columns of $\mathcal{P}$ such that it satisfies the following condition*

$$\left|m_{i,j}^{(k)}\right| \leq \left|m_{k,k}^{(k)}\right| \tag{4.27}$$

for all $k+1 \leq i, j \leq n$*, where* $m_{i,j}^{(0)} = \alpha_{ij}$ *for all* $1 \leq i, j \leq n$.

Proof. Let $|\overline{m}_{k+1}| = \max_{k+1 \leq i,j \leq n} \left| m_{i,j}^{(k)} \right|$. Then the equality (4.27) is equivalent to $m_{k,k}^{(k)} = \overline{m}_{k+1}$. To be unambiguous, we refer to the notation $m^{(k)}(\mathcal{P})$ as the principle minor of order k of $\mathcal{P}$ instead of $m^{(k)}$. We do similarly for $m_{i,j}^{(k-1)}(\mathcal{P})$ and $\overline{m}_{k+1}(\mathcal{P})$.

Let $I = \{1, \ldots, n\}$. Assume that $|\overline{m}_1(\mathcal{P})| = |a_{pq}| \neq |a_{11}|$. Let $\sigma_1 \colon I \to I$ be a permutation defined by

$$\sigma_1(j) = \begin{cases} j & \text{if } j \notin \{1, q\} \\ 1 & \text{if } j = q \\ q & \text{if } j = 1 \end{cases},$$

and $\tau_1 \colon I \to I$ be a permutation defined by

$$\tau_1(i) = \begin{cases} i & \text{if } i \notin \{1, p\} \\ 1 & \text{if } i = p \\ p & \text{if } i = 1 \end{cases}.$$

Let $\mathcal{P}^{(1)} \equiv \left[\alpha_{ij}^{(1)}\right] \equiv \left[\alpha_{\tau(i)\sigma(j)}\right]$. In fact, the matrix $\mathcal{P}^{(1)}$ is formed from the original matrix by two successive changes, starting by exchanging the q-th column and the first column in $\mathcal{P}$, and then by exchanging the p-th row and the first row. Consequently, $\overline{m}_1\left(\mathcal{P}^{(1)}\right) = a_{11}^{(1)} = a_{pq}$. Hence the condition (4.27) is satisfied for $k = 1$.

Suppose that we have constructed permutations $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_k$ such that the matrix $\mathcal{P}^{(k)} \equiv \left[\alpha_{ij}^{(k)}\right] \equiv \left[\alpha_{\tau_k(i)\sigma_k(j)\ldots\tau_1(i)\sigma_1(j)}\right]$ satisfies the condition $m_{i,j}^{(l)}\left(\mathcal{P}^{(k)}\right) \leq m^{(l+1)}\left(\mathcal{P}^{(k)}\right) = \overline{m}_{l+1}\left(\mathcal{P}^{(k)}\right)$ for all $1 \leq l \leq k$ and $1 \leq i, j \leq l$.

We now compare the terms $\overline{m}_{k+1}\left(\mathcal{P}^{(k)}\right)$ and $m^{(k+1)}\left(\mathcal{P}^{(k)}\right)$ of $\mathcal{P}^{(k)}$. Assume that $\left|\overline{m}_{k+1}(\mathcal{P}^{(k)})\right| = \left|m_{r,s}^{(k)}\left(\mathcal{P}^{k}\right)\right| \neq \left|m_{k+1,k+1}^{(k)}\left(\mathcal{P}^{k}\right)\right|$ for some $r, s > k+1$. Let $\sigma_{k+1} \colon I \to I$ be a permutation defined by

$$\sigma_{k+1}(j) = \begin{cases} j & \text{if } j \notin \{k+1, s\} \\ k+1 & \text{if } j = s \\ s & \text{if } j = k+1, \end{cases}$$

and $\tau_{k+1} \colon I \to I$ be a permutation defined by

$$\tau_{k+1}(i) = \begin{cases} i & \text{if } i \notin \{k+1, r\} \\ k+1 & \text{if } i = r \\ r & \text{if } i = k+1. \end{cases}$$

Let $\mathcal{P}^{(k+1)} \equiv \left[\alpha_{i'j'}^{(k+1)}\right] \equiv \left[\alpha_{\tau_{k+1}(i)\sigma_{k+1}(j)}^{(k)}\right]$. In other words, the matrix $\mathcal{P}^{(k+1)}$ resulted from $\mathcal{P}^{(k)}$ by two successive changes, starting by exchanging the s^{th} column and the $(k+1)^{th}$ column in $\mathcal{P}^{(k)}$, and then by exchanging the r^{th} row and the $(k+1)^{th}$ row.

Observe that the operation $\tau_{k+1}\sigma_{k+1}$ does not affect the submatrix

$$\mathcal{P}^{(k)} = \begin{bmatrix} \alpha_{11}^{(k)} & \cdots & \alpha_{1k}^{(k)} \\ \vdots & \ddots & \vdots \\ \alpha_{k1}^{(k)} & \cdots & \alpha_{kk}^{(k)} \end{bmatrix}.$$

Hence for $1 \leq l \leq k$ one has

$$\left| m_{i,j}^{(l-1)} \left(\mathcal{P}^{(k+1)} \right) \right| = \left| m_{i',j'}^{(l-1)} \left(\mathcal{P}^{(k)} \right) \right| \leq \left| \overline{m}_l \left(\mathcal{P}^{(k)} \right) \right| = \left| m^{(l)} \left(\mathcal{P}^{(k+1)} \right) \right|,$$

for all $l \leq i, j \leq n$, where $i' = \tau_{k+1}^{-1}(i) \in \{l, \ldots, n\}$ and $j' = \sigma_{k+1}^{-1}(j) \in \{l, \ldots, n\}$.

Also $\left| m_{k+1,k+1}^{(k)} \left(\mathcal{P}^{(k+1)} \right) \right| = \left| m_{r,s}^{(k)} \left(\mathcal{P}^{(k)} \right) \right|$. Because $k+1 \leq \tau_{k+1}^{-1}(i), \sigma_{k+1}^{-1}(j) \leq n$ for all $k+1 \leq i, j \leq n$, it follows that

$$\left| m_{i,j}^{(k)} \left(\mathcal{P}^{(k+1)} \right) \right| = \left| m_{\tau_{k+1}^{-1}(i), \sigma_{k+1}^{-1}(j)}^{(k)} \left(\mathcal{P}^{(k)} \right) \right| \leq \left| m_{r,s}^{(k)} \left(\mathcal{P}^{(k)} \right) \right| = \left| m_{k+1,k+1}^{(k)} \left(\mathcal{P}^{(k+1)} \right) \right|.$$

Hence $\left| m^{(k+1)} \left(\mathcal{P}^{(k+1)} \right) \right| = \left| m_{k+1,k+1}^{(k)} \left(\mathcal{P}^{(k+1)} \right) \right| = \left| \overline{m}_{k+1} \left(\mathcal{P}^{(k+1)} \right) \right|$.

Using (external) induction, we conclude that the matrix $\mathcal{P}^{(n)} \equiv [\alpha_{i'j'}^{(n)}] \equiv [\alpha_{\tau_n(i)\sigma_n(j)\ldots\tau_1(i)\sigma_1(j)}]$ obtained after carrying out n times of above operations satisfies the condition (4.27) for all $1 \leq k \leq n$. □

4.4.1 Explicit formulas for Gauss-Jordan elimination

The explicit formulas for the Gauss elimination, which transforms an arbitrary matrix into a triangular matrix, are given in some articles and books such as [16, 18, 32, 27]. In these works the authors represented elements of a matrix (system) after applying k steps of the Gauss elimination in terms of the ratio of two minors. In [33], an explicit formula of Gauss-Jordan elimination, which transforms an arbitrary matrix into the identity matrix was introduced. In all these studies the proofs tend to use advanced results in algebra. In [20], the Gauss-Jordan elimination formula was obtained by a process of successive multiplication of elementary matrices. The procedure transforms each column in a given matrix into a unit vector. They did not give a detailed proof. We will present here a proof for the explicit formula of Gauss-Jordan elimination based on some basic properties of determinants, induction and direct calculations. Then we will apply this formula to prove that the Gauss-Jordan elimination can be used to solve flexible systems under some suitable conditions.

Definition 4.4.3. A matrix $P = [a_{ij}]_{n \times n} \in \mathcal{M}_{n,n}(\mathbb{R})$ is called *Gauss-Jordan eliminable* if for $1 \leq k \leq n$,

$$m^{(k)} \equiv \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \neq 0.$$

Let $P=[a_{ij}]_{n\times n}\in\mathcal{M}_{n,n}(\mathbb{R})$ be a reduced Gauss-Jordan eliminable matrix. For every $p\in\{1,\dots,2n-1\}$ we define matrices $\mathcal{G}_p$ as follows

$$
\begin{aligned}
\mathcal{G}_1 &= \left[g_{ij}^{(1)}\right]_{n\times n}=\begin{bmatrix}1&0&\cdots&0\\-a_{21}&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\-a_{n1}&0&\cdots&1\end{bmatrix}\\
\mathcal{G}_{2k-2} &= \left[g_{ij}^{(2k-2)}\right]_{n\times n},\quad\text{where}\quad g_{ij}^{(2k-2)}=\begin{cases}1&\text{if } i=j\neq k\\0&\text{if } i\neq j\\\dfrac{m^{(k-1)}}{m^{(k)}}&\text{if } i=j=k,\end{cases}\qquad k\in\{2,...,n\},\\
\mathcal{G}_{2k-1} &= \left[g_{ij}^{(2k-1)}\right]_{n\times n},\quad\text{where}\\
g_{ij}^{(2k-1)} &= \begin{cases}0&\text{if } j\neq k, j\neq i\\1&\text{if } i=j\\(-1)^{k+i}\dfrac{m_{1\dots(k-1),1\dots(i-1)(i+1)\dots k}}{m^{(k-1)}}&\text{if } 1\leq i<k, j=k\\-\dfrac{m_{i,k}^{(k-1)}}{m^{(k-1)}}&\text{if } i>k, j=k,\end{cases}\qquad k\in\{2,...,n\}.
\end{aligned}
\tag{4.28}
$$

Notation 4.4.4. We denote by $\mathcal{G}(\cdot)$ the successive multiplications of matrices

$$\mathcal{G}_{2n-1}(\mathcal{G}_{2n-2}(\cdots(\mathcal{G}_1(\cdot)))).$$

We call $\mathcal{G}$ the *Gauss-Jordan procedure*, and for $1\leq p\leq 2n-1$, $\mathcal{G}_p$ the *Gaussian operation matrices* .

In addition, we write $\mathcal{G}_0=I_n$ the identity matrix of order n. Then $\mathcal{G}(.)=G_{2n-1}(\mathcal{G}_{2n-2}(\cdots(\mathcal{G}_1(\mathcal{G}_0(.))))).$

For each matrix $\mathcal{A}\in\mathcal{M}_n(\mathbb{E})$ we write $\mathcal{A}^{(k)}=\mathcal{G}_k(\mathcal{G}_{k-1}(\cdots(\mathcal{G}_1(\mathcal{G}_0(\mathcal{A})))))\equiv[\alpha_{ij}^{(k)}]_{n\times n}$.

Convention 4.4.5. Because of Proposition 4.4.2, from now on we always assume that $\mathcal{G}$ is the Gauss-Jordan procedure of a matrix which satisfies the condition (4.27). In case $\mathcal{A}\in\mathcal{M}_n(\mathbb{E})$ we choose a representative of $\mathcal{A}$ such that it satisfies this condition.

Theorem 4.4.6. *Let $P=[a_{ij}]_{n\times n}\in\mathcal{M}_n(\mathbb{R})$ be a reduced Gauss-Jordan eliminable matrix. Then*

$$P^{2k-1}=\mathcal{G}_{2k-1}\left(\mathcal{G}_{2k-2}\left(\cdots\left(\mathcal{G}_1(P)\right)\right)\right)=\begin{bmatrix}1&0&\cdots&0&a_{1(k+1)}^{(2k-1)}&\cdots&a_{1n}^{(2k-1)}\\\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&a_{(k-1)(k+1)}^{(2k-1)}&\cdots&a_{(k-1)n}^{(2k-1)}\\0&0&\cdots&1&a_{k(k+1)}^{2k-1}&\cdots&a_{kn}^{2k-1}\\0&0&\cdots&0&a_{(k+1)(k+1)}^{(2k-1)}&\cdots&a_{(k+1)n}^{(2k-1)}\\\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&a_{n(k+1)}^{(2k-1)}&\cdots&a_{nn}^{(2k-1)}\end{bmatrix}$$

for all $k+1 \le j \le n$, where

$$a_{ij}^{(2k-1)} = \begin{cases} \dfrac{m_{i,j}^{(k)}}{m^{(k)}} & \textit{if } i \ge k+1 \\ (-1)^{k+i}\dfrac{m_{1\dots k,1\dots(i-1)(i+1)\dots kj}}{m^{(k)}} & \textit{if } i < k+1, \end{cases} \tag{4.29a}$$

in particular

$$a_{(k+1)(k+1)}^{(2k-1)} = \frac{m^{(k+1)}}{m^{(k)}}. \tag{4.29b}$$

Proof. We will prove the theorem by induction. A short calculation shows that it is true for $k = 1$. Assume that it holds for k. Because of the inductive assumptions, we have

$$\begin{aligned} P^{(2k-1)} =& \mathcal{G}_{2k-1}(\dots\mathcal{G}_1(P)) \\ =& \begin{bmatrix} 1 & \cdots & 0 & (-1)^{k+1}\dfrac{m_{1\dots k,2\dots(k+1)}}{m^{(k)}} & a_{1(k+2)}^{(2k-1)} & \cdots & a_{1n}^{(2k-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & (-1)^{2k}\dfrac{m_{1\dots k,1\dots(k-1)(k+1)}}{m^{(k)}} & a_{k(k+2)}^{(2k-1)} & \cdots & a_{kn}^{(2k-1)} \\ 0 & \cdots & 0 & \dfrac{m^{(k+1)}}{m^{(k)}} & a_{(k+1)(k+2)}^{(2k-1)} & \cdots & a_{(k+1)n}^{(2k-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \dfrac{m_{n,k+1}^{(k)}}{m^{(k)}} & a_{n(k+2)}^{(2k-1)} & \cdots & a_{nn}^{(2k-1)} \end{bmatrix} \end{aligned}$$

and formulas (4.29a), (4.29b) hold. We need to prove that it holds for $k+1$, that is

$$P^{(2k+1)} = \mathcal{G}_{2k+1}(\mathcal{G}_{2k-2}(\cdots(\mathcal{G}_1(P)))) = \begin{bmatrix} 1 & 0 & \cdots & 0 & a_{1(k+2)}^{(2k+1)} & \cdots & a_{1n}^{(2k+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{k(k+2)}^{(2k+1)} & \cdots & a_{kn}^{(2k+1)} \\ 0 & 0 & \cdots & 1 & a_{(k+1)(k+2)}^{(2k+1)} & \cdots & a_{k+1n}^{(2k+1)} \\ 0 & 0 & \cdots & 0 & a_{(k+2)(k+2)}^{(2k+1)} & \cdots & a_{(k+2)n}^{(2k+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n(k+2)}^{(2k+1)} & \cdots & a_{nn}^{(2k+1)} \end{bmatrix},$$

where

$$a_{ij}^{(2k+1)} = \begin{cases} \dfrac{m_{i,j}^{(k+1)}}{m^{(k+1)}} & \text{if } i \ge k+2, j \ge k+2 \\ (-1)^{k+i+1}\dfrac{m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}}{m^{(k+1)}} & \text{if } i < k+2 \le j \le n, \end{cases} \tag{4.30a}$$

and

$$a_{(k+2)(k+2)}^{(2k+1)} = \frac{m^{(k+2)}}{m^{(k+1)}}. \tag{4.30b}$$

We do it in two steps. In the first step we show that the $(k+1)$-th column of the matrix $P^{(2k+1)}$ is a unit vector.

In the second step we verify formulas (4.30a) and (4.30b).

As for the first step, one has

$$
\begin{aligned}
P^{(2k)} &= \mathcal{G}_{2k}\left(P^{(2k-1)}\right) \\
&= \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \dfrac{m^{(k)}}{m^{(k+1)}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix} \cdot \\
&\quad \begin{bmatrix}
1 & \cdots & 0 & (-1)^{k+1}\dfrac{m_{1\ldots k,2\ldots(k+1)}}{m^{(k)}} & a_{1(k+2)}^{(2k-1)} & \cdots & a_{1n}^{(2k-1)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & (-1)^{2k}\dfrac{m_{1\ldots k,1\ldots(k-1)(k+1)}}{m^{(k)}} & a_{k(k+2)}^{(2k-1)} & \cdots & a_{kn}^{(2k-1)} \\
0 & \cdots & 0 & \dfrac{m^{(k+1)}}{m^{(k)}} & a_{(k+1)(k+2)}^{(2k-1)} & \cdots & a_{(k+1)n}^{(2k-1)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \dfrac{m_{n,k+1}^{(k)}}{m^{(k)}} & a_{n(k+2)}^{(2k-1)} & \cdots & a_{nn}^{(2k-1)}
\end{bmatrix} \\
&= \begin{bmatrix}
1 & \cdots & 0 & (-1)^{k+1}\dfrac{m_{1\ldots k,2\ldots(k+1)}}{m^{(k)}} & a_{1(k+2)}^{(2k)} & \cdots & a_{1n}^{(2k)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & (-1)^{2k}\dfrac{m_{1\ldots k,1\ldots(k-1)(k+1)}}{m^{(k)}} & a_{k(k+2)}^{(2k)} & \cdots & a_{kn}^{(2k)} \\
0 & \cdots & 0 & 1 & a_{(k+1)(k+2)}^{(2k)} & \cdots & a_{k+1n}^{(2k)} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \dfrac{m_{n,k+1}^{(k)}}{m^{(k)}} & a_{n(k+2)}^{(2k)} & \cdots & a_{nn}^{(2k)}
\end{bmatrix}.
\end{aligned}
$$

So

$$
\begin{aligned}
P^{(2k+1)} =& \mathcal{G}_{2k+1}(P^{(2k)}) \\
=& \begin{bmatrix}
1 & \cdots & 0 & (-1)^{k+2}\dfrac{m_{1\ldots k,2\ldots(k+1)}}{m^{(k)}} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & (-1)^{2k+1}\dfrac{m_{1\ldots k,1\ldots(k-1)(k+1)}}{m^{(k)}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\dfrac{m^{(k)}_{k+2,k+1}}{m^{(k)}} & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -\dfrac{m^{(k)}_{n,k+1}}{m^{(k)}} & 0 & \cdots & 1
\end{bmatrix} \cdot \\
& \begin{bmatrix}
1 & \cdots & 0 & (-1)^{k+1}\dfrac{m_{1\ldots k,2\ldots(k+1)}}{m^{(k)}} & a^{(2k)}_{1(k+2)} & \cdots & a^{(2k)}_{1n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & (-1)^{2k}\dfrac{m_{1\ldots k,1\ldots(k-1)(k+1)}}{m^{(k)}} & a^{(2k)}_{k(k+2)} & \cdots & a^{(2k)}_{kn} \\
0 & \cdots & 0 & 1 & a^{(2k)}_{(k+1)(k+2)} & \cdots & a^{(2k)}_{k+1n} \\
0 & \cdots & 0 & \dfrac{m^{(k)}_{k+2,k+1}}{m^{(k)}} & a^{(2k)}_{(k+2)(k+2)} & \cdots & a^{(2k)}_{(k+2)n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \dfrac{m^{(k)}_{n,k+1}}{m^{(k)}} & a^{(2k)}_{n(k+2)} & \cdots & a^{(2k)}_{nn}
\end{bmatrix} \\
=& \begin{bmatrix}
1 & \cdots & 0 & 0 & a^{(2k+1)}_{1(k+2)} & \cdots & a^{(2k+1)}_{1n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & a^{(2k+1)}_{k(k+2)} & \cdots & a^{(2k+1)}_{kn} \\
0 & \cdots & 0 & 1 & a^{(2k+1)}_{(k+1)(k+2)} & \cdots & a^{(2k+1)}_{k+1n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & a^{(2k+1)}_{n(k+2)} & \cdots & a^{(2k+1)}_{nn}
\end{bmatrix}.
\end{aligned}
$$

Hence the $(k+1)$-th column of $\mathcal{G}_{2k+1}(\cdots(\mathcal{G}_1(P)))$ is a unit vector.

As for the second step we compute $a^{(2k+1)}_{ij}(1 \le i \le n, k+2 \le j \le n)$ and show that they satisfy formulas (4.30a) and (4.30b). We consider three cases.

Case 1: $i = k+2$. Let

$$
T_{(k+2),j} = \begin{bmatrix}
1 & \cdots & 0 & a^{(2k+1)}_{1j} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & a^{(2k+1)}_{(k+1)j} \\
0 & \cdots & 0 & a^{(2k+1)}_{(k+2)j}
\end{bmatrix}.
$$

Then

$$\det\left(T_{(k+2),j}\right) = a^{(2k+1)}_{(k+2)j}.$$

Observe that T_{2k+1} is obtained from $P_{1\dots(k+2),1\dots(k+1)j}$ by multiplying it successively by $\mathcal{G}_1,\dots,\mathcal{G}_{2k+1}$ and that the $G_{2r+1}, 0 \le r \le k$ do not affect the determinants. In fact, they represent the operations of adding a multiple of one row to another. For each $r \in \{1,\dots,k+1\}$, the operation G_{2r-2} represents the multiplication of the r-th row by $\dfrac{m^{(r-1)}}{m^{(r)}}$. In addition $\det\left(P_{1\dots(k+2),1\dots(k+1)j}\right) = m^{(k+1)}_{k+2,j}$, so

$$\begin{aligned} a^{(2k+1)}_{(k+2)j} =& \det\left(T_{(k+2,j)}\right) = \det(\mathcal{G}_2)\det(\mathcal{G}_4)\dots\det(\mathcal{G}_{2k})\det\left(P_{1\dots(k+2),1\dots(k+1)j}\right) \\ =& \frac{1}{m^{(2)}}\cdot\frac{m^{(2)}}{m^{(3)}}\cdots\frac{m^{(k)}}{m^{(k+1)}}\cdot m^{(k+1)}_{k+2,j} = \frac{m^{(k+1)}_{k+2,j}}{m^{(k+1)}} \quad \text{for all} \quad k+2 \le j \le n. \end{aligned}$$

In particular for $j = k+2$, $a^{(2k+1)}_{(k+2)(k+2)} = \dfrac{m^{(k+1)}_{k+2,k+2}}{m^{(k+1)}} = \dfrac{m^{(k+2)}}{m^{(k+1)}}$.

Thus formula (4.30a) and (4.30b) hold for $i = k+2$ and $k+2 \le j \le n$.

Case 2: $i > k+2$ and $k+2 \le j \le n$. Let

$$U_{ij} \equiv \begin{bmatrix} 1 & \cdots & a_{1(k+1)} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{(k+1)1} & \cdots & a_{(k+1)(k+1)} & a_{(k+1)j} \\ a_{i1} & \cdots & a_{i(k+1)} & a_{ij} \end{bmatrix}.$$

Then $\det\left(U_{i,j}\right) = m^{(k+1)}_{i,j}$. Also operations $\mathcal{G}_1,\dots\mathcal{G}_{2k+1}$ transform the matrix $U_{i,j}$ into

$$U'_{i,j} \equiv \begin{bmatrix} 1 & \cdots & 0 & a^{(2k+1)}_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a^{(2k+1)}_{(k+1)j} \\ 0 & \cdots & 0 & a^{(2k+1)}_{ij} \end{bmatrix}.$$

With an analogous argument as in Case 1, one obtains

$$\begin{aligned} \det\left(U'_{i,j}\right) =& a^{(2k+1)}_{ij} = \det(\mathcal{G}_2)\det(\mathcal{G}_4)\cdots\det(\mathcal{G}_{2k})\det\left(U_{i,j}\right) \\ =& \frac{1}{m^{(2)}}\cdot\frac{m^{(2)}}{m^{(3)}}\cdots\frac{m^{(k)}}{m^{(k+1)}}\cdot m^{(k+1)}_{i,j} = \frac{m^{(k+1)}_{i,j}}{m^{(k+1)}}. \end{aligned}$$

Hence (4.30a) holds for $i > k+2$ and $k+2 \le j \le n$.

Case 3: $i < k+2$ and $k+2 \leq j \leq n$. Let

$$V_{i,j} = \begin{bmatrix} 1 & \cdots & a_{1(i-1)} & a_{1(i+1)} & \cdots & a_{1(k+1)} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(i-1)} & a_{(i-1)(i+1)} & \cdots & a_{(i-1)(k+1)} & a_{(i-1)j} \\ a_{i1} & \cdots & a_{i(i-1)} & a_{i(i+1)} & \cdots & a_{i(k+1)} & a_{ij} \\ a_{(i+1)1} & \cdots & a_{(i+1)(i-1)} & a_{(i+1)(i+1)} & \cdots & a_{(i+1)(k+1)} & a_{(i+1)j} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\vdots & \\ a_{(k+1)1} & \cdots & a_{(k+1)(i-1)} & a_{(k+1)(i+1)} & \cdots & a_{(k+1)(k+1)} & a_{(k+1)j} \end{bmatrix}.$$

Note that $V_{i,j} = P_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}$, so that

$$\det(V_{i,j}) = m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}.$$

Operations $\mathcal{G}_1, \dots, \mathcal{G}_{2k+1}$ transform the matrix $V_{i,j}$ into

$$V'_{i,j} \equiv \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 & a_{1j}^{(2k+1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 & a_{ij}^{(2k+1)} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a_{ij}^{(2k+1)} \\ 0 & \cdots & 0 & 1 & \cdots & 0 & a_{ij}^{(2k+1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & a_{(k+1)j}^{(2k+1)} \end{bmatrix}$$

Expanding the determinant along the i-th row we obtain that

$$\det(V'_{i,j}) = (-1)^{i+k+1} a_{ij}^{(2k+1)} \tag{4.31}$$

Once again, with analogous arguments as in Case 1, one obtains

$$\begin{aligned} \det(V'_{i,j}) =& \det(V_{i,j})\det(\mathcal{G}_2)\det(\mathcal{G}_4)\dots\det(\mathcal{G}_{2k}) \\ =& m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j} \cdot \frac{1}{m^{(2)}} \cdot \frac{m^{(2)}}{m^{(3)}} \cdots \frac{m^{(k)}}{m^{(k+1)}} = \frac{m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}}{m^{(k+1)}}. \end{aligned} \tag{4.32}$$

Formulas (4.31) and (4.32) imply that

$$(-1)^{i+k+1} a_{ij}^{(2k+1)} = \frac{m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}}{m^{(k+1)}}.$$

So

$$a_{ij}^{(2k+1)} = (-1)^{i+k+1} \frac{m_{1\dots(k+1),1\dots(i-1)(i+1)\dots(k+1)j}}{m^{(k+1)}}.$$

Hence formula (4.30a) holds for $1 \leq i < k+2$ and $k+2 \leq j \leq n$. □

In particular, when $k = n$ one has

Corollary 4.4.7. *Let $P = [a_{ij}]_{n\times n}$ be a reduced Gauss-Jordan eliminable matrix. We have*

$$\mathcal{G}_{2n-1}(\mathcal{G}_{2k}(\dots(\mathcal{G}_1(P)))) = I_n.$$

Let $n \in \mathbb{N}$ be standard and $P = [a_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ be a reduced Gauss-Jordan eliminable matrix. For $1 \leq p \leq 2n-1$, let $P^{(p)} \equiv \mathcal{G}_p(\mathcal{G}_{p-1}(\dots(\mathcal{G}_1(P)))) \equiv [a_{ij}^{(p)}]_{n\times n}$.

A limited number does not blow up neutrices. We will show that entries of the coefficient matrix obtaining after k Gauss-Jordan elimination steps with $k \in \mathbb{N}$ an odd number are limited. As a sequence, all entries of G_{2p-1} with $p \in \mathbb{N}$ are limited.

Lemma 4.4.8. *Let $n \in \mathbb{N}$ be standard and $P = [a_{ij}]_{n\times n} \in \mathcal{M}_n(\mathbb{R})$ be a reduced Gauss-Jordan eliminable matrix. Then $a_{ij}^{(p)}$ is limited for all $1 \leq i, j \leq n$ and $p = 2k-1$.*

Proof. We apply external induction. Because P is reduced, $|a_{ij}| \leq 1$ for all $1 \leq i, j \leq n$. This implies that $\left|a_{ij}^{(1)}\right| = |a_{ij} - a_{i1}\cdot a_{1j}| \leq |a_{ij}| + |a_{i1}|\cdot|a_{1j}| \leq 2$ for all $2 \leq i \leq n, 1 \leq j \leq n$. For $i = 1$, one has $\left|a_{1j}^{(1)}\right| = |a_{1j}| \leq 1$ for all $1 \leq j \leq n$. Hence $a_{ij}^{(1)}$ is limited for $1 \leq i, j \leq n$. Suppose that $a_{ij}^{(2k-1)}$ is limited for $k < n$ and for all $1 \leq i, j \leq n$. Because the q-th column of $a_{.j}^{(2k+1)}$ is a unit vector for $1 \leq j \leq k+1$, the entries of these columns are limited. We just need to show that $a_{ij}^{(2k+1)}$ is also limited for $1 \leq i \leq n, k+2 \leq j \leq n$. One has

$$a_{ij}^{(2k)} = \begin{cases} a_{ij}^{(2k-1)} & \text{if } i \neq k+1 \\ \dfrac{m_{i,j}^{(k+1)}}{m^{(k+1)}} & \text{if } i = k+1. \end{cases}$$

So $a_{ij}^{(2k)} = a_{ij}^{(2k-1)}$ is limited by the induction hypothesis, for $i \neq k+1, k+2 \leq j \leq n$. For $i = k+1$ and $k+2 \leq j \leq n$, one has $a_{ij}^{(2k)} = \dfrac{m_{i,j}^{(k+1)}}{m^{(k+1)}}$. By Convention 4.4.5 we have $\left|a_{ij}^{(2k)}\right| \leq 1$ for all $k+2 \leq j \leq n$. This implies that $a_{ij}^{(2k+1)} = a_{ij}^{(2k-1)} - a_{ik}^{(2k-1)} \cdot a_{(k+1)j}^{(2k)}$ is limited for all $k+2 \leq j \leq n, 1 \leq i \leq n, i \neq k+1$. For $i = k+1$ one has that $a_{(k+1)j}^{(2k+1)} = a_{(k+1)j}^{(2k)}$ is limited for $k+2 \leq j \leq n$. Hence $a_{ij}^{(2k+1)}$ is limited for $1 \leq i, j \leq n$. □

Corollary 4.4.9. *Let $\mathcal{A} = [a_{ij}]_{n\times n}$ be a reduced Gauss-Jordan eliminable matrix and $\mathcal{G}_m$ for $1 \leq m \leq 2n-1$ be the Gauss operation matrices of a representative of $\mathcal{A}$. Then all the entries of $\mathcal{G}_{2p-1}$ are limited for $1 \leq p \leq n$.*

Proof. It follows directly from the fact that $\left|g_{i(p+1)}^{(2p+1)}\right| = \left|a_{i(p+1)}^{(2p-1)}\right|$ for all $1 \leq i \leq n$, $1 \leq p \leq 2n-1$ and Lemma 4.4.8. □

4.4.2 Conditions for solvability of a non-singular flexible system by Gauss-Jordan elimination

We recall two facts of the Gauss-Jordan elimination in classical linear algebra:

(i) The Gaussian operations do not make any change on the set of solutions of the systems. That is, the system $A[x] = b$ is equivalent to $(\mathcal{G}A)[x] = \mathcal{G}(b)$, where $\mathcal{G}$ is the Gauss-Jordan procedure.

(ii) The Gauss-Jordan elimination determines the solution of every non-singular system. In fact, $\mathcal{G}(b)$ is the unique solution of the given system.

However, in general, these facts are not true for flexible systems. This means

(i) The non-singular flexible systems $\mathcal{A}\xi \subseteq \mathcal{B}$ may be not equivalent to $(\mathcal{G}\mathcal{A})\xi \subseteq \mathcal{G}\mathcal{B}$, and

(ii) The vector $\mathcal{G}(\mathcal{B})$ may be not equal to the set of all real admissible solutions of a given system.

The following example shows that $\mathcal{G}B$ is different from the solution obtained by Cramer's rule.

Example 4.4.10. Consider the system

$$\begin{cases} (2+\oslash)x_1 & + & \epsilon\oslash x_2 & \subseteq 1+\epsilon\oslash \\ (-1+\epsilon\oslash)\,x_1 & + & (1+\oslash)\,x_2 & \subseteq 2+\epsilon\oslash\,. \end{cases}$$

Using the Gauss operations we obtain

$$\begin{array}{c} \\ \\ 1/2R_1 \\ \longrightarrow \\ R_2+R_1 \\ \longrightarrow \end{array} \begin{array}{l} \begin{cases} (2+\oslash)x_1 & + & \epsilon\oslash x_2 & \subseteq 1+\epsilon\oslash \\ (-1+\epsilon\oslash)\,x_1 & + & (1+\oslash)\,x_2 & \subseteq 2+\epsilon\oslash \end{cases} \\ \begin{cases} (1+\oslash)x_1 & + & \epsilon\oslash x_2 & \subseteq 1/2+\epsilon\oslash \\ (-1+\epsilon\oslash)\,x_1 & + & (1+\oslash)\,x_2 & \subseteq 2+\epsilon\oslash \end{cases} \\ \begin{cases} (1+\oslash)x_1 & + & \epsilon\oslash x_2 & \subseteq 1/2+\epsilon\oslash \\ \epsilon\oslash x_1 & + & (1+\oslash)\,x_2 & \subseteq 5/2+\epsilon\oslash\,. \end{cases} \end{array}$$

Hence $\mathcal{G}(\mathcal{B}) = (1/2+\epsilon\oslash, 5/2+\epsilon\oslash)$ whereas Cramer's rule gives us $(1/2+\oslash, 5/2+\oslash)$.

Next we will present conditions in order to apply the Gauss-Jordan elimination to flexible systems. Flexible systems satisfying these conditions will be also called *Gauss-Jordan eliminable*.

Definition 4.4.11. Consider the flexible system (4.2). It is said to be *Gauss-Jordan eliminable* if it satisfies the following conditions.

(i) The system is non-singular,

(ii) $\overline{B} = \underline{B} = B$,

(iii) $R(\mathcal{A}) \subseteq P(\mathcal{B})$,

(iv) The entries $\alpha_{kk}^{(2k-1)}$ are zeroless for all $1 \leq k \leq n$ (see Notation 4.4.4),

(v) The determinant $\Delta/\overline{\alpha}^n$ is not an absorber of B.

Every flexible systems with $\overline{\alpha}$ zeroless is equivalent to a reduced system. So, from now on, we always assume that a flexible system is reduced.

Suppose a flexible system does not satisfy the condition (ii). The modified system, in the way that the neutrix parts of constant terms are always taking to be the smallest neutrix $\underline{B}$, does satisfy condition (ii). The set of solutions of the latter system is a subset of the solutions of the original system. However, both sets of solutions may very well be equal. Let consider the flexible system 4.25, i.e.

$$\begin{cases} 2x & - & y & \subseteq & 1+\oslash \\ -x & + & y & \subseteq & \epsilon £. \end{cases} \tag{4.33}$$

Using the method of Section 4.6 it is easy to verify that the exact solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \oslash \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \epsilon £ \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{4.34}$$

Yet if we modify the system (4.33) to

$$\begin{cases} 2x & - & y & \subseteq & 1+\epsilon £ \\ -x & + & y & \subseteq & \epsilon £, \end{cases} \tag{4.35}$$

the Gauss-Jordan elimination method yields the solution $(x, y) = (1 + \epsilon £, 1 + \epsilon £)$, which is strictly included in (4.34).

Observe that the conditions that for each $1 \leq k \leq n$, $|\alpha_{ij}^{(2k-1)}| \leq |\alpha_{kk}^{(2k-1)}|$ for all $k \leq i, j \leq n$ and that Δ is zeroless do not guarantee that $\alpha_{kk}^{(2k-1)}$ is zeroless. This is shown in the next example.

Example 4.4.12. Let $\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \oslash & \epsilon \\ 0 & \epsilon & 0 \end{bmatrix}$, where $\epsilon > 0$ is infinitesimal. Then $\Delta = \det(\mathcal{A}) = \epsilon^2$ is zeroless and $|\alpha_{ij}| \leq |\alpha_{22}|$ for all $2 \leq i, j \leq 3$. However $\alpha_{22} = \oslash$ is a neutrix.

It is not convenient to use Definition 4.4.11 to verify that a given flexible system is Gauss-Jordan eliminable or not. The reason is because we have to implement Gauss operations to calculate $\mathcal{A}^{(2p-1)}$, $1 \leq p \leq k$ to check if the pivot element of $A^{(2k-1)}$ is zeroless or not. Next, we will present some conditions to guarantee that $\alpha_{kk}^{(2k-1)}$ is zeroless without carrying out Gauss operations. This means that we can check a given flexible system is Gauss-Jordan eliminable or not without performing Gauss operations. To do that we first need to determine the neutrix part and a representative of $\alpha_{kk}^{(2k-1)}$.

The result below determines the neutrix parts of the entries of a matrix after applying $2k-1$ steps of the Gauss-Jordan elimination. It also gives an estimate for the neutrix part of the pivot element of $\mathcal{A}^{(2k-1)}$.

Notation 4.4.13. Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n \times n} \in \mathcal{M}_n(\mathbb{E})$. We write

$$L_{ij} = \max_{1 \leq p \leq i} \{A_{pj}\}, \quad \text{for all} \quad 1 \leq i, j \leq n. \tag{4.36}$$

Lemma 4.4.14. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a non-singular matrix over $\mathbb{E}$. Then for all $k+1 \le i, j \le n$, the neutrix parts of $\alpha_{ij}^{(2k-1)}$ are given by*

$$\begin{cases} A_{ij}^{(1)} = A_{ij} + a_{i1}A_{1j}, \\ A_{ij}^{(2k-1)} = A_{ij}^{(2k-3)} + \dfrac{m_{i,k}^{(k-1)}}{m^{(k)}} A_{k,j}^{(2k-3)}, \end{cases} \tag{4.37}$$

and

$$A_{(k+1)(k+1)}^{(2k-1)} \subseteq L_{(k+1)(k+1)}. \tag{4.38}$$

Proof. We first demonstrate formula (4.37). Clearly, $A_{ij}^{(1)} = A_{ij} + a_{i1}A_{1j}$. For the second equality, one has $A_{ij}^{(2k-1)} = \max\left\{A_{ij}^{(2k-3)}, \dfrac{a_{ik}^{(2k-3)}}{a_{kk}^{(2k-3)}} A_{kj}^{(2k-3)}\right\} = A_{ij}^{(2k-3)} + \dfrac{a_{ik}^{(2k-3)}}{a_{kk}^{(2k-3)}} A_{kj}^{(2k-3)}$ for all $k+1 \le i, j \le n$. Also by formula (4.29) we have $\dfrac{a_{ik}^{(2k-3)}}{a_{kk}^{(2k-3)}} = \dfrac{m_{i,k}^{(k-1)}/m^{(k-1)}}{m^{(k)}/m^{(k-1)}} = \dfrac{m_{i,k}^{(k-1)}}{m^{(k)}}$. Hence formula (4.37) is proved.

Next we prove formula (4.38). For $p = 1$, one has $A_{ij}^{(1)} = \max\{A_{ij}, a_{i1}A_{1j}\} \subseteq L_{ij}$, for all $2 \le i, j \le n$. In particular $A_{22}^{(1)} \subseteq L_{22}$. Suppose that $A_{ij}^{(2k-3)} \subseteq L_{ij}$ for all $k \le i, j \le n$. We will show that $A_{ij}^{(2k-1)} \subseteq L_{ij}$ for all $k+1 \le i, j \le n$. Indeed, by Convention 4.4.5 which implies $\left|\dfrac{m_{i,j}^{(k-1)}}{m^{(k)}}\right| \le 1$ for $i, j \ge k+1$, and by formula (4.37) we have

$$A_{ij}^{(2k-1)} = \max\left\{A_{ij}^{(2k-3)}, \frac{m_{i,j}^{(k-1)}}{m^{(k)}} A_{kj}^{(2k-3)}\right\} \subseteq \max\{L_{ij}, L_{kj}\} = L_{ij}.$$

In particular $A_{(k+1)(k+1)}^{(2k-1)} = N\left(\alpha_{(k+1)(k+1)}^{(2k-1)}\right) \subseteq L_{(k+1)(k+1)}$. □

The next result gives an estimation of a representative of the pivot element of $\mathcal{A}^{(2k-1)}$ so that, in some cases, it enables us to verify whether the pivot element of $\mathcal{A}^{(2k-1)}$ is zeroless or not.

Theorem 4.4.15. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced non-singular matrix over $\mathbb{E}$. Let $\Delta = \det(\mathcal{A}) \equiv d + D$. For each $1 \le k \le n$, if $\alpha_{kk}^{(2k-3)}$ is zeroless then $\left|\dfrac{m^{(k+1)}}{m^{(k)}}\right| > \oslash\Delta$.*

Proof. For each $1 \leq k \leq n-1$ we have

$$\begin{aligned}\mathcal{A}^{(2k-1)} \equiv& \mathcal{G}_{2k-1}((\mathcal{G}_{2k-2}(\cdots \mathcal{G}_1(\mathcal{A})))) \\ =& \begin{bmatrix} 1+A_{11} & A_{12}^{(2k-1)} & \cdots & A_{1k}^{(2k-1)} & \alpha_{1(k+1)}^{(2k-1)} & \cdots & \alpha_{1n}^{(2k-1)} \\ A_{21}^{(2k-1)} & 1+A_{22}^{(2k-1)} & \cdots & \alpha_{2k}^{(2k-1)} & \alpha_{2(k+1)}^{(2k-1)} & \cdots & \alpha_{1n}^{(2k-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{k1}^{(2k-1)} & A_{k2}^{(2k-1)} & \cdots & 1+A_{kk}^{(2k-1)} & \alpha_{k(k+1)}^{(2k-1)} & \cdots & \alpha_{kn}^{(2k-1)} \\ A_{(k+1)1}^{(2k-1)} & A_{(k+1)2}^{(2k-1)} & \cdots & A_{(k+1)k}^{(2k-1)} & \alpha_{(k+1)(k+1)}^{(2k-1)} & \cdots & \alpha_{(k+1)n}^{(2k-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{n1}^{(2k-1)} & A_{n2}^{(2k-1)} & \cdots & A_{nk}^{(2k-1)} & \alpha_{n(k+1)}^{(2k-1)} & \cdots & \alpha_{nn}^{(2k-1)} \end{bmatrix} \equiv [\alpha_{ij}^{(2k-1)}]_{n\times n}\end{aligned}$$

Suppose on contrary that $a_{(k+1)(k+1)}^{(2k-1)} = \frac{m^{(k+1)}}{m^{(k)}} \in \oslash\Delta$. From $\left|a_{ij}^{(2k-1)}\right| = \left|\frac{m_{ij}^{(k)}}{m^{(k)}}\right| \leq \left|\frac{m^{(k+1)}}{m^{(k)}}\right| = \left|a_{(k+1)(k+1)}^{(2k-1)}\right|$ for all $k+1 \leq i,j \leq n$ one derives that $a_{ij}^{(2k-1)} \in \oslash\Delta$ for all $k+1 \leq i,j \leq n$.

Let S_{n-k} be the set of all permutations of $\{k+1,\ldots,n\}$ and $\sigma \in S_{n-k}$. Put $\Delta^{(2k-1)} = \det\left(\mathcal{A}_{k+1\ldots n,k+1\ldots n}^{(2k-1)}\right) = d^{(2k-1)} + D^{(2k-1)}$, where

$$d^{(2k-1)} = \sum_{\sigma\in S_{n-k}} \mathrm{sgn}(\sigma) a_{k+1\sigma(k+1)}^{(2k-1)} \cdots a_{n\sigma(n)}^{(2k-1)} \in (\oslash\Delta)^{n-k} \subseteq \oslash\Delta. \tag{4.39}$$

On the other hand $a_{ii}^{(2k-1)} = 1$ for all $1 \leq i \leq k$, so $d^{(2k-1)}$ is also a representative of $\det(\mathcal{A}^{(2k-1)})$. Applying the successive Laplace expansions we obtain

$$\left|d^{(2k-1)}\right| = |\det(\mathcal{G}_{2k}) \cdot \det(\mathcal{G}_{2k-2}) \cdots \det(\mathcal{G}_2) \cdot d| = \left|\frac{m^{(k-1)}}{m^{(k)}} \frac{m^{(k-2)}}{m^{(k-1)}} \cdots \frac{m^{(2)}}{m^{(3)}} \frac{1}{m^{(2)}} d\right| = \left|\frac{d}{m^{(k)}}\right|.$$

By formual (4.39), it follows that $d \in m^{(k)} \cdot \oslash\Delta$. By Proposition 3.2.3 it holds that $d \in \oslash\Delta$. Hence $d \in \oslash\Delta \cap \Delta$. Because Δ is zeroless, one has a contradiction to Lemma 2.2.22. □

The two next results present conditions to know that the pivot element of $\mathcal{A}^{2k-1}$ is zeroless, without the need to effectuate Gauss operations. This means that the condition (iv) of Definition 4.4.11 is satisfied.

Theorem 4.4.16. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced non-singular matrix over $\mathbb{E}$. Assume that $\left|\frac{m^{(k+1)}}{m^{(k)}}\right| > L_{(k+1)(k+1)}$ for all $1 \leq k \leq n-1$. Then $\left|\alpha_{(k+1)(k+1)}^{(2k-1)}\right|$ is zeroless for all $1 \leq k \leq n-1$.*

Proof. By formula (4.38) one has $N\left(\alpha_{(k+1)(k+1)}^{(2k-1)}\right) \subseteq L_{(k+1)(k+1)}$ and formula (4.29) shows that $a_{(k+1)(k+1)}^{(2k-1)} =$

$\dfrac{m^{(k+1)}}{m^{(k)}}$. By the assumption we have

$$N\left(\alpha^{(2k-1)}_{(k+1)(k+1)}\right) \leq L_{(k+1)(k+1)} < \left|\frac{m^{(k+1)}}{m^{(k)}}\right| = \left|a^{(2k-1)}_{(k+1)(k+1)}\right|.$$

This means that $\alpha^{(2k-1)}_{(k+1)(k+1)}$ is zeroless for k with $1 \leq k \leq n-1$. $\square$

Theorem 4.4.17. *Let $n \in \mathbb{N}$ be standard and $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced non-singular matrix over $\mathbb{E}$. Let $\Delta = \det(\mathcal{A})$. If $\overline{A} \subseteq \oslash\Delta$ then $\left|\dfrac{m^{(k+1)}}{m^{(k)}}\right| > \oslash\Delta$. Moreover $\alpha^{(2k-1)}_{(k+1)(k+1)}$ is zeroless for all $1 \leq k \leq n-1$.*

Proof. Let $\mathcal{A}^{(1)} = \mathcal{G}_1\mathcal{A} = [\alpha^{(1)}_{ij}]_{n\times n}$. Then $a^{(1)}_{22} = m^{(2)}$ and $a^{(1)}_{ij} = m^{(1)}_{ij}$ for all $2 \leq i,j \leq n$. Suppose that $m^{(2)} \in \oslash\Delta$. Because $\left|a^{(1)}_{ij}\right| = \left|m^{(1)}_{ij}\right| \leq \left|m^{(2)}\right| = \left|a^{(1)}_{22}\right|$ for $2 \leq i,j \leq n$, it follows that $\left|a^{(1)}_{ij}\right| \in \oslash\Delta$ for all $2 \leq i,j \leq n$. Let S_{n-1} be the set of all permutations of $\{2,\ldots,n\}$ and $\sigma \in S_{n-1}$. Put $\Delta^{(1)} = \det\left(\mathcal{A}^{(1)}_{2\ldots n,2\ldots n}\right) \equiv d^{(1)} + D^{(1)}$, where

$$d^{(1)} = \sum_{\sigma\in S_{n-1}} \operatorname{sgn}(\sigma) a^{(1)}_{2\sigma(2)} \cdots a^{(1)}_{n\sigma(n)} \in (\oslash\Delta)^{n-1} \subseteq \oslash\Delta.$$

Also $d = d^{(1)}$ since $\mathcal{G}_1$ does not change the determinant of $P = [a_{ij}]_{n\times n}$. Hence $d^{(1)} = d \in \oslash\Delta \cap \Delta$, a contradiction. Thus $\left|\dfrac{m^{(2)}}{m^{(1)}}\right| = \left|m^{(2)}\right| > \oslash\Delta$. On the other hand, by formula (4.38) we have $N(\alpha^{(1)}_{22}) \subseteq \overline{A} \subseteq \oslash\Delta$. So $\alpha^{(1)}_{22}$ is zeroless. We now assume by induction that $\left|\dfrac{m^{(r+1)}}{m^{(r)}}\right| > \oslash\Delta$ for all $1 \leq r \leq k-1$. Because $N\left(\alpha^{(r)}_{(r+1)(r+1)}\right) \subseteq \overline{A} \subseteq \oslash\Delta$, by formula (4.38) it holds that $\alpha^{(r)}_{(r+1)(r+1)} = \dfrac{m^{(r+1)}}{m^{(r)}} + N\left(\alpha^{(r)}_{(r+1)(r+1)}\right)$ is zeroless for all $1 \leq r \leq k-1$. Then Theorem 4.4.15 implies that $\left|\dfrac{m^{(k+1)}}{m^{(k)}}\right| > \oslash\Delta$. Also by formula (4.38) we obtain that $N\left(\alpha^{(2k-1)}_{(k+1)(k+1)}\right) \subseteq \overline{A} \subseteq \oslash\Delta$ and hence $\alpha^{(2k-1)}_{(k+1)(k+1)} = \dfrac{m^{(k+1)}}{m^{(k)}} + N\left(\alpha^{(2k-1)}_{(k+1)(k+1)}\right)$ is zeroless. $\square$

For Gauss-Jordan eliminable flexible systems, entries of $\mathcal{G}_{2k}$ do not make any change to the neutrix parts of constant terms .

Lemma 4.4.18. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Then*

$$\frac{m^{(k+1)}}{m^{(k)}}B = \frac{m^{(k)}}{m^{(k+1)}}B = B \textit{ for all } 1 \leq k \leq n-1.$$

Proof. By Theorem 4.4.17 there exists $t \in [@, \not\oslash]$ such that $\left|\dfrac{m^{(k+1)}}{m^{(k)}}\right| \in t|\Delta|$ for all $1 \leq k \leq n-1$. On the other hand, Proposition 2.2.26 and the facts that Δ is not an absorber of B and that is limited yield that

$\Delta \cdot B = B$. Also $\dfrac{m^{(k+1)}}{m^{(k)}}$ is limited by Lemma 4.4.8. Hence, by Proposition 2.1.3(ii) we have

$$B = \Delta B \subseteq |t\Delta| B = \left|\frac{m^{(k+1)}}{m^{(k)}}\right| B \subseteq \pounds B = B.$$

So $\left|\dfrac{m^{(k+1)}}{m^{(k)}}\right| B = B$ for all $1 \leq k \leq n-1$. This implies that $\left|\dfrac{m^{(k)}}{m^{(k+1)}}\right| B = B$ for all $1 \leq k \leq n-1$. □

Similarly, Gauss operations do not change the neutrix part of the constant terms.

Lemma 4.4.19. *Assume that the system* (4.2) *is Gauss-Jordan eliminable. Then*

$$\mathcal{G}[B] = [B].$$

Proof. We will prove it by external induction. Due to Remark 2.2.27 and the fact that coefficients of reduced systems are limited, we have

$$\mathcal{G}_1[B] \equiv [B^{(1)}]_{n\times 1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

Suppose by induction that $\mathcal{G}_p(\mathcal{G}_{p-1}(\cdots(\mathcal{G}_1([B])))) = [B]$. We will prove that $\mathcal{G}_{p+1}(\mathcal{G}_p(\cdots(\mathcal{G}_1([B])))) = [B]$. We consider two cases.

Case 1: $p+1 = 2k$ for some $k \in \{1, \ldots, n-1\}$. Then, by the inductive hypothesis and Lemma 4.4.18 we have

$$\begin{aligned} \mathcal{G}_{p+1}(\mathcal{G}_p \cdots (\mathcal{G}_1([B]))) =& \mathcal{G}_{2k} \cdot [B] \\ =& \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{m^{(k)}}{m^{(k+1)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}. \end{aligned}$$

Case 2: $p+1 = 2k+1$ for some $k \in \{1, \ldots, n-1\}$. Then, due to Corollary 4.4.9 we have

$$(-1)^{k+i+1} \frac{m_{1\ldots k, 1\ldots(i-1)(i+1)\ldots k+1}}{m^{(k)}} \in \pounds$$

and $\dfrac{m^{(k)}_{i,k+1}}{m^{(k+1)}} \in £$ for all $i, \in \{1,\ldots,n\}, k \in \{1,\ldots,n-1\}$. So, by Remark 2.2.27 we obtain

$$\mathcal{G}_{p+1}(\mathcal{G}_p \cdots (\mathcal{G}_1([B]))) = \mathcal{G}_{2k+1}\cdot [B]$$

$$= \begin{bmatrix} 1 & \cdots & 0 & (-1)^{k+2}\dfrac{m_{1\ldots k,2\ldots(k+1)}}{m^{(k)}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & (-1)^{2k+1}\dfrac{m_{1\ldots k,1\ldots(k-1)(k+1)}}{m^{(k)}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\dfrac{m^{(k)}_{k+2,k+1}}{m^{(k)}} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\dfrac{m^{(k)}_{n,k+1}}{m^{(k)}} & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

Because $n \in \mathbb{N}$ is standard, by external induction we conclude that $\mathcal{G}[B] = \mathcal{G}_{2n-1}(\mathcal{G}_{2n-2}(\cdots(\mathcal{G}_1(B)))) = [B]$. □

As regards to flexible systems, equalities are replaced by inclusions. Next we investigate the relationship between the multiplication of matrices and inclusion relationships.

The result below shows that the multiplication between a matrix and a vector with external numbers preserves inclusion relationships on these vectors.

Lemma 4.4.20. *Let $\mathcal{A} = [\alpha_{ij}]$ be $n \times n$ matrix over $\mathbb{E}$ and $\gamma = (\gamma_1,\ldots,\gamma_n)^T, \beta = (\beta_1,\ldots,\beta_n)^T$ be two column vectors in $\mathbb{E}^n$ such that $\gamma_i \subseteq \beta_i$ for all $i = 1,\ldots,n$. Then*

$$\mathcal{A}\gamma \subseteq \mathcal{A}\beta.$$

Proof. One has $\displaystyle\sum_{j=1}^{n} \alpha_{ij}\gamma_i \subseteq \sum_{j=1}^{n} \alpha_{ij}\beta_i$ for all $i = 1,\ldots,n$. Hence $\mathcal{A}\gamma \subseteq \mathcal{A}\beta$. □

Gauss operations preserve inclusion relationships on vectors with external numbers.

Lemma 4.4.21. Let $\mathcal{G}$ be the Gauss-Jordan matrix of $A = [a_{ij}]_{n\times n}$ on $\mathbb{R}$. Let $\gamma = (\gamma_1,\cdots,\gamma_n)^T, \beta = (\beta_1,\cdots,\beta_n)^T$, where $\gamma_i, \beta_i \in \mathbb{E}$, for all $i \in \{1,\ldots,n\}$ such that $\gamma_i \subseteq \beta_i$. Then $\mathcal{G}(\gamma) \subseteq \mathcal{G}(\beta)$.

Proof. For each $p \in \{1,\ldots,2n-1\}$, let $\mathcal{U}_p = \mathcal{G}_p\mathcal{U}_{p-1} = [u_i]_{n\times 1}, \mathcal{V}_p = \mathcal{G}_p\mathcal{V}_{p-1}$, with $\mathcal{U}_0 = \gamma, \mathcal{V}_0 = \beta$. By Lemma 4.4.20, one has $\mathcal{U}_1 \subseteq \mathcal{V}_1$. Suppose now that $\mathcal{U}_p \subseteq \mathcal{V}_p$. By Lemma 4.4.20 we obtain $\mathcal{U}_{p+1} = \mathcal{G}_{p+1}\mathcal{U}_p \subseteq \mathcal{G}_{p+1}\mathcal{V}_p = \mathcal{V}_{p+1}$. By external induction we conclude that $\mathcal{U}_p \subseteq \mathcal{V}_p$ for all $p = \{1,\ldots,2n-1\}$. In particular for $p = 2n-1$ we obtain $\mathcal{G}(\gamma) \subseteq \mathcal{G}(\beta)$. □

4.4.3 Main results on the Gauss-Jordan elimination method

For Gauss-Jordan eliminable systems the Gauss operations do not change the set of solutions.

Theorem 4.4.22. *Assume that the flexible system* (4.2) *is Gauss-Jordan eliminable. Then the system* (4.2) *is equivalent to the system* $\Big(\mathcal{G}(\mathcal{A})\Big)\xi \subseteq \mathcal{G}(\mathcal{B})$.

We prove this theorem by showing that the set of all real admissible solutions of both systems are equal to the Cramer-solution of the given system. We do this in two steps. In the first step, we prove that the external set of all real admissible solutions of the system (4.2) equals the Cramer-solution. In the second step, we demonstrate that the set of all real admissible solutions of the system $(\mathcal{G}\mathcal{A})\xi \subseteq \mathcal{G}\mathcal{B}$ equals the Cramer-solution of the original system.

Theorem 4.4.23. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Then the set of all its real admissible solutions is equal to the Cramer-solution of the given system.*

Proof. Let S be the external set of all real admissible solutions of the system and $x = (x_1, \dots, x_n)^T \in S$. Let ξ be the Cramer-solution of the system. Because the Cramer-solution is maximal, we have $x \in \xi$ and hence $S \subseteq \xi$.

On the other hand, let $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$ be a representative of $\xi = (\xi_1, \dots, \xi_n)^T$. Then

$$\sum_{j=1}^{n} \alpha_{ij} y_j \subseteq \sum_{j=1}^{n} \alpha_{ij} \xi_j \subseteq \beta_i \quad \text{for all} \quad i \in \{1, \dots, n\}.$$

So y is a real admissible solution of the system and hence $\xi \subseteq S$. Combining these two facts we conclude that $S = \xi$. □

In the next step we will show that the Cramer-solution equals the set of all real admissible solutions of the system $(\mathcal{G}\mathcal{A})\xi \subseteq \mathcal{G}\mathcal{B}$. We call these solutions *Gauss-solutions.*

Definition 4.4.24 ([20]). Let $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ be a real column vector. The column vector x is called a *Gauss-solution* of the system (4.2) if for every representative of α_{ij}, $1 \le i \le m, 1 \le j \le n$, and corresponding matrices one has

$$(\mathcal{G}\mathcal{A})x \subseteq \mathcal{G}\mathcal{B}.$$

So a Gauss-solution is a real admissible solution of the system $(\mathcal{G}\mathcal{A})\xi \subseteq \mathcal{G}\mathcal{B}$.

The below theorem says that the external set of Gauss-solutions is equal to the Cramer-solution.

Theorem 4.4.25. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Then the Cramer-solution of the flexible system* (4.2) *is equal to the external set of all Gauss-solutions.*

To prove this theorem we first demonstrate some auxiliary results. The proposition below generalizes [20, Theorem 5.36, p. 82] for non-homogeneous non-singular flexible systems. Here we state not only for non-homogeneous but also for upper homogeneous flexible systems, in particular, for homogeneous flexible systems. We also give a new proof for it.

Proposition 4.4.26. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Let*

$$x = (x_1, \dots, x_n)^T \in \Big(\frac{\det(M_1)}{\Delta}, \dots, \frac{\det(M_n)}{\Delta}\Big)^T.$$

Then $x = (x_1, \dots, x_n)^T \in \mathcal{M}_{n,1}(\mathbb{R})$ *is a Gauss-solution of* (4.2).

Proof. Because $x_i \in \dfrac{\det(M_i)}{\Delta}$ for all $i \in \{1, \dots, n\}$, we have

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \subseteq \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

By Lemma 4.4.21, this implies

$$\mathcal{G}\left(\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

On the other hand, by Lemma 2.2.28 we have

$$\mathcal{G}\left(\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \left(\mathcal{G}\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Hence

$$\left(\mathcal{G}\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

So $x = (x_1, \dots, x_n)$ is a Gauss-solution of the system (4.2). □

Lemma 4.4.27. *Let* $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$, $[B] = (B, \dots, B)^T$ *be such that* A_{ij}, B *are neutrices for all* $i, j \in \{1, \dots, n\}$ *and* $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$. *If* $u + Au \subseteq B$ *then* $u_i \in B$.

Proof. The vector $Au = \begin{bmatrix} A_{11}u_1 + \cdots + A_{1n}u_n \\ \vdots \\ A_{n1}u_1 + \cdots + A_{nn}u_n \end{bmatrix}$ is a neutrix vector, so $0 \in A_{i1}u_1 + \cdots + A_{in}u_n$ for all $i \in \{1, \dots, n\}$. Hence $u \in u + Au \subseteq B$. □

The following result is stated in [20, Prop. 5.35, p. 80]. We here give a new proof for it.

Proposition 4.4.28. *Suppose that the system* (4.2) *is Gauss-Jordan eliminable. Let* $x = (x_1, \ldots, x_n)^T, y = (y_1, \ldots, y_n)^T$ *be two Gauss-solutions of the system and* $u_i = x_i - y_i$, *for all* $1 \le i \le n$. *Then* $u_i \in B$ *for all* $i = 1, \ldots, n$.

Proof. By subdistributivity, one has

$$\left(\mathcal{G}\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \left(\mathcal{G}\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix}x_1 - y_1\\ \vdots\\ x_n - y_n\end{bmatrix}$$
$$\subseteq \left(\mathcal{G}\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} - \left(\mathcal{G}\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$$
$$\subseteq \mathcal{G}\begin{bmatrix}b_1 + B\\ \vdots\\ b_n + B\end{bmatrix} - \mathcal{G}\begin{bmatrix}b_1 + B\\ \vdots\\ b_n + B\end{bmatrix} = \mathcal{G}\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix} + \mathcal{G}\begin{bmatrix}B\\ \vdots\\ B\end{bmatrix} - \mathcal{G}\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix} - \mathcal{G}\begin{bmatrix}B\\ \vdots\\ B\end{bmatrix} = \mathcal{G}\begin{bmatrix}B\\ \vdots\\ B\end{bmatrix} = \begin{bmatrix}B\\ \vdots\\ B\end{bmatrix},$$

by Lemma 4.4.19. On the other hand,

$$\left(\mathcal{G}\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \left(\mathcal{G}\begin{bmatrix}1 & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} + \left(\mathcal{G}\begin{bmatrix}A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$$
$$= \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} + \left(\mathcal{G}\begin{bmatrix}A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}.$$

It follows that

$$\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} + \left(\mathcal{G}\begin{bmatrix}A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} \subseteq \begin{bmatrix}B\\ \vdots\\ B\end{bmatrix}.$$

By Lemma 4.4.27, $u_i \in B$ for all $i \in \{1, \ldots, n\}$. □

We will extend Theorem 5.37 of [20] which is stated for non-homogeneous systems. We prove that it is true for both non-homogeneous and upper homogeneous systems.

Proposition 4.4.29. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Let* $x = (x_1, \ldots, x_n)^T$ *be a Gauss-solution of the system. Then*

$$x_i \in \frac{\det(M_i)}{\Delta} \quad \textit{for all} \quad 1 \le i \le n.$$

Proof. Let $a_{ij} \in \alpha_{ij}, 1 \leq i, j \leq n$, $b_i \in \beta_i, 1 \leq i \leq n$ and $x = (x_1, \ldots, x_n)$ be a Gauss-solution.

Put

$$d_j = \det \begin{bmatrix} 1 & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

for each $j \in \{1, \ldots, n\}$ and

$$d = \det \begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

By Proposition 4.4.26, the vector $x_0 = (\frac{d_1}{d}, \ldots, \frac{d_n}{d})^T$ is a Gauss-solution of the system (4.2). Let $x = (x_1, \ldots, x_n)^T$ be an arbitrary Gauss-solution of (4.2). Then by Proposition 4.4.28 and Lemma 4.3.15 one has $x_i \in \frac{d_i}{d} + B = \frac{d_i}{d} + N\Big(\frac{\det(M_i)}{\Delta}\Big)$ for all $i = 1, \ldots, n$. Moreover $\frac{d_i}{d} + N\Big(\frac{\det(M_i)}{\Delta}\Big) = \frac{\det(M_i)}{\Delta}$ for all $i = 1, \ldots, n$. Hence $x_i \in \frac{\det(M_i)}{\Delta}$ for all $i = 1, \ldots, n$. □

Proof of Theorem 4.4.25. It follows from Proposition 4.4.26 and Proposition 4.4.29. □

Proof of Theorem 4.4.22. By Theorem 4.4.25, the external set of all Gauss-solutions is the same to the Cramer-solution of the system (4.2). Moreover, by Theorem 4.4.23 the Cramer-solution exactly equals the set of real admissible solutions of the system (4.2). It follows that the sets of real admissible solutions of both systems (4.2) and $(\mathcal{GA})\xi \subseteq \mathcal{B}$ are the same. Therefore, the both systems are equivalent. □

Corollary 4.4.30. *Suppose that the flexible system* (4.2) *is Gauss-Jordan eliminable. Then a vector* $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ *is a real admissible solution of the system* (4.2) *if and only if it is a Gauss-solution.*

The theorem below gives an explicit formula for the set of all Gauss-solutions of the system 4.2. In fact, the vector $\mathcal{G}(\beta)$ is the external set of all Gauss solutions.

Theorem 4.4.31. *Suppose that the flexible system* $\mathcal{A}\xi \subseteq \mathcal{B}$*, where* $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ *with* $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ *and* $\mathcal{B} = [b_i + B]_{n\times 1}$ *with* $b_i + B \in \mathbb{E}$*, is Gauss-Jordan eliminable. Then* $\mathcal{GB}$ *is the external set of all Gauss-solutions of the given flexible system.*

To prove Theorem 4.4.31 we first prove the following lemmas. In two the next lemmas we use following notions

Notation 4.4.32. Consider the Gauss-Jordan eliminable flexible system (4.2) with coefficient matrix $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_n(\mathbb{E})$. We write $A = [A_{ij}]_{n\times n}$, where A_{ij} is the neutrix part of α_{ij} for all $1 \leq i, j \leq n$ and $b = (b_1, \ldots, b_n)^T$ as a representative of $\mathcal{B}$. Let $\mathcal{B}' = \mathcal{GB}$.

Lemma 4.4.33 ([20, prop. 5.32, p. 75])**.** *Assume that the flexible system* (4.2) *is Gauss-Jordan eliminable and non-homogeneous. Then*

$$(\mathcal{G}A) \cdot [B] \subseteq [B].$$

Proof. The condition $R(\mathcal{A}) \subseteq P(B)$ and Proposition 2.2.26 imply that $\overline{A} \subseteq B/\overline{\beta}$. As a consequence

$$A_{ij} \subseteq \overline{A} \subseteq B/\overline{\beta} \subseteq \oslash,$$

for all $1 \leq i, j \leq n$. Due to Lemma 4.4.19 we have

$$\begin{aligned}\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} &\subseteq \mathcal{G}\begin{bmatrix} B/\overline{\beta} & \cdots & B/\overline{\beta} \\ \vdots & \ddots & \vdots \\ B/\overline{\beta} & \cdots & B/\overline{\beta} \end{bmatrix} = \mathcal{G}\begin{bmatrix} B/b & \cdots & B/b \\ \vdots & \ddots & \vdots \\ B/b & \cdots & B/b \end{bmatrix} \\ &= \frac{1}{b}\mathcal{G}\begin{bmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} = \frac{1}{b}\begin{bmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} = \begin{bmatrix} B/b & \cdots & B/b \\ \vdots & \ddots & \vdots \\ B/b & \cdots & B/b \end{bmatrix}.\end{aligned}$$

By Lemma 4.4.19 we obtain that

$$\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \subseteq \begin{bmatrix} B/b & \cdots & B/b \\ \vdots & \ddots & \vdots \\ B/b & \cdots & B/b \end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \subseteq \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash \end{bmatrix} \cdot \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \subseteq \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

□

Lemma 4.4.34. *Assume that the flexible system* (4.2) *is homogeneous and Gauss-Jordan eliminable. Let A as in Notation 4.4.32. We choose $b_i \in \beta_i, 1 \leq i \leq n$ such that $\left|\overline{b}^0\right| = |\overline{b}| = \max_{1\leq i\leq n} |b_i| \in \overline{\beta}$. Put*

$$\mathcal{G}_p \cdots (\mathcal{G}_0(A))) = [A_{ij}^{(p)}]_{1\leq i,j\leq n}, \tag{4.40a}$$

$$\mathcal{G}_p(\mathcal{G}_{p-1} \ldots (\mathcal{G}_0([b]))) = [b_i^{(p)}]_{1\leq i\leq n}, \tag{4.40b}$$

for $p = 0, \ldots, 2n-1$, where $A_{ij}^{(0)} = A_{ij}$, $b_i^0 = b_i$ and $b = (b_1, \ldots, b_n)^T$. Put $\left|\overline{b}^{(p)}\right| = \max_{1\leq i\leq n} \left|b_i^{(p)}\right|$. Then

(i) $A_{ij}^{(p)}\overline{b}^{(p)} \subseteq B$, *for all* $p \in \{0, \ldots, 2n-1\}$.

(ii) $R(\mathcal{G}\mathcal{A}) \subseteq P(\mathcal{B}')$.

Proof. (i) We will prove this part by external induction. For $p = 0$, $\mathcal{G}_0(A) = IA = A$. The condition $R(\mathcal{A}) = \dfrac{\overline{A}}{\Delta} \subseteq P(\mathcal{B})$ and formula (4.40b) yield $A_{ij}\overline{b} \subseteq \overline{A}\,\overline{b} \subseteq \dfrac{\overline{A}}{\Delta}\overline{b} \subseteq \dfrac{\overline{A}}{\Delta}\overline{\beta} \subseteq B$ for all $i, j \in \{1, \ldots, n\}$. Thus the claim is true for $p = 0$. Assume that the claim is true for p. That is $A_{ij}^{(p)}\overline{b}^{(p)} \subseteq B$. We will prove that it is true for $p+1$. This means we need to show that the entries of the matrix $\mathcal{G}_{p+1}[A_{ij}^{(p)}]$ satisfy the condition of (i). Because $\overline{b}^{(p+1)} = \max_{1\leq i\leq n} \left|b_i^{(p+1)}\right| = \left|b_q^{(p+1)}\right|$ for some $q \in \{1, \ldots, n\}$, and by formulas (4.40b), (4.40a) we obtain

$$\overline{b}^{(p+1)} = b_q^{(p+1)} = \left|\sum_{j=1}^{n} g_{qj}^{(p+1)} b_j^{(p)}\right| \leq \sum_{j=1}^{n} \left|g_{qj}^{(p+1)}\right| \left|b_j^{(p)}\right| \leq \sum_{j=1}^{n} \left|g_{qj}^{(p+1)}\right| \overline{b}^{(p)},$$

and

$$A_{ij}^{(p+1)} = g_{i1}^{(p+1)} A_{1j}^{(p)} + \cdots + g_{in}^{(p+1)} A_{nj}^{(p)}. \tag{4.41}$$

If $p+1 = 2k+1$ for some $k \in \{1, \ldots, n-1\}$, by the induction hypotheses and Corollary 4.4.9 which says that $g_{ij}^{(p+1)} \in £$ for all $1 \leq i, j \leq n$, one has

$$\begin{aligned}
A_{ij}^{(p+1)} \overline{b}^{(p+1)} &\subseteq \Big(g_{i1}^{(p+1)} A_{1j}^{(p)} + \cdots + g_{in}^{(p+1)} A_{nj}^{(p)} \Big) \Big(\sum_{j=1}^{n} \Big| g_{ij}^{(p+1)} \Big| \, \overline{b}^{(p)} \Big) \\
&= \Big(g_{i1}^{(p+1)} A_{1j}^{(p)} \overline{b}^{(p)} + \cdots + g_{in}^{(p+1)} A_{nj}^{(p)} \overline{b}^{(p)} \Big) \left(\sum_{j=1}^{n} \Big| g_{ij}^{(p+1)} \Big| \right) \\
&\subseteq \Big(g_{i1}^{(p+1)} B + \cdots + g_{in}^{(p+1)} B \Big) \left(\sum_{j=1}^{n} |g_{ij}^{(p+1)}| \right) \subseteq B.
\end{aligned}$$

If $p+1 = 2k$ for some $k \in \{1, \ldots, n-1\}$, we verify the condition (i) in two separate cases: $i \neq k+1$ and $i = k+1$.

Case 1: For $i \neq k+1$ and $1 \leq i \leq n$, the row $g_i^{(p+1)}$ is a unit vector, so the i-th row in $\mathcal{A}^{(p+1)}$ satisfies $A_i^{(p+1)} = A_i^{(p)}$ and

$$b^{(p+1)} = \left(b_1^{(p)}, \ldots, b_k^{(p)}, \frac{m^{(k)}}{m^{(k+1)}} b_{k+1}^{(p)}, b_{k+2}^{(p)}, \ldots, b_n^{(n)} \right).$$

- If $\overline{b}^{(p+1)} = b_r^{(p)}$ for some $r \in \{1, \ldots, n\} \setminus \{k+1\}$ then for all $i \neq k+1, 1 \leq i \leq n$ and $1 \leq j \leq n$ one has

  $$A_{ij}^{(p+1)} \overline{b}^{(p+1)} = A_{ij}^{(p)} b_r^{(p)} \subseteq A_{ij}^{(p)} \overline{b}^{(p)} \subseteq B,$$

  by the induction hypothesis.

- If $\overline{b}^{(p+1)} = \dfrac{m^{(k)}}{m^{(k+1)}} b_{k+1}^{(p)}$, then for all $i \neq k+1, 1 \leq i \leq n$ and $1 \leq j \leq n$ one has, using Lemma 4.4.18

  $$A_{ij}^{(p+1)} \overline{b}^{(p+1)} = A_{ij}^{(p)} \frac{m^{(k)}}{m^{(k+1)}} b_{k+1}^{(p+1)} \subseteq \frac{m^{(k)}}{m^{(k+1)}} A_{ij}^{(p)} \overline{b}^{(p)} \subseteq \frac{m^{(k)}}{m^{(k+1)}} B = B$$

  .

Case 2: For $i = k+1$, by formula (4.41) one has

$$A_{(k+1)j}^{(p+1)} = A_{ij}^{(p)} \frac{m^{(k)}}{m^{(k+1)}} \quad \text{for all} \quad 1 \leq j \leq n.$$

- If $\overline{b}^{(p+1)} = b_r^{(p)}$ for some $r \in \{1, \ldots, n\} \setminus \{k+1\}$, due to Lemma 4.4.18 for all $1 \leq j \leq n$ one has

$$A_{(k+1)j}^{(p+1)} \overline{b}^{(p+1)} = \frac{m^{(k)}}{m^{(k+1)}} A_{(k+1)j}^{(p)} b_r^{(p)} \subseteq \frac{m^{(k)}}{m^{(k+1)}} A_{(k+1)j}^{(p)} \overline{b}^{(p)} \subseteq \frac{m^{(k)}}{m^{(k+1)}} B = B.$$

- If $\overline{b}^{(p+1)} = \dfrac{m^{(k)}}{m^{(k+1)}} b_{k+1}^{(p)}$ then for all $1 \leq j \leq n$, again using Lemma 4.4.18 one has

$$A_{(k+1)j}^{(p+1)} \overline{b}^{(p+1)} = \frac{m^{(k)}}{m^{(k+1)}} A_{(k+1)j}^{(p)} \frac{m^{(k)}}{m^{(k+1)}} b_{k+1}^{(p+1)} \subseteq \left(\frac{m^{(k)}}{m^{(k+1)}}\right)^2 A_{(k+1)j}^{(p)} \overline{b}^{(p)} \subseteq \left(\frac{m^{(k)}}{m^{(k+1)}}\right)^2 B = B.$$

Hence the claim holds for all $p \in \{0, \ldots, 2n-1\}$.

(ii) Let $\mathcal{G}(A) \equiv A' = [A'_{ij}]$ and $\overline{A'} = \max\limits_{1 \leq i,j \leq n} A'_{ij}$. We consider two cases. If $\overline{b}' + B$ is zeroless, applying Part (i) with $p = 2n-1$ we have

$$A_{ij}^{(2n-1)} \overline{b}^{(2n-1)} = A'_{ij} \overline{b'} \subseteq B \quad \text{for all} \quad 1 \leq i, j \leq n.$$

So $\overline{A'}.\overline{b'} \subseteq B$. Also $\Delta' \equiv \det(\mathcal{GA}) \equiv 1 + D'$ and $\overline{a'} = 1$, hence $R(\mathcal{GA}) = R(\mathcal{A}') = \dfrac{\overline{A'}}{\Delta'} = \overline{A'} \subseteq B/\overline{b'} = P(\mathcal{B}')$. If $\overline{\beta}' = \overline{b}' + B$ is neutricial, by Lemma 4.4.33 we have $A' \cdot [B] = \mathcal{G}(A) \cdot [B] \subseteq [B]$. As a consequence, $\overline{A}' \cdot B \subseteq B$, and since $\Delta' = 1 + D$ we have $R(\mathcal{A}') = \dfrac{\overline{A}'}{\Delta'} = \overline{A}' \subseteq B : B = P(\mathcal{B}')$. $\square$

Proof of Theorem 4.4.31. If the system is homogeneous then $\mathcal{GB} = [B]$. Also, by Theorem 4.3.14, the Cramer-solution of the system is the vector $\xi = (B, \ldots, B)^T$. So $\mathcal{GB}$ is the external set of all Gauss-solution of the given system.

We now assume that the system is non-homogeneous. Let a_{ij}^0 be fixed representatives of α_{ij} for $1 \leq i, j \leq n$ with $a_{11}^0 = 1$. Consider the flexible system

$$(\mathcal{GA})\xi \subseteq \mathcal{GB}. \tag{4.42}$$

Note that $\mathcal{G}(\mathcal{A}) = I_A$ is a near identity matrix and $\mathcal{G}[B] = [B]$ by Lemma 4.4.19. So $N(\mathcal{GB}) = [B]$. Put

$$\mathcal{GB} = [b' + B]. \tag{4.43}$$

Because $\Delta' \equiv \det(\mathcal{GA}) \equiv 1 + D' \subseteq 1 + \oslash$ is zeroless, the system (4.42) is non-singular. Also, obviously, Δ' is not an absorber of B. By Lemma 4.4.34 one has $R(\mathcal{GA}) \subseteq P(\mathcal{GB})$.

Hence the system (4.42) satisfies all the Cramer conditions. In addition, by Proposition 4.3.15 we have $N\left(\dfrac{\det(M'_j)}{\Delta'}\right) = B$ for all $1 \leq j \leq n$, where

$$M'_j = \begin{bmatrix} 1 + A'_{11} & \cdots & A'_{1(j-1)} & b'_1 + B & A'_{1(j+1)} & \cdots & A'_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A'_{n1} & \cdots & A'_{n(j-1)} & b'_n + B & A'_{n(j+1)} & \cdots & 1 + A'_{nn} \end{bmatrix}.$$

Applying Cramer's rule to the system (4.42) the vector $\xi = (\xi_1, \ldots, \xi_n)^T$ with

$$\xi_j = \frac{\det(M'_j)}{\Delta'} = \frac{\det\big(M'_j(a', b')\big)}{1} + N\left(\frac{\det(M'_j)}{\Delta'}\right) = b'_j + B, 1 \leq j \leq n. \tag{4.44}$$

is the Cramer-solution of the system. From formulas (4.43) and (4.44) we obtain $\xi = \mathcal{GB}$. Moreover, because of Theorem 4.4.25, the vector $\xi = (\xi_1, \ldots, \xi_n)$ is the external set of all Gauss-solutions of the system (4.42). By Theorem 4.4.22 we conclude that $\xi = \mathcal{GB}$ is the external set of all Gauss-solutions of the system (4.2). $\square$

4.5 Singular flexible systems

In this section we investigate singular flexible systems of the form

$$\begin{cases} \alpha_{11}\xi_1 + & \alpha_{12}\xi_2 + & \cdots & +\alpha_{1n}\xi_n & \subseteq b_1 + B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}\xi_1 + & \alpha_{m2}\xi_2 + & \cdots & +\alpha_{mn}\xi_n & \subseteq b_m + B_m \end{cases} \tag{4.45}$$

where $m, n \in \mathbb{N}$ are standard and $\alpha_{ij}, \beta_i \in \mathbb{E}$, for all $1 \leq i \leq m, 1 \leq j \leq n$.

Note that in this case we do not require that $m = n$. Even if $m = n$ it may not be necessary that det$(\mathcal{A})$ is zeroless.

In classical linear algebra we know that if a linear system has rank r, the system has exactly r independent equations. So we can reduce the given system to an equivalent system with exact r equations. Then some variables are seen as parameters and the solutions of the system are expressed through these parameters. Here we use a similar technique to deal with singular flexible systems.

This section has the following structure.

In Subsection 4.5.1 we give a necessary condition such that a flexible system has a solution. For a classical system of linear equations we know that if the rank of the coefficient matrix is not equal to the rank of the augmented matrix, the system has no solution. We will generalize this result to a flexible system by using the strict rank of the coefficient matrix and the augmented matrix of a given flexible system. If both strict ranks are equal to each other, we call it simply the *strict rank* of a flexible system.

In Subsection 4.5.2 we will show that a flexible system with identical neutrix parts in the constant term vector can be transformed into an equivalent system such that the entries in each column of the augmented matrix have the same neutrix parts.

In Subsection 4.5.3 we will investigate the relationship between the solutions of a given flexible system and its associated homogeneous system. Recall that in classical linear algebra the set of all solutions of a non-homogeneous linear system equals the sum of the set of solutions of its associated homogeneous linear system and a particular solution of the original system. We will prove that it is still true for flexible systems.

In the next subsections we will consider several special cases of flexible systems. We will provide sufficient conditions such that a flexible system has a solution. A solution formula corresponding to each case is given. In fact, in Subsection 4.5.4 we deal with flexible systems such that the coefficients have the same neutrix parts. In Subsection 4.5.5 we study flexible systems with the strict rank equal to the number of rows. In Subsection 4.5.6 we investigate flexible systems with the strict rank not equal to the numbers of rows.

4.5.1 Necessary condition for the existence of solutions of a flexible system

A flexible system has a solution only if the strict rank of the coefficient matrix is equal to the strict rank of the augmented matrix.

Theorem 4.5.1. *Consider the flexible system* (4.45)*. If* $\mathrm{sr}(\mathcal{A}) \neq \mathrm{sr}\big([\mathcal{A}|\mathcal{B}]\big)$*, the system* (4.45) *has no solution.*

Proof. Assume that $\mathrm{sr}\big([\mathcal{A}|\mathcal{B}]\big) = r$ and $\mathrm{sr}(\mathcal{A}) = s < r$. Let $\alpha_i' = (\alpha_{i1}, \dots, \alpha_{in}, \beta_i) \in \mathbb{E}^{n+1}$ and $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{E}^n$ for $1 \leq i \leq m$. By Theorem 3.4.9 there are exactly r linearly independent row vectors. Without loss of generality, we assume that $V_1 = \{\alpha_1', \dots, \alpha_r'\}$ are linearly independent. Similarly, since $\mathrm{sr}(\mathcal{A}) = s$, we assume that $V_2 = \{\alpha_1, \dots, \alpha_s\}$ is linearly independent with $s < r$. Then there are real numbers $t_1, \dots, t_s$, with at least one of them is not zero, and a neutrix vector $D = (D_1, \dots, D_n)$ such that

$$\alpha_{s+1} + t_1\alpha_1 + \cdots + t_s\alpha_s = (D_1, \dots, D_n). \tag{4.46}$$

On the other hand, vectors $\alpha_1', \cdots, \alpha_{(s+1)}' \subseteq V_1$ is linearly independent, so $\beta_{s+1} + \sum_{i=1}^{s} t_i\beta_i$ is not a neutrix vector. Otherwise, combining with (4.46), one derives that $\alpha_{s+1}' + \sum_{i=1}^{s} t_i\alpha_i'$ is a neutrix vector, and therefore the vectors $\{\alpha_1', \dots, \alpha_{s+1}'\} \subseteq V_1$ are linearly dependent, a contradiction.

Suppose on contrary that the system (4.45) has a solution. Let $x = (x_1, \dots, x_n)$ be a solution of the system. So

$$\sum_{j=1}^{n} \alpha_{ij}x_j \subseteq \beta_i \quad \text{for all} \quad 1 \leq i \leq m.$$

It follows that

$$\sum_{j=1}^{n} \Big(\alpha_{sj} + \sum_{i=1}^{s} t_i\alpha_{ij}\Big)x_j \subseteq \beta_{s+1} + \sum_{i=1}^{s} t_i\beta_i \quad \text{for all} \quad 1 \leq i \leq m. \tag{4.47}$$

However, by formula (4.46), the left side of condition (4.47) is a neutrix vector, while the right side is a zeroless vector, which is a contradiction. Hence the system has no solution. □

In case of the strict ranks of both coefficient and augmented matrices are identical we call it the *rank* of a flexible system.

Definition 4.5.2. Consider the flexible system (4.45). We say that the system has the *strict rank* r if $\mathrm{sr}(\mathcal{A}) = \mathrm{sr}(\mathcal{A}|\mathcal{B}) = r$.

Example 4.5.3. Let $\epsilon > 0$ be infinitesimal. Consider the following system

$$\begin{cases} (1+\oslash)\xi_1 + & (1+\epsilon+\epsilon^2\oslash)\xi_2 + & (1+\oslash)\xi_3 & \subseteq & 1 & + & \epsilon £ \\ & \xi_2 + & \xi_3 & \subseteq & -1/2 & + & \epsilon £. \end{cases}$$

Then the augmented matrix of the system is

$$[\mathcal{A}|\mathcal{B}] = \begin{bmatrix} 1+\oslash & 1+\epsilon+\epsilon^2\oslash & 1+\oslash & 1+\epsilon£ \\ 0 & 1 & 1 & -1/2+\epsilon£ \end{bmatrix}.$$

So $\mathrm{sr}\left([\mathcal{A}|\mathcal{B}]\right) = 2 = \mathrm{sr}(\mathcal{A})$ and hence the strict rank of the given system is 2.

4.5.2 Equivalent flexible systems

We can simplify singular flexible systems by transforming them into equivalent systems in which all the entries in each column of the coefficient matrices have the same neutrix parts.

Definition 4.5.4. A flexible system

$$\begin{cases} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_m & \subseteq b_1+B_1 \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}x_1+ & \cdots & +\alpha_{mn}x_m & \subseteq b_m+B_m \end{cases}$$

is said to be *Gaussian equivalent* to a system

$$\begin{cases} \alpha'_{11}x_1+ & \cdots & +\alpha'_{1n}x_m & \subseteq b'_1+B'_1 \\ \vdots & \ddots & \vdots & \vdots \\ \alpha'_{m1}x_1+ & \cdots & +\alpha'_{mn}x_m & \subseteq b'_m+B'_m \end{cases}$$

if the sets of Gauss-solutions of the two systems are the same.

A flexible system with a constant term vector with identical neutrix parts can be transformed to an equivalent system where neutrix parts of the entries in each column are the same.

Theorem 4.5.5. *Consider the flexible system*

$$\begin{cases} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_m & \subseteq b_1+B \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}x_1+ & \cdots & +\alpha_{mn}x_m & \subseteq b_m+B. \end{cases} \tag{4.48}$$

Then the flexible system

$$\begin{cases} (a'_{11}+\overline{A}_1)x_1+ & \cdots & +(a'_{1n}+\overline{A}_n)x_m & \subseteq b'_1+B \\ \vdots & \ddots & \vdots & \vdots \\ (a'_{m1}+\overline{A}_1)x_1+ & \cdots & +(a'_{m1}+\overline{A}_n)x_m & \subseteq b'_m+B, \end{cases} \tag{4.49}$$

where $\overline{A}_j = \max\limits_{1\le i\le m}\{A_{ij}\}$ *for* $1 \le j \le n$*, is Gaussian equivalent to system* (4.48).

Before we give a proof of this theorem we will illustrate how the theorem works by the following example.

Example 4.5.6. Consider the following flexible system

$$\begin{cases} (1+\oslash)\xi_1+ & (\epsilon £)\xi_2+ & (-2+\epsilon\oslash)\xi_3 & \subseteq 2+\oslash \\ (-3+\epsilon\oslash)\xi_1+ & (2+\oslash)\xi_2+ & (3+\epsilon\oslash)\xi_3 & \subseteq 5+\oslash. \end{cases} \tag{4.50}$$

In the first column the neutrix part in the second row is smaller than in the first row, so we add the first row to the second row and we obtain the equivalent system

$$\begin{cases} (1+\oslash)\xi_1+ & (\epsilon £)\xi_2+ & (-2+\epsilon\oslash)\xi_3 & \subseteq 2+\oslash \\ (-2+\oslash)\xi_1+ & (2+\oslash)\xi_2+ & (-1+\epsilon\oslash)\xi_3 & \subseteq 7+\oslash. \end{cases}$$

Now the neutrix parts in the first column are the same. Next we do it for the second column. The neutrix part in the first row is smaller than in the second row so we add the second row to the first row. Once again, we obtain the equivalent system

$$\begin{cases} (-1+\oslash)\xi_1+ & (2+\oslash)\xi_2+ & (-1+\epsilon\oslash)\xi_3 & \subseteq 9+\oslash \\ (-2+\oslash)\xi_1+ & (2+\oslash)\xi_2+ & (1+\epsilon\oslash)\xi_3 & \subseteq 7+\oslash. \end{cases}$$

Note that the neutrix parts of all entries in each column are now the same.

The following lemma says that adding one row to another does not change the set of real admissible solutions.

Lemma 4.5.7. *Consider the flexible system*

$$\begin{cases} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_n & \subseteq & b_1 & + & B \\ \alpha_{21}x_1+ & \cdots & +\alpha_{2n}x_n & \subseteq & b_2 & + & B. \end{cases} \tag{4.51}$$

Then system (4.51) *is Gaussian equivalent to the following system:*

$$\begin{cases} \alpha_{11}x_1+ & \cdots+ & \alpha_{1n}x_n & \subseteq & b_1 & + & B \\ (\alpha_{21}+\alpha_{11})x_1+ & \cdots+ & (\alpha_{2n}+\alpha_{1n})x_n & \subseteq & b_2+b_1 & + & B. \end{cases} \tag{4.52}$$

Proof. Let $x=(x_1,\dots,x_n)$ be a Gauss-solution of the system (4.51). To prove that x is a Gauss-solution of the system (4.52), we just need to show that x satisfies the second row. Because x is a Gauss-solution of the system (4.51),

$$\begin{cases} \alpha_{11}x_1+\cdots+\alpha_{1n}x_n & \subseteq & b_1 & + & B \\ \alpha_{21}x_1+\cdots+\alpha_{2n}x_n & \subseteq & b_2 & + & B. \end{cases}$$

It follows that

$$\begin{aligned}(\alpha_{21}+\alpha_{11})x_1+\cdots+(\alpha_{2n}+\alpha_{1n})x_n =& (\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)+(\alpha_{21}x_1+\cdots+\alpha_{2n}x_n) \\ \subseteq& b_1+B+b_2+B=b_1+b_2+B.\end{aligned}$$

Conversely, suppose that $x=(x_1,\dots,x_n)$ is a Gauss-solution of the system (4.52). We will show that

$$\alpha_{21}x_1+\cdots+\alpha_{2n}x_n \subseteq (\alpha_{21}+\alpha_{11})x_1+\cdots+(\alpha_{2n}+\alpha_{1n})x_n-(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n). \tag{4.53}$$

Indeed,

$$\begin{aligned}
&(\alpha_{21}+\alpha_{11})x_1+\cdots+(\alpha_{2n}+\alpha_{1n})x_n-(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)\\
=&(\alpha_{21}x_1+\cdots+\alpha_{2n}x_n)+(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)-(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)\\
=&\alpha_{21}x_1+\cdots+\alpha_{2n}x_n+(A_{11}x_1+\cdots+A_{1n}x_n)=(a_{21}+\overline{A}_{.1})x_1+\cdots+(a_{2n}+\overline{A}_{.n})x_n. \qquad (4.54)
\end{aligned}$$

where $\overline{A}_{.j}=\max\{A_{1j},A_{2j}\}$ for $j=1,\dots,n$. Also

$$\alpha_{21}x_1+\cdots+\alpha_{2n}x_n=(a_{21}+A_{21})x_1+\cdots+(a_{2n}+A_{2n})x_n\subseteq(a_{21}+\overline{A}_{.1})x_1+\cdots+(a_{2n}+\overline{A}_{.n})x_n. \qquad (4.55)$$

It follows by formulas (4.54) and (4.55) that formula (4.53) holds.

Because x is a Gauss-solution of the system (4.52), and by formula (4.53) we have

$$\alpha_{21}x_1+\cdots+\alpha_{2n}x_n\subseteq b_1+B+b_2+B-(b_1+B)=b_2+B.$$

So x satisfies the second equation of the system (4.51). Obviously, x satisfies the first equation of the system (4.51). Hence x is a Gauss-solution of the system (4.51).

Thus the two systems (4.51) and (4.52) are Gaussian equivalent. □

Because of the lemma above we can transform every flexible system into a system which has the same neutrix parts in each column. For those systems we can generalize the result above by showing that adding to a row a limited scalar multiple of another row does not change the set of solutions.

Lemma 4.5.8. *Consider the flexible system*

$$\begin{cases}\alpha_{11}x_1 & + & \cdots & + & \alpha_{1n}x_n & \subseteq & b_1+B\\ \alpha_{21}x_1 & + & \cdots & + & \alpha_{2n}x_n & \subseteq & b_2+B,\end{cases} \qquad (4.56)$$

where $A_{ij}=A_j$ for all $i=1,2$. Let $t\in\mathbb{R}$ be limited. Then the system (4.56) *is equivalent to*

$$\begin{cases}\alpha_{11}x_1 & + & \cdots & + & \alpha_{1n}x_n & \subseteq & b_1+B\\ (\alpha_{21}+t\alpha_{11})x_1 & + & \cdots & + & (\alpha_{2n}+t\alpha_{1n})x_n & \subseteq & b_2+tb_1+B.\end{cases} \qquad (4.57)$$

Proof. Let $x=(x_1,\dots,x_n)$ be a Gauss-solution of the system (4.56). To prove that x is a Gauss-solution of the system (4.57), we just need to show that x satisfies the second row. Since x is a Gauss-solution of the system (4.56), we have

$$\begin{cases}\alpha_{11}x_1+\cdots+\alpha_{1n}x_n & \subseteq b_1+B\\ \alpha_{21}x_1+\cdots+\alpha_{2n}x_n & \subseteq b_2+B.\end{cases}$$

It follows that

$$\begin{aligned}(\alpha_{21}+t\alpha_{11})x_1+\cdots+(\alpha_{2n}+t\alpha_{1n})x_n =& t\big(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n\big)+\big(\alpha_{21}x_1+\cdots+\alpha_{2n}x_n\big)\\ \subseteq& tb_1+tB+b_2+B=tb_1+b_2+B,\end{aligned}$$

since $t \in £$.

Conversely, suppose that $x=(x_1,\dots,x_n)$ is a Gauss-solution of the system (4.57). We will show that

$$\alpha_{21}x_1+\cdots+\alpha_{2n}x_n=(\alpha_{21}+t\alpha_{11})x_1+\cdots+(\alpha_{2n}+t\alpha_{1n})x_n-t(\alpha_{11}x_1+\cdots+t\alpha_{1n}x_n). \tag{4.58}$$

Indeed, because $t\in £$, one has $tA_j \subseteq A_j$ for all $1\le j\le n$. It follows that

$$\begin{aligned}&(\alpha_{21}+t\alpha_{11})x_1+\cdots+(\alpha_{2n}+t\alpha_{1n})x_n-t(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)\\ =&\big(\alpha_{21}x_1+\cdots+\alpha_{2n}x_n\big)+t\big(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n-t(\alpha_{11}x_1+\cdots+\alpha_{1n}x_n)\big)\\ =&\alpha_{21}x_1+\cdots+\alpha_{2n}x_n+(tA_{11}x_1+\cdots+tA_{1n}x_n)=\alpha_{21}x_1+\cdots+\alpha_{2n}x_n.\end{aligned}$$

Because x is a Gauss-solution of the system (4.57) and by formula (4.58),

$$\alpha_{21}x_1+\cdots+\alpha_{2n}x_n\subseteq tb_1+b_2+B-(tb_1+B)=b_2+B.$$

It shows that x satisfies the second equation of the system (4.56). Obviously, x satisfies the first equation of the system (4.56). Hence x is a Gauss-solution of the system (4.56).

So systems (4.56) and (4.57) are equivalent. $\square$

Proof of Theorem 4.5.5. For each column j, let $\overline{A}_j=\max\limits_{1\le i\le m}\{A_{ij}\}=A_{kj}$ for some $k\in\{1,\dots,m\}$. For all $i\neq k, i\in\{1,\dots,m\}$, if $A_{ij}\subset A_{kj}=\overline{A}_j$, we add the k-th row to i-th row. The new transformed system is equivalent to the given system by Lemma 4.5.7. Also $A_{ij}=\overline{A}_j$ for all $i=1,\dots,m$. Applying this process for all $j\in\{1,\dots,n\}$ the system will take the form (4.49), and this last system is equivalent to the system (4.48). $\square$

Convention 4.5.9. From now on, we consider flexible systems in the form

$$\begin{cases}\alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_n & \subseteq & \beta_1=b_1+B\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ \alpha_{m1}x_1+ & \cdots & +\alpha_{mn}x_n & \subseteq & \beta_m=b_m+B.\end{cases} \tag{4.59}$$

Note that in this form we have $\overline{B}=\underline{B}=B$. By Theorem 4.5.5, we can always assume that $\alpha_{ij}=a_{ij}+A_{ij}$ with $A_{ij}=A_{kj}=A_j$ for all $i\neq k; i,k\in\{1,\dots,m\}$ and for all $j=1,\dots,n$. This means the neutrix parts of the entries on each column in $\mathcal{A}$ are the same.

Suppose that the strict rank of the system is r, where $r\le \min\{m,n\}$. Hence, without loss of generality, we may assume that $\mathcal{A}_l = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix}$ is non-singular with $\Delta = \det(\mathcal{A}_l)$ zeroless.

We also suppose that the flexible system (4.59) satisfies the following assumptions:

(i) The submatrix A_l is reduced.

(ii) Let $\overline{A_l} = \max\limits_{1 \le j \le r} A_j$ and $\mathcal{A}_r = [\alpha_{ij}]$ be the submatrix of $\mathcal{A}$ formed by all entries of $\mathcal{A}$ which do not belong to $\mathcal{A}_l$. Let $\underline{A_r} = \min\limits_{r+1 \le j \le n} A_j$ be the minimum neutrix part of the entries in $\mathcal{A}_r$. Then $\overline{A_l} \subseteq \underline{A_r}$.

(iii) $R(\mathcal{A}_l) \subseteq P(\mathcal{B})$.

(iv) Δ is not an absorber of B.

Definition 4.5.10. A flexible system of the form (4.59) satisfying all the assumptions above is said to be *solvable*.

Remark 4.5.11. If all the coefficients of a given system have the same neutrix parts, the second assumption is satisfied automatically.

4.5.3 An associated homogeneous system

In this subsection we will present a relationship between the sets of solutions of a given flexible system and its homogeneous system.

Definition 4.5.12. The system of the form

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \subseteq \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} \tag{4.60}$$

is called the *associated homogeneous flexible system* of the system (4.45).

We denote by S_O, S_H the sets of all Gauss-solutions of the system (4.45) and (4.60), respectively.

Proposition 4.5.13. *Consider the system* (4.45). *Suppose that* $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T$ *are two Gauss-solutions of the system. Put* $u_i = x_i - y_i$, *for* $1 \le i \le n$. *Then*

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \subseteq \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}. \tag{4.61}$$

Proof. Because both x, y are two Gauss-solutions of the given system, we have

$$\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n \subseteq \beta_i$$

and

$$\alpha_{i1}y_1 + \cdots + \alpha_{in}y_n \subseteq \beta_i$$

for all $1 \le i \le m$. It follows from the subdistributivity that for all $i \in \{1, \ldots, m\}$,

$$\begin{aligned}\alpha_{i1}u_1 + \cdots + \alpha_{in}u_n =& \alpha_{i1}(x_1 - y_1) + \cdots + \alpha_{in}(x_n - y_n)\\ \subseteq& \big(\alpha_{i1}x_1 + \cdots \alpha_{in}x_n\big) - \big(\alpha_{i1}y_1 + \cdots \alpha_{in}y_n\big)\\ \subseteq& \beta_i - \beta_i = B_i.\end{aligned}$$

Hence u satisfies (4.61). □

We show that the set of all solutions of a non-homogeneous flexible system can be expressed as the sum of the set of solutions of its associated homogeneous system and a particular solution of the original system.

Theorem 4.5.14. *Consider the two systems* (4.45) *and* (4.60)*. We have*

$$S_O = x_0 + S_H,$$

where $x_0 = (x_1^0, \ldots, x_n^0)^T$ is a particular Gauss-solution of the system (4.45)*.*

Proof. Let $x = (x_1, \cdots, x_n)^T \in S_O$. By Proposition 4.5.13, it holds that $u = (u_1, \ldots, u_n)^T$, with $u_i = x_i - x_i^0$ for $1 \le i \le n$, is a solution of the system (4.60). This means that $u \in S_H$ and hence $x = x_0 + u \in x_0 + S_H$. So $S_O \subseteq x_0 + S_H$.

On the other hand, let $u = (u_1, \ldots, u_n)^T \in S_H$. Then the system (4.60) is satisfied by u. Let $x = x_0 + u$. One has for each $i \in \{1, \ldots, n\}$,

$$\begin{aligned}\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n =& \alpha_{i1}(x_1^0 + u_1) + \cdots + \alpha_{in}(x_n^0 + u_n)\\ \subseteq& \big(\alpha_{i1}x_1^0 + \cdots \alpha_{in}x_n^0\big) + \big(\alpha_{i1}u_1 + \cdots \alpha_{in}u_n\big)\\ \subseteq& \beta_i + B_i = \beta_i.\end{aligned}$$

Hence $x \in S_O$ and therefore $x_0 + S_H \subseteq S_O$. □

According to the proposition above, to solve a system $\mathcal{A}.\xi \subseteq \mathcal{B}$ we just need to find a concrete Gauss-solution and solve the associated homogeneous flexible system of the given system. Also note that if $x_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in \mathbb{R}^n$ is a solution of the linear system

$$\begin{cases} a_{11}x_1 + & \cdots & +a_{1n}x_n & = b_1\\ \vdots & \ddots & \vdots & \vdots\\ a_{m1}x_1 + & \cdots & +a_{mn}x_n & = b_m,\end{cases} \tag{4.62}$$

where $a_{ij} \in \alpha_{ij}, b_i \in \beta_i.1 \le i \le n, 1 \le j \le m$, it is also a solution of the flexible system (4.45). This guides us how to find a concrete Gauss-solution of a given system.

4.5.4 The flexible system with identical neutrix parts

In this subsection we consider a special form of flexible systems in which the strict rank of the system is equal to the number of equations and all neutrix parts of the coefficients are the same. To be more detailed, we investigate the reduced system of the form:

$$\begin{cases} \alpha_{11}x_1 + & \alpha_{12}x_2 + & \cdots & +\alpha_{1n}x_n & \subseteq b_1 + B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}x_1 + & \alpha_{m2}x_2 + & \cdots & +\alpha_{mn}x_n & \subseteq b_m + B, \end{cases} \tag{4.63}$$

where $\alpha_{ij} = a_{ij} + A$ and $|\alpha_{ij}| = |a_{ij} + A| \leq 1 + \oslash$; $A \subseteq \oslash$ and $\mathrm{sr}(\mathcal{A}) = m \leq n$, here $\mathcal{A} = [\alpha_{ij}] \in \mathcal{M}_{m,n}(\mathbb{E})$ is the coefficient matrix of the system (4.63). Since $\mathrm{sr}(\mathcal{A}) = m$, without loss of generality, we assume that

$$\Delta = \det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix} = d + D$$

is zeroless. Let

$$\mathcal{A}_l = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix},$$

and

$$|\alpha_l| = \max_{1 \leq i,j \leq m} |\alpha_{ij}|.$$

We write

$$M_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & b_1 + B - \sum_{k=m+1}^{n} a_{1k}x_k & \alpha_{1(j+1)} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{m(j-1)} & b_m + B - \sum_{k=m+1}^{n} a_{mk}x_k & \alpha_{m(j+1)} & \cdots & \alpha_{mm} \end{bmatrix} (m+1 \leq j \leq n). \tag{4.64}$$

Note first that although the system (4.63) is reduced the matrix $\mathcal{A}_l$ may be not.

Theorem 4.5.15. *Assume that the system* (4.63) *satisfies following conditions:*

(i) $\mathcal{A}_l$ *is an* $m \times m$ *reduced matrix,*

(ii) Δ *is not an absorber of* B*,*

(iii) $R(\mathcal{A}_l) \subseteq P(\mathcal{B})$.

Then the set of all Gauss-solutions of the system (4.63) *is given by*

$$S=\left\{(x_1,\dots,x_n)^T\middle| x_1\in\frac{\det(M_1)}{\Delta},\dots,x_m\in\frac{\det(M_m)}{\Delta},x_j\in B:A,\ \forall j=m+1,\dots,n\right\}. \quad (4.65)$$

Proof. Observe that if $x=(x_1,x_2,\dots,x_n)^T$ is a Gauss-solution of the system (4.63) then $A.x_j\subseteq B$ for all $1\le j\le n$, that is

$$x_j\in\big(B:A\big)\quad\text{for all}\quad j=1,\dots,n. \quad (4.66)$$

Let $x_j\in\big(B:A\big)$ for all $j=m+1,\dots,n$. Then the system (4.63) is equivalent to

$$\begin{cases}\alpha_{11}x_1 & + & \cdots & + & \alpha_{1m}x_m & \subseteq & b_1+B & - & \alpha_{1m+1}x_m & - & \cdots & - & \alpha_{1n}x_n\\ \alpha_{21}x_1 & + & \cdots & + & \alpha_{2m}x_m & \subseteq & b_2+B & - & \alpha_{2m+1}x_m & - & \cdots & - & \alpha_{2n}x_n\\ \vdots & & \ddots & & \vdots & & \vdots & & \vdots & & \ddots & & \vdots\\ \alpha_{m1}x_1 & + & \cdots & + & \alpha_{mm}x_m & \subseteq & b_m+B & - & \alpha_{mm+1}x_m & - & \cdots & - & \alpha_{mn}x_n.\end{cases}$$

Because $\sum_{j=m+1}^{n}A_{ij}x_j\subseteq B$ for all $1\le i\le m$, the system above becomes

$$\begin{cases}\alpha_{11}x_1 & + & \cdots & + & \alpha_{1m}x_m & \subseteq & b_1+B & - & a_{1m+1}x_m & - & \cdots & - & a_{1n}x_n\\ \alpha_{21}x_1 & + & \cdots & + & \alpha_{2m}x_m & \subseteq & b_2+B & - & a_{2m+1}x_m & - & \cdots & - & a_{2n}x_n\\ \vdots & & \ddots & & \vdots & & \vdots & & \vdots & & \ddots & & \vdots\\ \alpha_{m1}x_1 & + & \cdots & + & \alpha_{mm}x_m & \subseteq & b_m+B & - & a_{mm+1}x_m & - & \cdots & - & a_{mn}x_n.\end{cases} \quad (4.67)$$

We will show that the system (4.67) satisfies all the Cramer conditions in Definition 4.3.3 for all values $x_j\in B:A, m+1\le j\le n$. To do this we just need to verify that for $m+1\le j\le n, 1\le i\le m$ and for all $x_j\in(B:A)$,

$$\frac{A\left(b_i+B-\sum_{j=m+1}^{n}a_{ij}x_j\right)}{d}\subseteq B.$$

Because $|a_{ij}|\le 1+\oslash$, one has for all $i=1,\dots,m$,

$$\begin{aligned}A\left(b_i+B-\sum_{j=m+1}^{n}a_{ij}x_j\right) &= AB+\left(b_i-\sum_{j=m+1}^{n}a_{ij}x_j\right)A\\ &\subseteq AB+b_iA-\sum_{j=m+1}^{n}(a_{ij}x_j\cdot A)\subseteq AB+b_iA-\sum_{j=m+1}^{n}a_{ij}B\\ &\subseteq AB+b_iA-\sum_{j=m+1}^{n}(1+\oslash)B\subseteq B+b_iA-B=b_iA+B.\end{aligned}$$

Also $R(\mathcal{A}_l)\subseteq P(\mathcal{B})$. So for all $1\le i\le m$,

$$b_iA/\Delta\subseteq B.$$

Indeed, if $\overline{\beta}$ is zeroless then $\overline{\beta}A/\Delta = \overline{b}A/\Delta \subseteq B$. It follows that $b_iA/\Delta \subseteq B, 1 \leq i \leq m$. Similarly, if $\overline{\beta}$ is a neutrix then $\overline{\beta} = B$ and $b_i \subseteq \overline{\beta} = B$. So $b_iA/\Delta \subseteq BA/\Delta \subseteq B$ for $1 \leq i \leq m$.

Moreover $|\Delta| = |d + D| \leq £$ is zeroless and not an absorber of B, so $\dfrac{B}{\Delta} = \dfrac{B}{d} = B$ by Proposition 2.2.26. Consequently, for all $i = 1, \ldots, m$,

$$\frac{A\left(b_i + B - \displaystyle\sum_{j=m+1}^{n} a_{ij}x_j\right)}{d} \subseteq \frac{b_iA + B}{d} \subseteq B + B = B.$$

Applying Theorem 4.3.8 to the system (4.67), one has $\xi_j = \dfrac{\det(M_j)}{\Delta}$ for $1 \leq j \leq m$ is the maximal solution of the system (4.67) with respect to $x_j \in B : A$ $(m+1 \leq j \leq n)$. So $\left\{\left(\dfrac{\det(M_1)}{\Delta}, \ldots, \dfrac{\det(M_m)}{\Delta}\right)\right\}$ is the set of all Gauss-solutions of the system (4.67) by Theorem 4.4.25. That is, the set S given by (4.65) is the set of all Gauss-solutions of the system (4.63). $\square$

Example 4.5.16. Let $\epsilon > 0$ be an infinitesimal. Consider the following homogeneous flexible system

$$\begin{cases} (1+\oslash)x_1 & + & (\epsilon+\oslash)x_2 & + & \oslash x_3 & \subseteq \oslash \\ (-1+\oslash)x_1 & + & (1+\oslash)x_2 & + & (\frac{1}{2}+\oslash)x_3 & \subseteq \oslash. \end{cases}$$

The system is reduced with $\overline{\alpha} = 1 + \oslash$ and $B = \oslash$. The determinant

$$\Delta = \det \begin{bmatrix} 1+\oslash & \epsilon+\oslash \\ -1+\oslash & 1+\oslash \end{bmatrix} = 1 - \epsilon + \oslash = 1 + \oslash \in @$$

is not an absorber of B. Let $\mathcal{A}_l = \begin{bmatrix} 1+\oslash & \epsilon+\oslash \\ -1+\oslash & 1+\oslash \end{bmatrix}$. Then $\mathcal{A}_l$ is a reduced matrix, $R(\mathcal{A}_l) = \dfrac{A}{\Delta} = \oslash$ and $P(\mathcal{B}) = \oslash : \oslash = £$, so $R(\mathcal{A}_l) \subseteq P(\mathcal{B})$ and hence all of the conditions of the Theorem 4.5.15 are satisfied. For $x_3 \in \oslash : \oslash = £$, the system is equivalent to

$$\begin{cases} (1+\oslash)x_1 & + & (\epsilon+\oslash)x_2 & \subseteq & & \oslash \\ (-1+\oslash)x_1 & + & (1+\oslash)x_2 & \subseteq & -\frac{1}{2}x_3 + & \oslash. \end{cases}$$

Applying Cramer′s rule to this system, one has

$$\xi_1 = \det \begin{bmatrix} \oslash & \epsilon+\oslash \\ -\frac{1}{2}x_3+\oslash & 1+\oslash \end{bmatrix} /\Delta = \oslash$$
$$\xi_2 = \det \begin{bmatrix} 1+\oslash & \oslash \\ -1+\oslash & -\frac{1}{2}x_3+\oslash \end{bmatrix} /\Delta = -\frac{1}{2}x_3 + \oslash.$$

Hence the set of all Gauss-solutions of the system is given by

$$S = \left\{ \left(\oslash, -\frac{1}{2}x_3 + \oslash, x_3 \right) \middle| x_3 \in £ \right\}.$$

As for flexible systems whose strict rank equals the number of rows but the neutrix parts are not identical, we can apply the theorem to them by using upper and lower neutrix of the entries in $\mathcal{A}$ allowing to find the upper and lower bounds of the set of all Gauss-solutions. The theorem below shows the relationships between these two sets of solutions.

Theorem 4.5.17. *Consider the following reduced system*

$$\begin{cases} \alpha_{11}x_1 + & \alpha_{12}x_2 + & \cdots & +\alpha_{1n}x_n & \subseteq b_1 + B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}x_1 + & \alpha_{m2}x_2 + & \cdots & +\alpha_{mn}x_n & \subseteq b_m + B, \end{cases} \tag{4.68}$$

where $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $|\alpha_{ij}| = |a_{ij} + A_{ij}| \leq 1 + \oslash$. Suppose that the strict rank of the system is m. Let $\overline{A} = \max\limits_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} A_{ij}$ and $\underline{A} = \min\limits_{\substack{1 \leq j \leq n \\ 1 \leq i \leq m}} A_{ij}$. Consider the two following systems, respectively

$$\begin{cases} \lambda_{11}x_1 + & \lambda_{12}x_2 + & \cdots & +\lambda_{1n}x_n & \subseteq b_1 + B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{m1}x_1 + & \lambda_{m2}x_2 + & \cdots & +\lambda_{mn}x_n & \subseteq b_m + B, \end{cases} \tag{4.69}$$

where $\lambda_{ij} = a_{ij} + \underline{A}$ and

$$\begin{cases} \gamma_{11}x_1 + & \gamma_{12}x_2 + & \cdots & +\gamma_{1n}x_n & \subseteq b_1 + B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{m1}x_1 + & \gamma_{m2}x_2 + & \cdots & +\gamma_{mn}x_n & \subseteq b_m + B, \end{cases} \tag{4.70}$$

where $\gamma_{ij} = a_{ij} + \overline{A}$.

Let S_1, S, S_2 be the sets of all Gauss solutions of the systems (4.70)*,* (4.68)*,* (4.69)*, respectively. Then*

$$S_1 \subseteq S \subseteq S_2.$$

Furthermore, the conclusion does not depend on choosing $a_{ij} \in \alpha_{ij}$.

Proof. As for the first inclusion, let $x = (x_1, x_2, \ldots, x_n)^T \in S_1$. Then $\sum\limits_{j=1}^{n} a_{ij}x_j \in b_i + B$, for all $a_{ij} \in \alpha_{ij}$ and $x_j \cdot \overline{A} \subseteq B$ for all $i = 1, \ldots, m$. As a consequence, $x_j \cdot A_{ij} \subseteq x_j\overline{A} \subseteq B$. So

$$\sum_{j=1}^{n} a_{ij}x_j + x_j \cdot A_{ij} \subseteq b_i + B \text{ for } i = 1, \ldots, m.$$

That means $\sum_{j=1}^{n} \alpha_{ij}x_j \subseteq \beta_i$, for all $i \in \{1,\dots,m\}$ and hence $x \in S$. Since $x \in S_1$ is arbitrary, this implies $S_1 \subseteq S$.

As for the second inclusion, let $y = (y_1, y_2, \dots, y_n)^T \in S$ be arbitrary. Then $\sum_{j=1}^{n} a_{ij}y_j \in b_i + B$ for all $a_{ij} \in \alpha_{ij}$ and $y_j \cdot \underline{A} \subseteq y_j \cdot A_{ij} \subseteq B$ for all $i \in \{1,\dots,m\}$. It follows that $\sum_{j=1}^{n} (a_{ij}y_j + y_j \cdot \underline{A}) \subseteq b_i + B$ for all $i = 1,\dots,m$. So $y = (y_1,\dots,y_n)^T \in S_2$ and hence $S \subseteq S_2$. □

4.5.5 The flexible system with the strict rank equal to the number of rows

In this subsection we consider flexible systems of the from (4.59) such that the strict rank of a given system is equal to the number of equations. Because of Convention 4.5.9 one has $\mathcal{A}_l = [\alpha_{ij}]_{m\times m}$ is non-singular. We will give some conditions to guarantee that singular flexible systems can be solved.

Put

$$M_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & b_1 + B - \sum_{k=m+1}^{n} a_{1k}x_k & \alpha_{1(j+1)} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{m(j-1)} & b_m + B - \sum_{k=m+1}^{n} a_{mk}x_k & \alpha_{m(j+1)} & \cdots & \alpha_{mm} \end{bmatrix},$$

$$\text{for } j \in \{1,\dots,m\} \text{ and } N_j = B : A_j \text{ for } j \in \{m+1,\dots,n\}. \tag{4.71}$$

Theorem 4.5.18. *Assume that the flexible system* (4.59) *is solvable and* $r = m$, *where* r *is the strict rank of the system. Then the external set of all Gauss-solutions of the system is given by*

$$S = \left\{ (x_1,\dots,x_n)^T \;\middle|\; x_j \in \frac{\det(M_j)}{\Delta} \textit{ for } 1 \le j \le m \textit{ and } x_j \in N_j \quad for \quad m+1 \le j \le n \right\}. \tag{4.72}$$

Remark 4.5.19. There is an analogy between formula (4.72) and parameter presentations of a solution of a given system of equation in classical linear algebra. The first part of formula (4.72) expresses a particular solution. As for the second part, classically the parameters range over $\mathbb{R}$ and here over neutrices.

Proof of Theorem 4.5.18. Suppose that $x = (x_1,\dots,x_n)^T$ is a Gauss-solution of the system (4.59). Then $x_j \in B : A_j$ for all $j = 1,\dots,n$. For each $j = m+1,\dots,n$, we choose $x_j \in B : A_j$. Then the system is equivalent to the following one:

$$\begin{cases} \alpha_{11}x_1 + & \cdots & +\alpha_{1m}x_m & \subseteq b_1 + B & -a_{1(m+1)}x_{m+1} - & \cdots & -a_{1n}x_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}x_1 + & \cdots & +\alpha_{mm}x_m & \subseteq b_m + B & -a_{m(m+1)}x_{m+1} - & \cdots & -a_{mn}x_n. \end{cases} \tag{4.73}$$

We will show that all the Cramer conditions in Definition 4.3.3 are satisfied for the system (4.73). We just need to verify the condition $R(\mathcal{A}_m) \subseteq P(\mathcal{B})$, where $P(\mathcal{B}) = B/\overline{\beta}$ with

$$|\overline{\beta}| = \max_{1\leq i\leq m} \left\{ \left| b_i - \left(a_{i(m+1)}x_{m+1} + \cdots + a_{in}x_n \right) \right| \right\},$$

i.e. we need to verify the following condition

$$\left(b_i - \sum_{j=m+1}^{n} a_{ij}x_j \right) \cdot \frac{\overline{A}_l}{\Delta} \subseteq B$$

for all $i = 1, \ldots, m$ and for all $x_j \in B : A_j, j = m+1, \ldots, n$. For each $x_j \in B : A_j, m+1 \leq j \leq n$, since $|a_{ij}| \leq 1 + \oslash$, it follows that

$$a_{ij}x_j \in a_{ij}\left(B : A_j\right) \subseteq B : \underline{A}_r, \quad \text{for all} \quad 1 + m \leq j \leq n, 1 \leq i \leq m.$$

Then

$$a_{ij}x_j \cdot \frac{\overline{A}_l}{\Delta} \subseteq (B : \underline{A}_r) \cdot \frac{\overline{A}_l}{\Delta} = \frac{B}{\Delta} \cdot \left(\overline{A}_l : \underline{A}_r \right) \quad \text{for all} \quad 1 + m \leq j \leq n, 1 \leq i \leq m. \tag{4.74}$$

Moreover, by the second hypothesis $\overline{A}_l \subseteq \underline{A}_r$, one has $\overline{A}_l : \underline{A}_r \subseteq \pounds$. It follows that

$$\frac{B}{\Delta} \cdot \left(\overline{A}_l : \underline{A}_r \right) \subseteq \pounds \cdot \frac{B}{\Delta} = B. \tag{4.75}$$

By formulas (4.74), (4.75) we have

$$a_{ij}x_j . \frac{\overline{A}_l}{\Delta} \subseteq B \text{ for all } i = 1, \ldots, m \text{ and } j = m+1, \ldots, n.$$

Also, by condition $R(\mathcal{A}_l) \subseteq P(\mathcal{B})$, it follows that $\dfrac{\overline{A}_l}{\Delta} b_i \subseteq B$ for all $i = 1, \ldots, m$. Because of subdistributivity, we have

$$\left(b_i - \sum_{j=m+1}^{n} a_{ij}x_j \right) \cdot \frac{\overline{A}_l}{\Delta} \subseteq b_i \frac{\overline{A}_l}{\Delta} - \sum_{j=m+1}^{n} \left(a_{ij}x_j . \frac{\overline{A}_l}{\Delta} \right) \subseteq B + (n-m)B = B, \text{ for all } i = 1, \ldots, m.$$

Applying Theorem 4.3.8 to the system (4.73), the vector $\left(\dfrac{\det(M_n)}{\Delta}, \ldots, \dfrac{\det(M_m)}{\Delta} \right)^T$ is the Cramer-solution, and hence the set of all Gauss-solutions of the system (4.73) corresponds to each $x_j \in B : A_j$, for $j = m+1, \ldots, n$.

Thus the external set of all Gauss-solutions of the system (4.59) is given by (4.72). □

Remark 4.5.20. Although the arguments in the proof base on Cramer's rule, we can also apply Gauss-Jordan elimination to solve the non-singular system (4.73) for $x_j \in N_j, j = m+1, \ldots, n$ if $\mathcal{A}_l$ is Gauss-Jordan eliminable.

Example 4.5.21. Consider the flexible system, with $\epsilon > 0$ infinitesimal,

$$\begin{cases} (1+\epsilon\oslash)x_1 & + & \epsilon^2£x_2 & + & \epsilon£x_3 & + & (1/2+\oslash)x_4 & \subseteq & \oslash \\ (-1+\epsilon\oslash)x_1 & + & (1+\epsilon^2£)x_2 & + & (1+\epsilon£)x_3 & + & (-1+\oslash)x_4 & \subseteq & \oslash. \end{cases}$$

The system is reduced and $\mathcal{A}_l = \begin{bmatrix} 1+\epsilon\oslash & \epsilon^2£ \\ -1+\epsilon\oslash & 1+\epsilon^2£ \end{bmatrix}$ is a reduced matrix such that $\Delta = \det \begin{bmatrix} 1+\epsilon\oslash & \epsilon^2£ \\ -1+\epsilon\oslash & 1+\epsilon^2£ \end{bmatrix} = 1+\epsilon\oslash$ is zeroless.

(i) $A_2 = \epsilon^2£ \subset A_1 = \epsilon\oslash \subset A_3 = \epsilon£ \subset \oslash = A_4$, hence the second assumption is satisfied with $\overline{A}_l = \epsilon\oslash \subseteq \underline{A}_r = \epsilon£$.

(ii) the strict rank of the system equals the number of equations which is 2.

(iii) $R(\mathcal{A}_l) = \epsilon\oslash \subseteq P(\mathcal{B}) = \oslash : \epsilon\oslash = \frac{1}{\epsilon}£$.

(iv) $\Delta \in @$ is not an absorber of $B = \oslash$.

Thus the system satisfies all conditions above. Due to Theorem 4.5.18, one has

$$\det(M_1) = \det \begin{bmatrix} -1/2x_4+\oslash & \epsilon^2£ \\ -x_3+x_4+\oslash & 1+\epsilon£ \end{bmatrix} = -\frac{1}{2}x_4+\oslash$$
$$\det(M_2) = \det \begin{bmatrix} 1+\epsilon\oslash & -1/2x_4+\oslash \\ -1+\epsilon\oslash & -x_3+x_4+\oslash \end{bmatrix} = -x_3+\frac{1}{2}x_4+\oslash$$

and

$$\begin{cases} x_1 & \in \dfrac{\det(M_1)}{\Delta} = -\dfrac{1}{2}x_4+\oslash \\ x_2 & \in \dfrac{\det(M_2)}{\Delta} = -x_3+\dfrac{1}{2}x_4+\oslash. \end{cases}$$

Hence the set of all Gauss-solutions of the system is

$$S = \left\{ \left(-\frac{1}{2}x_4+\oslash, -x_3+\frac{1}{2}x_4+\oslash, x_3, x_4 \right) \middle| x_3 \in \frac{1}{\epsilon}\oslash, x_4 \in £ \right\}. \tag{4.76}$$

Now we solve the system by Gauss-Jordan elimination.

The augmented matrix of the system is of the form

$$[\mathcal{A}|\mathcal{B}] = \begin{bmatrix} 1+\epsilon\oslash & \epsilon^2£ & \epsilon£ & 1/2+\oslash & \oslash \\ -1+\epsilon\oslash 1 & 1+\epsilon^2£ & 1+\epsilon£ & -1+\oslash & \oslash \end{bmatrix}.$$

It is easy to see that the both principle minors of $[\mathcal{A}|\mathcal{B}]$ are appreciable, also the other conditions are satisfied as above. Hence we can apply the Gauss-Jordan elimination to solve this system. One has

$$\begin{aligned}
[\mathcal{A}|\mathcal{B}] &= \begin{bmatrix} 1+\epsilon\oslash & \epsilon^2 £ & \epsilon £ & 1/2+\oslash & \oslash \\ -1+\epsilon\oslash 1 & 1+\epsilon^2 £ & 1+\epsilon £ & -1+\oslash & \oslash \end{bmatrix} \\
\underset{\longrightarrow}{L_2+L_1} &\begin{bmatrix} 1+\epsilon\oslash & \epsilon^2 £ & \epsilon £ & 1/2+\oslash & \oslash \\ \epsilon\oslash & 1+\epsilon^2 £ & 1+\epsilon £ & -\dfrac{1}{2}+\oslash & \oslash \end{bmatrix}.
\end{aligned}$$

So we obtain

$$S=\left\{\left(-\frac{1}{2}x_4+\oslash, -x_3+\frac{1}{2}x_4+\oslash, x_3, x_4\right)\middle| x_3\in\frac{1}{\epsilon}\oslash, x_4\in £\right\},$$

which corresponds to the Gauss solutions of (4.76).

4.5.6 The flexible system with the strict rank not equal to the number of rows

We now deal with flexible systems of the form (4.59) in which $r<m$. To study this kind of systems, like in linear algebra, we first prove that it is equivalent to a system which has exactly r rows, and then applying Theorem 4.5.18 we find the external set of solutions of the given system.

Theorem 4.5.22. *Consider the flexible system* (4.59)*. Assume that the strict rank of the system is r, where $r<\min\{m,n\}$. Let $\Delta=\det(\mathcal{A}_l)=\det\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1r}\\ \vdots & \ddots & \vdots\\ \alpha_{r1} & \cdots & \alpha_{rr}\end{bmatrix}$. We also assume that Δ is zeroless and not an absorber of B. Then the system* (4.59) *is equivalent to the following one*

$$\begin{cases} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_n & \subseteq & b_1+B \\ \vdots & \ddots & \vdots & \vdots & \\ \alpha_{r1}x_1+ & \cdots & +\alpha_{rn}x_n & \subseteq & b_r+B. \end{cases} \tag{4.77}$$

The theorem follows from the following lemmas.

Lemma 4.5.23. *Consider the flexible system* (4.59)*. For $1\le i\le m$, let*

$$\alpha_{in+1}=b_i+B=a_{i(n+1)}+A_{i(n+1)}$$

be the $(n+1)^{th}$ column in the augmented matrix $[\mathcal{A}|\mathcal{B}]$. Assume that the strict rank of the system is r and $\Delta=\det(\mathcal{A}_l)=\det\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1r}\\ \vdots & \ddots & \vdots\\ \alpha_{r1} & \cdots & \alpha_{rr}\end{bmatrix}$ is zeroless and not an absorber of B. Taking $a_{ij}\in\alpha_{ij}$ for $1\le i\le m$

and $1 \leq j \leq n+1$. *For all* $j \in \{1, \ldots, n+1\}$ *and* $k \in \{r+1, \ldots, n\}$ *let*

$$W(j) = \begin{bmatrix} a_{kj} & a_{k1} & \cdots & a_{kr} \\ a_{1j} & a_{11} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{rj} & a_{r1} & \cdots & a_{rr} \end{bmatrix},$$

and, for all $i \in \{1, \ldots, r, k\}$ *by removing the first column and the* $(i+1)^{th}$ *row of* $W(j)$*, we denote*

$$W_i(j) = \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kr} \\ a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)r} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix}.$$

Let

$$\begin{cases} d_{ki} = (-1)^{i+2}\det\left(W_i(j)\right), \ \textit{for } i \in \{1, \ldots, r\} \\ d = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix}. \end{cases} \tag{4.78}$$

Then

(i) For $j \in \{1, \ldots, n+1\}$ *one has*

$$a_{kj} = -\frac{1}{d}(d_{k1}.a_{1j} + \cdots + d_{kr}.a_{rj}). \tag{4.79}$$

(ii) Vector

$$\alpha_k + \frac{1}{d}(d_{k1}\alpha_1 + \cdots + d_{kr}\alpha_r) = (A_1, \ldots, A_n, B) \tag{4.80}$$

is a neutrix vector.

Remark 4.5.24. By changing rows, we can choose d_{ki} and d such that $|d_{ki}| \leq |d|$.

Proof of Lemma 4.5.23. (i) Let $\alpha_i = \left(\alpha_{i1}, \cdots, \alpha_{i(n+1)}\right) \in \mathbb{E}^{n+1}$ for all $i \in \{1, \ldots, m\}$. Since sr$(\mathcal{A})$ = sr$[\mathcal{A}|\mathcal{B}] = r$ and det$(\mathcal{A}_l)$ is zeroless, the vector system $E = \{\alpha_1; \ldots; \alpha_r; \alpha_k\}$ is linearly dependent for all $k \in \{r+1, \ldots, m\}$. By the definition of linear dependence, there exists a set of vectors

$$V' = \{a_1; \cdots; a_r; a_k\}$$

which is linearly dependent, where $a_i = (a_{i1}, \ldots, a_{in+1}) \in \alpha_i, i \in \{1, \ldots, r, k\}$.

One has $\det\big(W(j)\big) = 0$, for all $j = 1, \ldots, n+1$. Indeed, if $1 \leq j \leq r$ then $W(j)$ has the two equal columns a_j and hence $\det\big(W(j)\big) = 0$. If $r+1 \leq j \leq n+1$, because the set of vectors $V' = \{a_1; \cdots; a_r; a_k\}$ is linearly dependent, then $\det\big(W(j)\big) = 0$.

Expanding the determinant $\det\big(W(j)\big)$ along the first column we obtain

$$\det\big(W(j)\big) = d_{k1}.a_{1j} + d_{k2}.a_{2j} + \cdots + d_{kr}.a_{rj} + d.a_{kj} = 0 \text{ for all } 1 \leq j \leq n+1.$$

It follows that

$$d_{k1}.a_1 + d_{k2}.a_2 + \cdots + d_{kr}.a_r + d.a_k = 0.$$

Moreover, by (4.78) and the definition of $W_i(j)$, one has $d \in \Delta$ and $d \neq 0$. Hence

$$a_k = -\frac{1}{d}\left(d_{k1}.a_1 + \cdots + d_{kr}.a_r\right), \tag{4.81}$$

i.e., for all $j \in \{1, \ldots, n+1\}$,

$$a_{kj} = -\frac{1}{d}\left(d_{k1}.a_{1j} + \cdots + d_{kr}.a_{rj}\right).$$

(ii) Let

$$\alpha'_k = \alpha_k + \frac{1}{d}(d_{k1}\alpha_1 + \cdots + d_{kr}\alpha_r) = (a'_{k1} + A'_{k1}, \ldots, a'_{k(n+1)} + B'). \tag{4.82}$$

By formula (4.79), one can choose $a'_{kj} = 0$ for all $j = 1, \ldots, n+1$ and hence $\alpha'_k = (A'_{k1}, \ldots, A'_{kn}, B')$, where

$$A'_{kj} = A_j + \frac{1}{d}\sum_{i=1}^{r}(d_{ki}.A_j)$$

and

$$B' = B + \sum_{i=1}^{r}\left(\frac{d_{ki}}{d}B\right) \tag{4.83}$$

for all $j = 1, \ldots, n+1$. Since $|d_{ki}| \leq |d|$ by Remark 4.5.24, we have

$$\left|\frac{d_{ki}}{d}\right| \leq 1. \tag{4.84}$$

By formulas (4.84), (4.80), (4.83) and r being standard, one has $A'_{kj} = A_j$ and $B' = B$ for all $k = r+1, \ldots, m$. $\square$

Lemma 4.5.25. *Consider the flexible system* (4.59)*. Assume that the strict rank of the system is $r < \min\{m, n\}$ and $\Delta = \det(\mathcal{A}_l) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix}$ is zeroless and not an absorber of B. Then the system* (4.59) *is* *equivalent to*

$$\begin{cases} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_n & \subseteq & b_1 & + & B \\ \vdots & \ddots & \vdots & & \vdots & & \\ \alpha_{r1}x_1+ & \cdots & +\alpha_{rn}x_n & \subseteq & b_r & + & B \\ A_1x_1+ & \cdots & +A_nx_n & \subseteq & & & B. \end{cases} \tag{4.85}$$

Proof. Let $x = (x_1, \ldots, x_n)^T$ is a Gauss-solution of the system (4.59). We will prove that x is also a Gauss-solution of the system (4.85). To do it we just need to verify that x satisfies the $(r+1)^{th}$ row of the system (4.85), because the other rows are automatically satisfied.

Since $x = (x_1, \ldots, x_n)^T$ is a solution of the system (4.59), one has $\sum\limits_{j=1}^{n} \alpha_{ij}x_j \subseteq b_i + B$ for all $i = 1, \ldots, m$. Also $d_{ki} \in$ £, and d is not an absorber of B, so for all $i \in \{1, \ldots, r\}$,

$$\sum_{j=1}^{n} \frac{d_{ki}}{d}\alpha_{ij}x_j = \frac{d_{ki}}{d}\sum_{j=1}^{n}\alpha_{ij}x_j \subseteq \frac{d_{ki}}{d}(b_i + B) \subseteq \frac{d_{ki}}{d}b_i + B. \tag{4.86}$$

Applying formula (4.79) with $j = n+1$, because for all $1 \leq i \leq r$, $a_{i(n+1)} = b_i$, one obtains that

$$b_k + \sum_{i=1}^{r}\frac{d_{ki}}{d}b_i = 0. \tag{4.87}$$

Also by formula (4.80),

$$A_j = \alpha_{kj} + \sum_{i=1}^{r}\frac{d_{ki}}{d}\alpha_{ij}. \tag{4.88}$$

Combining (4.88) and (4.87) with formula (4.86) one obtains

$$\begin{aligned} A_1.x_1 + \cdots + A_n.x_n &= \sum_{j=1}^{n}\left(\alpha_{kj} + \sum_{i=1}^{r}\frac{d_{ki}}{d}\alpha_{ij}\right)x_j \\ &= \sum_{j=1}^{n}\alpha_{kj}x_j + \sum_{j=1}^{n}\sum_{i=1}^{r}\frac{d_{ki}}{d}\alpha_{ij}x_j = \sum_{j=1}^{n}\alpha_{kj}x_j + \sum_{i=1}^{r}\sum_{j=1}^{n}\frac{d_{ki}}{d}\alpha_{ij}x_j \\ &\subseteq b_i + B + \sum_{i=1}^{r}\left(\frac{d_{ki}}{d}b_i + B\right) = \left(b_k + \sum_{i=1}^{r}\frac{d_{ki}}{d}b_i\right) + (r+1)B \\ &= (r+1)B = B. \end{aligned}$$

So x satisfies the $(r+1)^{th}$ row, and hence x is a solution of the system (4.85).

Conversely, suppose $x = (x_1, \ldots, x_n)^T$ is a solution of the system (4.85). Then

$$\sum_{j=1}^{n}\alpha_{ij}x_j \subseteq b_i + B,$$

for all $1 \leq i \leq r$ and formula (4.86) is still true.

For each $k > r$ arbitrary, using formulas (4.79), (4.87) and the fact that $\sum\limits_{j=1}^{n} a_{ij} \cdot x_j \in b_i + B$ one has

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{kj} x_j =& \sum_{j=1}^{n} \left(\sum_{i=1}^{r} -\frac{d_{ki}}{d} a_{ij} \right) x_j + \sum_{j=1}^{n} A_j x_j = \sum_{j=1}^{n} A_j x_j + \sum_{i=1}^{r} \left(-\frac{d_{ki}}{d} \left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \right) \\ \subseteq& B + \sum_{i=1}^{r} -\frac{d_{ki}}{d}(b_i + B) \ = B + \sum_{i=1}^{r} \frac{d_{ki}}{d} b_i + \sum_{i=1}^{r} \frac{d_{ki}}{d} B = b_k + B.\end{aligned}$$

Hence x is a solution of the system (4.59).

We conclude that the system (4.59) is equivalent to the system (4.85). $\square$

For flexible systems which have the same neutrix parts in each column then we can remove the rows whose coefficients are neutrices.

Lemma 4.5.26. *Consider the following system*

$$\begin{cases} \alpha_{11}x_1 & + & \cdots & + & \alpha_{1n}x_n & \subseteq & b_1 & + & B \\ \vdots & & \ddots & & \vdots & & \vdots & & \\ \alpha_{r1}x_1 & + & \cdots & + & \alpha_{rn}x_n & \subseteq & b_r & + & B \\ A_1x_1 & + & \cdots & + & A_nx_n & \subseteq & & & B, \end{cases} \tag{4.89}$$

where $\alpha_{ij} = a_{ij} + A_j$ for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n\}$. Assume that the strict rank of the system is $r < \min\{m, n\}$, and

$$\Delta = \det(\mathcal{A}_l) = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix}$$

is zeroless and not an absorber of B. Then the given system is equivalent to

$$\begin{cases} \alpha_{11}x_1 + & \cdots & +\alpha_{1n}x_n & \subseteq & b_1 + B \\ \vdots & \ddots & \vdots & & \vdots \\ \alpha_{r1}x_1 & + \cdots & +\alpha_{rn}x_n & \subseteq & b_r + B. \end{cases} \tag{4.90}$$

Proof. It is clear that all solutions of the system (4.89) are solutions of the system (4.90).

Conversely, if $x = (x_1, \ldots, x_n)^T$ is a solution of (4.90) then $A_j \cdot x_j \subseteq B$. It follows $\sum\limits_{j=1}^{n} A_j x_j \subseteq B$. Hence x is a solution of the system (4.89). $\square$

Proof of Theorem 4.5.22. The theorem is a direct consequence of the three lemmas aforementioned. $\square$

The theorem below gives a formula for the Gauss-solutions of a general system.

Theorem 4.5.27. *Suppose that a flexible system of the form* (4.59) *is solvable.*

Then the set of Gauss-solutions of the system is given by

$$S=\left\{(x_1,x_2,\dots,x_n)\,\middle|\,x_j\in\frac{\det(M_j)}{\Delta},\ \textit{for } j\in\{1,\dots,r\}\,;x_j\in N_j,\ \textit{for } j\in\{r+1,\dots,n\}\right\},$$

where $N_j=B:A_j$ *and* M_j *is defined by* (4.71).

Proof. By Theorem 4.5.22 the given system is equivalent to the following

$$\left\{\begin{array}{ccccc} \alpha_{11}x_1+ & \cdots & +\alpha_{1n}x_n & \subseteq & b_1+B \\ \vdots & \ddots & \vdots & \vdots & \\ \alpha_{r1}x_1+ & \cdots & +\alpha_{rn}x_n & \subseteq & b_r+B. \end{array}\right. \tag{4.91}$$

Since the matrix $\mathcal{A}$ satisfies all the basic assumptions, applying Theorem 4.5.18 to the system (4.91), one has

$$S=\left\{(x_1,x_2,\dots,x_n)\,\middle|\,x_j\in\frac{\det(M_j)}{\Delta},\ \text{for } j\in\{1,\dots,r\};x_j\in N_j,\ \text{for } j\in\{r+1,\dots,n\}\right\},$$

where $N_j=B:A_j$ and M_j is defined as (4.71). □

Remark 4.5.28. Note that interchanging two rows, and multiply each element of a row by a non-zero, appreciable scalar do not change the set of solutions of a flexible system. Combining with Lemma 4.5.8 we see that the following Gauss operations do not affect the set of solutions of a flexible system of the form (4.59).

(i) Interchange any two rows.

(ii) Multiply a row by a non-zero, appreciable scalar.

(iii) Add to a row a limited scalar multiple of another.

We call these operations the *restricted Gauss operations*, , the process of using these operations to transform a given matrix to an upper triangular matrix is called *restricted Gauss elimination*.

Moreover, we can apply the restricted Gauss elimination not only for reduced systems but also for every systems satisfying the conditions $\overline{B}=\underline{B}=B$ and $A_{ij}=A_{kj}=A_j$ for all $1\le i,k\le m, 1\le j\le n$.

The example below illustrates the restricted Gauss operations. It is also can be seen as an application of Theorem 4.5.27.

Example 4.5.29. Consider the flexible system, with $\epsilon>0$ infinitesimal,

$$\left\{\begin{array}{ccccccccc} (1+\epsilon\oslash)\,x_1 & + & \left(\frac{1}{2}+\epsilon\pounds\right)x_2 & + & \oslash x_3 & \subseteq & 2 & + & \oslash \\ \left(\frac{1}{2}+\epsilon\oslash\right)x_1 & + & (-\frac{1}{3}+\epsilon\pounds)x_2 & + & (-\frac{1}{4}+\oslash)x_3 & \subseteq & -1 & + & \oslash \\ \left(-\frac{1}{2}+\epsilon\oslash\right)x_1 & + & \left(-\frac{5}{6}+\epsilon\pounds\right)x_2 & + & (-\frac{1}{4}+\oslash)x_3 & \subseteq & -3 & + & \oslash. \end{array}\right.$$

Applying the restricted Gauss elimination to $[\mathcal{A}|\mathcal{B}]$ of the system, ons has

$$
\begin{aligned}
[\mathcal{A}|\mathcal{B}] &= \begin{bmatrix} 1+\epsilon\oslash & \frac{1}{2}+\epsilon £ & \oslash & 2+\oslash \\ \frac{1}{2}+\epsilon\oslash & -\frac{1}{3}+\epsilon £ & -\frac{1}{4}+\oslash & -1+\oslash \\ -\frac{1}{2}+\epsilon\oslash & -\frac{5}{6}+\epsilon £ & -\frac{1}{4}+\oslash & -3+\oslash \end{bmatrix} \\
\begin{matrix} L_2-\frac{1}{2}L_1 \\ \longrightarrow \\ L_3+\frac{1}{2}L_1 \end{matrix} & \begin{bmatrix} 1+\epsilon\oslash & \frac{1}{2}+\epsilon £ & \oslash & 2+\oslash \\ \epsilon\oslash & -\frac{7}{12}+\epsilon £ & -\frac{1}{4}+\oslash & -2+\oslash \\ \epsilon\oslash & -\frac{7}{12}+\epsilon £ & -\frac{1}{4}+\oslash & -2+\oslash \end{bmatrix} \\
\begin{matrix} L_3-L_2 \\ \longrightarrow \\ -\frac{12}{7}L_2 \end{matrix} & \begin{bmatrix} 1+\epsilon\oslash & \frac{1}{2}+\epsilon £ & \oslash & 2+\oslash \\ \epsilon\oslash & 1+\epsilon £ & \frac{3}{7}+\oslash & \frac{24}{7}+\oslash \\ \epsilon\oslash & \epsilon £ & \oslash & \oslash \end{bmatrix} \\
\begin{matrix} L_1-\frac{1}{2}L_2 \\ \longrightarrow \\ L_3+\frac{4}{7}L_2 \end{matrix} & \begin{bmatrix} 1+\epsilon\oslash & \epsilon £ & \frac{3}{14}+\oslash & \frac{2}{7}+\oslash \\ \epsilon\oslash & 1+\epsilon £ & \frac{3}{7}+\oslash & \frac{24}{7}+\oslash \\ \epsilon\oslash & \epsilon £ & \oslash & \oslash \end{bmatrix}.
\end{aligned}
$$

Hence the given system is equivalent to

$$
\begin{cases} (1+\epsilon\oslash)x_1 & + & \epsilon £ x_2 & + & \left(\frac{3}{14}+\oslash\right)x_3 & \subseteq & \frac{2}{7} & + & \oslash \\ \epsilon\oslash x_1 & + & (1+\epsilon £)\,x_2 & + & \left(\frac{3}{7}+\oslash\right)x_3 & \subseteq & \frac{24}{7} & + & \oslash. \end{cases}
$$

So the strict rank of the system is 2.

Let

$$
\mathcal{A}_l = \begin{bmatrix} 1+\epsilon\oslash & \epsilon £ \\ \epsilon\oslash & 1+\epsilon £ \end{bmatrix}.
$$

So $\Delta \equiv \det(\mathcal{A}_l) = 1+\epsilon £$ is zeroless. The matrix $\mathcal{A}_l$ also satisfies

(i) $\mathcal{A}_l$ is a reduced matrix,

(ii) $\overline{A_l} = \epsilon £ \subseteq \underline{A_r} = \oslash$,

(iii) $R(\mathcal{A}_2) = \epsilon £ \subseteq P(B) = \oslash$,

(iv) $\underline{B} = \overline{B} = B = \oslash$.

Observe that the given system reduces to

$$
\begin{cases} (1+\epsilon\oslash)x_1 + & \epsilon £ . x_2 & \subseteq -\frac{3}{14}.x_3 + \frac{2}{7} + \oslash \\ \epsilon\oslash x_1 + & (1+\epsilon £)x_2 & \subseteq -\frac{3}{7}.x_3 + \frac{24}{7} + \oslash \end{cases}, x_3 \in £
$$

Applying Theorem 4.5.27, the external set of Gauss-solutions of the system is given by

$$S = \left\{ \left(\frac{2}{7} - \frac{3}{14}.x_3 + \oslash, \frac{24}{7} - \frac{3}{7}x_3 + \oslash, x_3 \right) \middle| x_3 \in £ \right\}.$$

The following example emphasizes the operation of interchange two rows.

Example 4.5.30. Consider the following system, with $\epsilon > 0$ infinitesimal,

$$\begin{cases} (1+\epsilon£)x_1 & + & (2+\epsilon\oslash)x_2 & + & (-1+\oslash)x_3 & \subseteq & 1 & + & \oslash \\ \epsilon£x_1 & + & (-4\epsilon+\epsilon\oslash)x_2 & + & (\epsilon+\oslash)x_3 & \subseteq & 3\epsilon & + & \oslash \\ \epsilon£x_1 & + & (-4+\epsilon\oslash)x_2 & + & (1+\oslash)x_3 & \subseteq & 3 & + & \oslash. \end{cases}$$

Because $\alpha_{22} = -4\epsilon + \epsilon\oslash$ is an absorber of $B = \oslash$, the restricted Gauss operation of multiplying to the second row by the scalar $\dfrac{1}{a_{22}}$ is not true.

However, if we change the second row and the third row then we obtain the equivalent system

$$\begin{cases} (1+\epsilon£)x_1 & + & (2+\epsilon\oslash)x_2 & + & (-1+\oslash)x_3 & \subseteq & 1 & + & \oslash \\ \epsilon£x_1 & + & (-4+\epsilon\oslash)x_2 & + & (1+\oslash)x_3 & \subseteq & 3 & + & \oslash \\ \epsilon£x_1 & + & (-4\epsilon+\epsilon\oslash)x_2 & + & (\epsilon+\oslash)x_3 & \subseteq & 3\epsilon & + & \oslash. \end{cases}$$

Using the restricted Gauss elimination to the latter system, one has

$$\begin{aligned}
[\mathcal{A}|\mathcal{B}] &= \begin{bmatrix} 1+\epsilon£ & 2+\epsilon\oslash & -1+\oslash & 1+\oslash \\ \epsilon£ & -4+\epsilon\oslash & 1+\oslash & 3+\oslash \\ \epsilon£ & -4\epsilon+\epsilon\oslash & \epsilon+\oslash & 3\epsilon+\oslash \end{bmatrix} \\
\underset{L_3-\epsilon L_2}{\longrightarrow} &\begin{bmatrix} 1+\epsilon£ & 2+\epsilon\oslash & -1+\oslash & 1+\oslash \\ \epsilon£ & -4+\epsilon\oslash & 1+\oslash & 3+\oslash \\ \epsilon£ & \epsilon\oslash & \oslash & \oslash \end{bmatrix} \\
\underset{L_1+\frac{1}{2}L_2}{\longrightarrow} &\begin{bmatrix} 1+\epsilon£ & \epsilon\oslash & -\frac{1}{2}+\oslash & \frac{5}{2}+\oslash \\ \epsilon£ & -4+\epsilon\oslash & 1+\oslash & 3+\oslash \\ \epsilon£ & \epsilon\oslash & \oslash & \oslash \end{bmatrix} \\
\overset{-\frac{1}{4}L_2}{\longrightarrow} &\begin{bmatrix} 1+\epsilon£ & \epsilon\oslash & -\frac{1}{2}+\oslash & \frac{5}{2}+\oslash \\ \epsilon£ & 1+\epsilon\oslash & -\frac{1}{4}+\oslash & -\frac{3}{4}+\oslash \\ \epsilon£ & \epsilon\oslash & \oslash & \oslash \end{bmatrix}.
\end{aligned}$$

Then the given system is equivalent to

$$\begin{cases} (1+\epsilon£)x_1 + & \epsilon\oslash x_2 + & (-\frac{1}{2}+\oslash)x_3 & \subseteq & \frac{5}{2}+\oslash \\ \epsilon£x_1 + & (1+\epsilon\oslash)x_2 + & (-\frac{1}{4}+\oslash)x_3 & \subseteq & -\frac{3}{4}+\oslash. \end{cases}$$

So the set of solutions is given by

$$S=\left\{\left(\frac{5}{2}+\frac{1}{2}x_3+\oslash,-\frac{3}{4}+\frac{1}{4}x_3+\oslash,x_3\right)\middle|x_3\in\pounds\right\}.$$

The restricted Gauss operations are not applicable to the next example.

Example 4.5.31. Consider the following system

$$\begin{cases}(1+\epsilon\pounds)x_1 & + & (1+\epsilon\oslash)x_2 & + & (-1+\oslash)x_3 & \subseteq & 1 & + & \oslash\\ \epsilon\pounds x_1 & + & (\epsilon+\epsilon\oslash)x_2 & + & (2+\oslash)x_3 & \subseteq & 3\epsilon & + & \oslash\\ \epsilon\pounds x_1 & + & (-2\epsilon+\epsilon\oslash)x_2 & + & (-4+\oslash)x_3 & \subseteq & -6\epsilon & + & \oslash.\end{cases}$$

Since $a_{22}=\epsilon$ and $a_{23}=-2\epsilon$ are absorbers of $B=\oslash$, we can not implement the Gauss operations for this system. This is because that $\Delta=\det\begin{bmatrix}1+\epsilon\pounds & 1+\epsilon\oslash\\ \epsilon\pounds & \epsilon+\epsilon\oslash\end{bmatrix}$ is an absorber of B. The same is true for originating from first two columns.

If we consider $\mathcal{A}_l=\begin{bmatrix}1+\epsilon\pounds & -1+\oslash\\ \epsilon\pounds & 2+\oslash\end{bmatrix}$ then det$(\mathcal{A}_l)$ is zeroless. However, it does not satisfies the condition $\overline{A}_l\subseteq\underline{A}_r$. Hence we can not apply the theorem above for this system.

4.6 A parameter method to solve flexible systems

In this section we reconsider flexible systems of the form

$$\begin{cases}\alpha_{11}\xi_1 & + & \alpha_{12}\xi_2 & + & \cdots & + & \alpha_{1n}\xi_n & \subseteq & b_1+B_1\\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots\\ \alpha_{m1}\xi_1 & + & \alpha_{m2}\xi_2 & + & \cdots & + & \alpha_{mn}\xi_n & \subseteq & b_m+B_m,\end{cases}\tag{4.92}$$

where $\alpha_{ij},\beta_j\in\mathbb{E}$ for $1\le i\le m, 1\le j\le n$.

The neutrices in the constant terms may be seen as the sets of parameters. Then we can treat it as a system of linear equations with parameters. Formulas of solutions depending on parameters are given respectively for non-singular system, singular system with the strict rank equal to the numbers of equation, and singular system with the strict rank not equal to the number of equations. These formulas write the neutrix part of the solutions in terms of n-dimensional neutrices. It is shown in [3] that neutrices in n-dimensional space have always such a representation.

Notation 4.6.1. Let α be a external number and $u=(u_1,\dots,u_n)^T\in\mathcal{M}_{n,1}(\mathbb{R})$ be a column vector for $1\le j\le n$, here n is standard. We refer to αu as $xu=(xu_1,\dots,xu_n)^T$ with $x\in\alpha$.

We denote $N_j=\min\limits_{1\le i\le m}\{B_i:A_{ij}\}$ for $1\le j\le n$ and $\alpha_j^T=(\alpha_{1j},\dots,\alpha_{mj})^T$ be the j-th column of $\mathcal{A}$, a_j^T be a representative of α_j^T.

The the system (4.92) is transformed to a system such that the coefficient matrix of the latter system are a real matrix.

Theorem 4.6.2. *The system* (4.92) *is equivalent to*

$$\begin{cases} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & \in & b_1 & + & B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \\ a_{m1}\xi_1 & + & a_{m2}\xi_2 & + & \cdots & + & a_{mn}\xi_n & \in & b_m & + & B_m, \\ \xi_j \in N_j, j \in \{1,\dots,n\}, \end{cases} \tag{4.93}$$

where $a_{ij} \in \alpha_{ij}, 1 \le i, j \le n$.

Proof. The vector $\xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$ is a solution of system (4.92) if and only if

$$\begin{cases} (a_{11}+A_{11})\xi_1 & + & (a_{12}+A_{12})\xi_2 & + & \cdots & + & (a_{1n}+A_{1n})\xi_n & \subseteq & b_1 & + & B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \\ (a_{m1}+A_{m1})\xi_1 & + & (a_{m2}+A_{m2})\xi_2 & + & \cdots & + & (a_{mn}+A_{mn})\xi_n & \subseteq & b_m & + & B_m, \end{cases}$$

This is equivalent to

$$\begin{cases} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & \in & b_1 & + & B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \\ a_{m1}\xi_1 & + & a_{m2}\xi_2 & + & \cdots & + & a_{mn}\xi_n & \in & b_m & + & B_m, \end{cases}$$

and $A_{ij}\xi_j \subseteq B_i$ for all $1 \le i \le m$ and $1 \le j \le n$, the latter is equivalent to $\xi_j \in N_j$ for all $1 \le j \le n$. $\square$

Let $P = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be a representative of the coefficient matrix such that the rank of P is equal to the strict rank of the system.

4.6.1 Non-singular systems

Theorem 4.6.3. *Assume that System* (4.92) *is non-singular. Then the set of solutions of the system is given by*

$$S = \Big(P^{-1}b + \sum_{i=1}^{n} B_i P^{-1} e_i\Big) \bigcap \Big(\oplus_{i=1}^{n} N_i e_i \Big) \tag{4.94}$$

where $N_i e_i = (0, \dots, 0, N_i, 0, \dots, 0)^T$ *and* $b = (b_1, \dots, b_n)$.

Proof. By Theorem 4.6.2, a vector $\xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$ is a solution of system (4.92) if and only if it is a solution of

$$\left\{\begin{array}{ccccccccc} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & \in & b_1+B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}\xi_1 & + & a_{m2}\xi_2 & + & \cdots & + & a_{nn}\xi_n & \in & b_n+B_n \end{array}\right. \tag{4.95}$$

and satisfies the condition $\xi_j \in N_j, 1 \leq j \leq n$.

On the other hand, the point ξ is a solution of the system (4.95) if and only if there exists $\epsilon_i \in B_i$ for all $i \in \{1,\ldots,n\}$ such that

$$\left\{\begin{array}{ccccccccc} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & = & b_1+\epsilon_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}\xi_1 & + & a_{n2}\xi_2 & + & \cdots & + & a_{nn}\xi_n & = & b_n+\epsilon_n. \end{array}\right.$$

Using Notation 4.6.1 this implies that

$$\begin{aligned} \Big[\xi\Big] &= \Big(P^{-1}b+\sum_{i=1}^{n}\epsilon_i P^{-1}e_i\Big), \epsilon_i \in B_i \\ &= \Big(P^{-1}b+\sum_{i=1}^{n}B_i P^{-1}e_i\Big), \end{aligned}$$

where $[\xi]=(\xi_1,\ldots,\xi_n)^T \in \mathcal{M}_{n,1}(\mathbb{R})$. Because $\xi_i \in N_i$, that is $\Big[\xi\Big] \in \oplus_{i=1}^n (N_i e_i)$, we obtain

$$\Big[\xi\Big] \in \Big(P^{-1}b+\sum_{i=1}^{n}B_i P^{-1}e_i\Big)\bigcap\Big(\oplus_{i=1}^n N_i e_i\Big).$$

□

Example 4.6.4. Consider the flexible system

$$\left\{\begin{array}{ccccc} (1+\oslash)x & + & (1+\epsilon\oslash)y & \subseteq & \oslash \\ (1+\epsilon\pounds)x & - & (1+\epsilon\pounds)y & \subseteq & \epsilon\pounds. \end{array}\right.$$

The system is equivalent to

$$\left\{\begin{array}{ccccc} x & + & y & \in & \oslash \\ x & - & y & \in & \epsilon\pounds \end{array}\right. \tag{4.96}$$

with

$$x \in \pounds, y \in \pounds. \tag{4.97}$$

The vector $\xi=(x,y) \in \mathbb{R}^2$ is a solution of the system (4.96) if and only if

$$\left\{\begin{array}{ccccc} x & + & y & = & \epsilon_1 \\ x & - & y & = & \epsilon_2, \end{array}\right.$$

where $\epsilon_1 \in \oslash, \epsilon_2 \in \epsilon£$. A short calculation shows that $\begin{pmatrix} x \\ y \end{pmatrix} = \oslash \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \epsilon£ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$. Combining with (4.97) we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\oslash \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \epsilon£ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right) \bigcap \left(£ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + £ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \oslash \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \epsilon£ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

4.6.2 Singular systems with the strict rank equal to the number of equations

Next we consider the flexible system (4.92) with the strict rank equal to the number of equations. This means $\text{sr}(\mathcal{A}) = \text{sr}[\mathcal{A}|\mathcal{B}] = m$. So a representative

$$P = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

of $\mathcal{A}$ has strict rank m. We assume without restriction that

$$\det(M) = \det \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \neq 0. \tag{4.98}$$

Theorem 4.6.5. *Assume that the system* (4.92) *satisfies the condition that the strict rank of the system is equal to the number of equations. Then the set of solutions of the system is given by*

$$S = \left\{ (\xi_1, \ldots, \xi_n) \middle| \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix} = \left(\begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} B_i M^{-1} e_i \\ 0 \end{bmatrix} + \sum_{k=m+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1} a_k^T \\ e_k^{(n-m)} \end{bmatrix} \right) \bigcap \left(\oplus_{i=1}^{n} N_i e_i \right) \right\},$$

where $e_k^{(n-m)}$ is the k-th unit vector in $\mathbb{R}^{n-m}$.

Proof. A vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ is a solution of the system (4.93) if and only if there exist $\epsilon_i \in B_i, 1 \leq i \leq m$ such that it is a solution of the following linear system

$$\begin{cases} a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1n}\xi_n = b_1 + \epsilon_1 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m1}\xi_1 + a_{m2}\xi_2 + \cdots + a_{mn}\xi_n = b_m + \epsilon_m. \end{cases} \tag{4.99}$$

and satisfies the condition $\xi_j \in N_j$ for $1 \leq j \leq n$. The system (4.99) is equivalent to

$$\begin{cases} a_{11}\xi_1 + \cdots + a_{1m}\xi_m = b_1 + \epsilon_1 - a_{1(m+1)}\xi_{m+1} - \cdots - a_{1n}\xi_n \\ \vdots \qquad \ddots \qquad \vdots \qquad \qquad \vdots \\ a_{m1}\xi_1 + \cdots + a_{mm}\xi_m = b_m + \epsilon_m - a_{m(m+1)}\xi_{m+1} - \cdots - a_{mn}\xi_n. \end{cases} \tag{4.100}$$

By seeing $\xi_k, m+1 \leq k \leq n$ as parameters, the set of solutions of the system (4.100) is given by

$$\begin{aligned} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} &= M^{-1}b + \epsilon_1 M^{-1}e_1 + \cdots + \epsilon_m M^{-1}e_m - \xi_{m+1}M^{-1}a_{m+1}^T - \cdots - \xi_n M^{-1}a_n^T \\ &= M^{-1}b + B_1 M^{-1}e_1 + \cdots + B_m M^{-1}e_m - \mathbb{R}M^{-1}a_{m+1}^T - \cdots - \mathbb{R}M^{-1}a_n^T \\ &= M^{-1}b + \sum_{i=1}^{m} B_i M^{-1}e_i - \sum_{k=m+1}^{n} \mathbb{R}M^{-1}a_k^T. \end{aligned}$$

Hence we obtain that

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} B_i M^{-1}e_i \\ 0 \end{bmatrix} + \sum_{k=m+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1}a_k^T \\ e_k^{(n-m)} \end{bmatrix}.$$

Because $\xi_i \in N_i$, one has $\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \left(\oplus_{i=1}^n N_i e_i \right)$. By Theorem 4.6.2 we conclude that the set of solutions of the system (4.92) is given by

$$S = \left\{ (\xi_1, \ldots, \xi_n) \middle| \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \left(\begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} B_i M^{-1}e_i \\ 0 \end{bmatrix} + \sum_{k=m+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1}a_k^T \\ e_k^{(n-m)} \end{bmatrix} \right) \bigcap \left(\oplus_{i=1}^n N_i e_i \right) \right\}.$$

$\square$

Example 4.6.6. Let $\epsilon > 0$ be an infinitesimal. Consider the flexible system

$$\begin{cases} (2+\epsilon\oslash)x + (1+\epsilon\oslash)y + (1+\oslash)z \subseteq 1 + \oslash \\ (1+\epsilon\oslash)x + (1+\epsilon £)y + (3+\epsilon\oslash)z \subseteq 2 + \epsilon\oslash. \end{cases}$$

A short calculation shows that the system has strict rank 2. Let $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},\ M^{-1}b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \\ M^{-1}a_3^T &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}. \end{aligned}$$

Using Theorem 4.6.5 we find that the solutions $\xi = (x, y, z) \in \mathbb{R}^3$ of the system are given by

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \left(\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \epsilon \oslash \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right) \bigcap \left(\pounds \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \pounds \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \epsilon \oslash \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \left(\frac{3}{5} + \oslash \right) \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}. \end{aligned}$$

4.6.3 Flexible systems with strict rank less than the number of equations

In case the strict rank of system is less than the number of equations we need some conditions such that a parameter method can be applied.

Because the strict rank of system is $r < \min\{m, n\}$, without restriction of generality, we assume that $\Delta = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix}$ is zeroless and $\det(M) = \det \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \neq 0$.

Theorem 4.6.7. *Assume that the strict rank of system* (4.92) *is* $r < \min\{m, n\}$, $\underline{B} = \overline{B} = B$ *and* Δ *is not an absorber of* B. *Then the set of solutions of the system is given by*

$$S = \left\{ (\xi_1, \ldots, \xi_n) \middle| \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \xi_{r+1} \\ \vdots \\ \xi_n \end{bmatrix} = \left(\begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{r} \begin{bmatrix} BM^{-1}e_i \\ 0 \end{bmatrix} + \sum_{k=r+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1}a_k^T \\ e_k^{(n-r)} \end{bmatrix} \right) \bigcap \left(\oplus_{i=1}^{n} N_i e_i \right) \right\},$$

where $e_k^{(n-r)}$ *is the k-th unit vector in* $\mathbb{R}^{(n-r)}$.

Proof. By Theorem 4.5.22 the system is equivalent to

$$\begin{cases} \alpha_{11}\xi_1 & + & \alpha_{12}\xi_2 & + & \cdots & + & \alpha_{1n}\xi_n & \subseteq & b_1 + B \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \alpha_{r1}\xi_1 & + & \alpha_{r2}\xi_2 & + & \cdots & + & \alpha_{rn}\xi_n & \subseteq & b_r + B. \end{cases} \tag{4.101}$$

Applying Theorem 4.6.5 to system (4.101), the set of solutions of the given system is

$$S = \left\{ (\xi_1, \dots, \xi_n) : \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \xi_{r+1} \\ \vdots \\ \xi_n \end{bmatrix} = \left(\begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{r} \begin{bmatrix} BM^{-1}e_i \\ 0 \end{bmatrix} + \sum_{k=r+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1}a_k^T \\ e_k^{(n-r)} \end{bmatrix} \right) \bigcap \left(\oplus_{i=1}^{n} N_i e_i \right) \right\}.$$

$\square$

Let $M_{rk}^{(i)} = \begin{pmatrix} a_{11} & \dots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)r} \\ a_{k1} & \cdots & a_{kr} \\ a_{(i+1)1} & \cdots & a_{(i+1)r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$ be a submatrix of P formed by remaining the first r columns and the r rows $\{1, \dots, i-1, k, i+1, \dots, r\}$ of P, with $r+1 \leq k \leq m$ and $1 \leq i \leq r$.

Theorem 4.6.8. *Consider the flexible system* (4.92)*. Assume that the strict rank of system is* $r < \min\{m, n\}$ *and that*

(i) $\left(\dfrac{\det\left(M_{rk}^{(i)}\right)}{\det(M)} \right)^{-1}$ *is not an absorber of* B_k *for all* $1 \leq i \leq r$*,* $r+1 \leq k \leq m$*,*

(ii) $B_i \subseteq B_k$ *for* $r+1 \leq k \leq m, 1 \leq i \leq r$*.*

Then the set of solutions of the system is given by

$$S = \left\{ (\xi_1, \dots, \xi_n) \middle| \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \xi_{r+1} \\ \vdots \\ \xi_n \end{bmatrix} = \left(\begin{bmatrix} M^{-1}b \\ 0 \end{bmatrix} + \sum_{i=1}^{r} \begin{bmatrix} B_i M^{-1}e_i \\ 0 \end{bmatrix} + \sum_{k=r+1}^{n} \mathbb{R} \begin{bmatrix} -M^{-1}a_k^T \\ e_k^{(n-r)} \end{bmatrix} \right) \bigcap \left(\oplus_{i=1}^{n} N_i e_i \right) \right\},$$

where $e_k^{(n-r)}$ is the k-th unit vector in $\mathbb{R}^{(n-r)}$.

Lemma 4.6.9. *Consider the flexible system* (4.92)*. Assume that the strict rank of system is $r < \min\{m, n\}$ and satisfies assumptions* (i) *and* (ii) *in Theorem 4.6.8. Then the system* (4.92) *is equivalent to*

$$\begin{cases} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & \in & b_1 + B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{r1}\xi_1 & + & a_{r2}\xi_2 & + & \cdots & + & a_{rn}\xi_n & \in & b_r + B_r \\ \xi_i \in N_i, 1 \le i \le n. \end{cases} \tag{4.102}$$

Proof. Obviously, a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a solution of the system (4.92), then it is a solution of the system (4.102).

Conversely, assume that $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ is a solution of system (4.102). We only need to prove that ξ satisfies equation k of the system (4.92) for $r + 1 \le k \le m$. For $r + 1 \le k \le n$, let $\eta_k = (a_{k1}, \dots, a_{kn}, b_k) \in \mathbb{R}^{n+1}$. Since the matrix $[P|b]$ has the strict rank r, there exist real numbers $t_1, \dots, t_r$ such that the row k-th of the matrix $[P|b]$ can be expressed by

$$\eta_k = t_1\eta_1 + \cdots + t_r\eta_r. \tag{4.103}$$

Also we have $|t_i| = \dfrac{\det\left(M_{rk}^{(i)}\right)}{\det(M)}$ for $1 \le i \le r$. By assumption (i), the t_i is not an exploder of B_k for all $1 \le i \le r$, we have $t_iB_k \subseteq B_k$. By assumption (ii) we have

$$t_1B_1 + \cdots t_rB_r \subseteq t_1B_k + \cdots + t_rB_k \subseteq rB_k = B_k. \tag{4.104}$$

Because $\xi = (\xi_1, \dots, \xi_n)$ is a solution of the system (4.102), we have

$$\begin{cases} a_{11}\xi_1 & + & a_{12}\xi_2 & + & \cdots & + & a_{1n}\xi_n & \in & b_1 + B_1 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{r1}\xi_1 & + & a_{r2}\xi_2 & + & \cdots & + & a_{rn}\xi_n & \in & b_r + B_r \end{cases}$$

It follows that

$$\begin{cases} t_1a_{11}\xi_1 & + & t_1a_{12}\xi_2 & + & \cdots & + & t_1a_{1n}\xi_n & \in & t_1(b_1 + B_1) \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ t_ra_{r1}\xi_1 & + & t_ra_{r2}\xi_2 & + & \cdots & + & t_ra_{rn}\xi_n & \in & t_r(b_r + B_r) \end{cases}$$

Consequently,

$$\begin{aligned} &\big(t_1a_{11}\xi_1 + t_1a_{12}\xi_2 + \cdots + t_1a_{1n}\xi_n\big) + \cdots + \big(t_ra_{r1}\xi_1 + t_ra_{r2}\xi_2 + \cdots + t_ra_{rn}\xi_n\big) \\ \subseteq &t_1(b_1 + B_1) + \cdots + t_r(b_r + B_r). \end{aligned}$$

It is equivalent to

$$\begin{aligned}&\big(t_1a_{11}+\cdots+t_ra_{r1}\big)\xi_1+\big(t_1a_{12}+\cdots+t_ra_{r2}\big)\xi_2+\cdots+\big(t_1a_{1n}+\cdots+t_ra_{rn}\big)\xi_n\\ \subseteq&\big(t_1b_1+\cdots+t_rb_r\big)+\big(t_1B_1+\cdots+t_rB_r\big).\end{aligned}$$

By formulas (4.103) and (4.104) this implies that $a_{k1}\xi_1+\cdots a_{kn}\xi_n\subseteq b_k+\big(t_1B_1+\cdots+t_rB_r\big)\subseteq b_k+B_k$ for all $r+1\leq k\leq n$.

We conclude that two systems (4.102) and (4.92) are equivalent. □

Proof of Theorem 4.6.8. By Lemma 4.6.9, the system is equivalent to system (4.102). Applying Theorem 4.6.5 to this system, the set of solutions of the given system is

$$S=\left\{(\xi_1,\ldots,\xi_n):\begin{bmatrix}\xi_1\\ \vdots\\ \xi_r\\ \xi_{r+1}\\ \vdots\\ \xi_n\end{bmatrix}=\left(\begin{bmatrix}M^{-1}b\\ 0\end{bmatrix}+\sum_{i=1}^{r}\begin{bmatrix}B_iM^{-1}e_i\\ 0\end{bmatrix}+\sum_{k=r+1}^{n}\mathbb{R}\begin{bmatrix}-M^{-1}a_k^T\\ e_k^{(n-r)}\end{bmatrix}\right)\bigcap\big(\oplus_{i=1}^{n}N_ie_i\big)\right\}.$$

□

Remark 4.6.10. In many cases, we can interchange rows of P such that assumptions (i) and (ii) of Theorem 4.6.8 are satisfied. In particular, we can always interchange rows of a flexible system such that $|\det(M)|\geq|\det\left(M_{rk}^{(i)}\right)|$. As a consequence, the assumption (i) of Theorem 4.6.8 is satisfied.

In case Δ and $M_{rk}^{(i)}$ are appreciable numbers, the assumptions (i) and (ii) are always satisfied.

Example 4.6.11. Let $\epsilon>0$ be an infinitesimal. Consider the following system

$$\begin{cases}(1+\oslash)x & + & (1+\epsilon\oslash)y & + & 3z & \subseteq & 1 & + & \oslash\\ (1+\epsilon£)x & - & (1+\epsilon\oslash)y & + & 2z & \subseteq & 2 & + & \epsilon£\\ (2\epsilon+\epsilon\oslash)x & + & \epsilon£y & + & 5\epsilon z & \subseteq & 3\epsilon & + & \oslash.\end{cases}$$

A short calculation shows that the system has strict rank 2. Let $P=\begin{pmatrix}1&1&3&1\\1&-1&2&2\\2\epsilon&0&5\epsilon&3\epsilon\end{pmatrix}$ and $M=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$.

We have $\Delta=\det M=2$, $M_{23}^{(1)}=-2\epsilon, M_{23}^{(2)}=2\epsilon$. So assumption (i) of Theorem 4.6.8 is satisfied. Clearly, assumption (ii) of Theorem 4.6.8 is also satisfied.

The system is equivalent to

$$\begin{cases}x & + & y & + & 3z & \in & 1 & + & \oslash\\ x & - & y & + & 2z & \in & 2 & + & \epsilon£\end{cases}\tag{4.105}$$

with $x\in£, y\in£, z\in\mathbb{R}$.

We also have

$$\begin{aligned} M^{-1} =& \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, M^{-1}b = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ M^{-1}a_3^T =& \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \end{pmatrix}. \end{aligned}$$

Using Theorem 4.6.9, the solutions $\xi = (x, y, z)$ of the system are given by

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \left(\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \epsilon £ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right) \bigcap \left(£ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + £ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \epsilon £ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} + £ \begin{pmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}. \end{aligned}$$

5

Flexible sequences

This chapter is devoted to the study of sequences with uncertainties in terms of external numbers. We will call this kind of sequences *flexible sequences*.

The chapter has the following structure.

In Section 5.1 we introduce the notion of a flexible sequence and give some examples.

We will consider two types of convergence for flexible sequences. Firstly, in Section 5.2, we generalize the notion of convergence of a classical sequence to flexible sequences in terms of approaching external numbers. Secondly, basing on a well-known result in non standard analysis which says that every standard sequence converges to a, the n-th term of the sequence belongs to $a + \oslash$ with n unlimited, in Section 5.3 we will define another notion of convergence for flexible sequences. We call it *strongly convergent*. The relationship between two notions of convergences are studied. In fact, if the neutrix of the limit is not zero we prove that the two notions of convergences are equivalent. Properties of and operations on these kinds of limits are considered. We also study the relationship between the convergence of subsequences and of a given flexible sequence. Because, in general, induction may not apply to external formulas, we can not use it to construct subsequences like in classical mathematics. To overcome this drawback we use the notion of cofinal set to define the subsequences of a flexible sequence. Then we will show that every flexible sequence has an internal subsequence which satisfies all properties of a conventional subsequence. The Cauchy criterion for the convergences of flexible sequences is also presented.

In the last section we will introduce notions of convergences of vector flexible sequences.

Acknowledgement: The idea to study flexible sequences comes from Bruno Dinis (University of Lisboa) who also prove some elementary properties of convergence.

5.1 Definition and example

Definition 5.1.1. A mapping $u\colon \mathbb{N} \longrightarrow \mathbb{E}$, of the form $\bigcup_{st(u)\in U} \bigcap_{st(v)\in V} I_{uv}$ where U, V are standard sets and $I\colon U \times V \rightrightarrows \mathbb{N} \times \mathcal{P}(\mathbb{R})$ is an internal set-valued mapping, is called a *flexible sequence*; we denote such a flexible sequence usually by $\{u_n\}$.

Example 5.1.2. (a) The sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ given by $u_n = \dfrac{1}{n} + \oslash$ for all $n \in \mathbb{N}$ $\{u_n\}$ is a flexible sequence. Observe that we can write

$$u = \bigcap_{\mathrm{st}(m)\in\mathbb{N}} \left\{ \{n\} \times \left\{ \frac{1}{n} + \left[-\frac{1}{m}, \frac{1}{m} \right] \right\}, n \in \mathbb{N} \right\}.$$

(b) Let $\epsilon > 0$ be infinitesimal. Let $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be given by $u_n = n + n\epsilon\oslash$ for all $n \in \mathbb{N}$ and $v\colon \mathbb{N} \longrightarrow \mathbb{E}$ be given by

$$v_n = \begin{cases} \oslash, & \text{if } n \in \pounds \\ 1 + \oslash, & \text{if } n \notin \pounds. \end{cases}$$

Then $\{u_n\}, \{v_n\}$ are two flexible sequences. Indeed, in the first case we can express

$$u = \bigcap_{\mathrm{st}(m)\in\mathbb{N}} \left\{ \{n\} \times \left\{ n + \epsilon \left[-\frac{n}{m}, \frac{n}{m} \right] \right\}, n \in \mathbb{N} \right\}$$

and

$$v = \left(\bigcup_{\mathrm{st}(q)\in\mathbb{N}} \bigcap_{\mathrm{st}(p)\in\mathbb{N}} \left\{ \{n\} \times \left\{ \left[-\frac{1}{p}, \frac{1}{p} \right] \right\}, -q \le n \le q \right\} \right)$$
$$\bigcup \left(\bigcap_{\mathrm{st}(q)\in\mathbb{N}} \bigcap_{\mathrm{st}(p)\in\mathbb{N}} \left\{ \{n\} \times \left\{ 1 + \left[-\frac{1}{p}, \frac{1}{p} \right] \right\}, n \notin [-q, q] \right\} \right).$$

Using the Reduction Algorithm v can be expressed by only one union and one intersection, meaning that v has the form as in the definition.

5.2 N-convergence

Motivated by the fact that a neutrix may be seen as generalized zero. In this section we will introduce a notion of N-convergence for flexible sequences, reduces to the classical notion of limit when the neutrix N is zero. Then we will show that properties on limits still hold for the notion of N-convergence on flexible sequences with some modifications. The relationship between N-convergence of a subsequence and a given flexible sequence are investigated. Finally the Cauchy criterion for N-convergence of flexible sequence is studied.

5.2.1 Definition and example

Definition 5.2.1. Let N be a neutrix. A sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ is said to N-*converge* to $\alpha \in \mathbb{E}$ if

$$\forall \epsilon \in \mathbb{E}\, \epsilon > N\, \exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}(n \geq n_0 \Rightarrow |u_n - \alpha| < \epsilon).$$

Then we also call α a N-limit of $\{u_n\}$. We write $u_n \underset{N}{\longrightarrow} \alpha$ or N-$\lim u_n = \alpha$.

A flexible sequence which is not N-convergent to any element $\alpha \in \mathbb{E}$ is called N-*divergent*.

One could replace the condition $\epsilon \in \mathbb{E}$ in the definition above by the condition $\epsilon \in \mathbb{R}$.

Theorem 5.2.2. *Let N be a neutrix and $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a flexible sequence. Then $u_n \underset{N}{\longrightarrow} \alpha$ if and only if*

$$\forall \epsilon \in \mathbb{R}\, \epsilon > N\, \exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}(n \geq n_0 \Rightarrow |u_n - \alpha| < \epsilon). \tag{5.1}$$

Proof. If $u_n \underset{N}{\longrightarrow} \alpha$, formula (5.1) is satisfied since $\mathbb{R} \subset \mathbb{E}$.

Conversely, assume that formula (5.1) holds. For every $\epsilon \in \mathbb{E}, \epsilon > N$ there exists $\nu \in \mathbb{R}, N < \nu < \epsilon$. By formula (5.1), there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has $|u_n - \alpha| < \nu < \epsilon$. Hence $u_n \underset{N}{\longrightarrow} \alpha$. $\square$

Clearly every sequence is $\mathbb{R}$-convergent. So, in the remainder of this chapter we always assume that $N \neq \mathbb{R}$. The other extreme case is $N = 0$ corresponding to the usual notion of convergence. In this case we adopt the usual notation and terminology, i.e. we say that $\{u_n\}$ converges to α (the real number) and write $u_n \to \alpha$ or $\lim u_n = \alpha$.

Example 5.2.3. (a) Consider the flexible sequence $u\colon \mathbb{N} \to \mathbb{E}$ given by $u_n = \dfrac{n}{n+1} + \dfrac{1}{n}\oslash$. Then $u_n \longrightarrow 1$ and $u_n \underset{\oslash}{\longrightarrow} 1 + \oslash$.

(b) Let $\epsilon > 0$ be an infinitesimal and $u\colon \mathbb{N} \to \mathbb{E}$ be a sequence defined by $u_n = (-1)^n\epsilon$ for all $n \in \mathbb{N}$. We know that the sequence is divergent in the classical sense. However, the terms always belong to $\oslash$, so $\oslash$-$\lim u_n = \oslash$-$\lim(-1)^n\epsilon = \oslash$.

(c) Let N be a fixed neutrix in $\mathbb{E}$ and $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a sequence defined by $u_n = s_n + N$, where $\{s_n\}$ is a real sequence which converges to $a \in \mathbb{R}$. We show that $\{u_n\}$ is N-convergent to $a + N$. For $\epsilon > N$, because

$s_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $|s_m - a| < \epsilon/2$ for $m > n_0$. So for $m > n_0$ we have

$$|u_m - \alpha| = |s_m + N - \alpha| \leq |s_m - \alpha| + N < \frac{\epsilon}{2} + N < \epsilon.$$

5.2.2 Some elementary properties

Proposition 5.2.4. *Let N be a neutrix and $\{u_n\}$ be a sequence that N-converges to some element $\alpha = a + N(\alpha) \in \mathbb{E}$. Let M be a neutrix such that $N \leq M$. Then $\{u_n\}$ also M-converges to α. Moreover, there exists the smallest neutrix M_0 such that $\{u_n\}$ is M_0-convergent.*

Proof. Firstly, let $\epsilon > M$. Then $\epsilon > N$. Since $\{u_n\}$ N-converges to α, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|u_n - \alpha| < \epsilon$. Hence $\{u_n\}$ M-converges to α.

Secondly, without loss of generality, we may assume that u_n is N-convergent to a neutrix. Let $L = \{M \in \mathcal{N} \mid \{u_n\}$is M-convergent$\}$ and M_0 be the infimum of L. By Proposition 2.1.6 the neutrix $M_0 = p \cdot I$, where I is an idempotent neutrix and $p \in \mathbb{R}$ is a positive. Upon dividing by p we may assume that M_0 itself is idempotent. If $M_0 = \oslash$ then $\oslash \in L$, so the conclusion is trivial. Otherwise, M_0 is not isomorphic to $\oslash$. Suppose that the sequence is not M_0-convergent. Let $\epsilon > M_0$. For all $n \in \mathbb{N}$ there exists a element $m \geq n$ such that $\epsilon \leq u_m$. However $\epsilon\oslash < \epsilon$, so u_n is not $\epsilon\oslash$-convergent. Also $M_0 \subset \epsilon\oslash$. So $\epsilon\oslash \notin L$. By the first part for every neutrix $M \in L$ we have $\epsilon\oslash \subset M$. It follows that $\epsilon\oslash \subseteq M_0$, a contradiction. Hence u_n is M_0-convergent. $\square$

In practice, obviously, we prefer to work with neutrices which are as small as possible.

Note that for a given flexible sequence its N-limits are not unique. In fact, N-convergence is unable to distinguish between elements whose "distance" is less than or equal to the neutrix N. That is, N-limits are unique up to modulo N in the sense that if $\{u_n\}$ N-converges to two (possibly distinct) elements $\alpha, \beta \in \mathbb{E}$ then the absolute value of their difference must be less than N.

Proposition 5.2.5. *Let $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a flexible sequence and let $\alpha, \beta \in \mathbb{E}$. Assume that $u_n \underset{N_0}{\longrightarrow} \alpha$ and $u_n \underset{N_0}{\longrightarrow} \beta$. Then*

(i) $N(\alpha) \subseteq N_0$.

(ii) $|\alpha - \beta| \leq N_0$.

(iii) $u_n \underset{N_0}{\longrightarrow} \alpha + N_0$.

(iv) $u_n \underset{N_0}{\longrightarrow} \gamma$ *for all* $\gamma \subseteq \alpha + N_0$.

Proof. (i) Suppose on the contrary that $N_0 \subset N(\alpha)$. Let $\epsilon \in \mathbb{R}$ be such that $N_0 < \epsilon \leq N(\alpha)$. Then there exists $n_0 \in$ such that for $n \geq n_0$ we have

$$N(\alpha) \leq N(|u_n - \alpha|) \leq |u_n - \alpha| < \epsilon \leq N(\alpha),$$

which is a contradiction.

(ii) Suppose $u_n \underset{N_0}{\longrightarrow} \alpha$ and $u_n \underset{N_0}{\longrightarrow} \beta$ with $|\alpha-\beta| > N_0$. For $\epsilon_0 = \frac{|\alpha-\beta|}{2} > N_0$, there exists $n_0 \in \mathbb{N}$ such that $|u_n - \alpha| < \epsilon_0$ for $n \geq n_0$ and there exists $n_1 \in \mathbb{N}$ such that $|u_n - \beta| < \epsilon_0$ for $n \geq n_1$. Let $k = \max\{n_0, n_1\}$. Then for $n \geq k$,

$$|\alpha-\beta| \leq |\alpha - u_n + u_n - \beta| \leq |\alpha - u_n| + |u_n - \beta| < 2\epsilon_0 = |\alpha-\beta|\,,$$

a contradiction. Hence $|\alpha-\beta| \leq N_0$.

(iii) Let $\epsilon > N_0$. Then $\epsilon/2 > N_0/2 = N_0$. Because $u_n \underset{N}{\longrightarrow} \alpha$, there exists n_0 such that $|u_n - \alpha| < \epsilon/2$ for all $n \geq n_0$. It follows that

$$|u_n - \alpha + N_0| = |u_n - \alpha| + N_0 < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq n_0$. Hence $u_n \underset{N_0}{\longrightarrow} \alpha + N_0$.

(iv) For $\gamma \subseteq \alpha + N_0$ we have $|u_n - \gamma| \leq |u_n - \alpha + N_0| < \epsilon$ for all $n \geq n_0$, and hence $u_n \underset{N_0}{\longrightarrow} \gamma$. □

In the previous proposition if $N_0 = 0$, it follows from (ii) that $\alpha = \beta$. Hence in this case we get the same conclusion as the classical notion of limit.

Assume α, β are two N-limits of a given flexible sequence. From Proposition 5.2.5 we conclude that $\beta \subseteq \alpha + N$ and $\alpha \subseteq \beta + N$.

Remark 5.2.6. A flexible sequence $\{u_n\}$ is N-*divergent* if and only if for each $\alpha = a + A \in \mathbb{E}, A \subseteq N$ there is $\epsilon_0 > N$ satisfying that for every $n \in \mathbb{N}$, there exists $n_0 \geq n$ such that

$$\epsilon_0 \leq |u_{n_0} - \alpha|. \tag{5.2}$$

From Remark 2.2.31 the inequality (5.2) can not be substituted by $|u_{n_0} - \alpha| \geq \epsilon_0$.

Convention 5.2.7. Because of Proposition 5.2.5, for the sake of simplicity, from now on we always take N as the neutrix part of a N-limit of a given flexible sequence. This means that if $N\text{-}\lim u_n = \alpha$ then $N(\alpha) = N$.

It is easy to see that a flexible sequence which is eventually a constant to some $\alpha \in \mathbb{E}$, the sequence is $N(\alpha)$-convergent to that constant.

A flexible sequence N-converges to α if and only if every real part of the sequence N-converges to a real part of α and the neutrix part of the sequence N-converges to the neutrix part of α.

Proposition 5.2.8. *Let* $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ *be a sequence such that* $u_n = a_n + A_n$ $a_n \in \mathbb{R}$ *for all* $n \in \mathbb{N}$. *Let* $\alpha = a + N \in \mathbb{E}$, $a \in \mathbb{R}$. *Then* $u_n \underset{N}{\longrightarrow} \alpha$ *if and only if* $a_n \underset{N}{\longrightarrow} a$ *and* $A_n \underset{N}{\longrightarrow} N$.

Proof. Assume first that $a_n \underset{N}{\longrightarrow} a$ and $A_n \underset{N}{\longrightarrow} N$. For $\epsilon > N$, there exist $k, l \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for $n \geq k$ and $|A_n + N| = |A_n - N| < \epsilon/2$ for all $n \geq l$. Let $m = \max\{k, l\}$. Then for $n \geq m$ it holds that

$$|u_n - \alpha| = |a_n + A_n - a + N| \leq |a_n - a| + |A_n - N| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $u_n \underset{N}{\longrightarrow} \alpha$.

Assume that $u_n \underset{N}{\longrightarrow} \alpha$. Let $\epsilon > N$. Then there exists $n_0 \in \mathbb{N}$ such that $|u_n - \alpha| < \epsilon/2$ for all $n \in \mathbb{N}, n \geq n_0$. It follows that $|a_n - a| \leq |a_n + A_n - a + N| = |u_n - \alpha| < \epsilon$ for all $n \in \mathbb{N}, n \geq n_0$. So $a_n \underset{N}{\longrightarrow} a$. On the other hand for all $n \geq n_0, n \in \mathbb{N}$ we have

$$|N(u_n) - N| = |a_n - a_n + a - a + A_n + N| \leq |u_n - \alpha| + |a_n - a| \leq |u_n - \alpha| + |u_n - \alpha| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $A_n \underset{N}{\longrightarrow} N$. □

5.2.3 Boundedness

Definition 5.2.9. A sequence $u \colon \mathbb{N} \longrightarrow \mathbb{E}$ is said to be

(i) *bounded* if there exists $\alpha \in \mathbb{E}$ such that $\alpha \neq \mathbb{R}$ and $|u_n| \leq \alpha$, for all $n \in \mathbb{N}$. Otherwise, we call it *unbounded*.

(ii) *eventually bounded* if there exists n_0 and $\alpha \neq \mathbb{R}$ such that for all $n \geq n_0$, $|u_n| \leq \alpha$.

Example 5.2.10. Consider the flexible sequences

(a) $u_n = \dfrac{1}{n} + \dfrac{1}{n}\oslash$ for all $n \in \mathbb{N}$,

(b) $v_n = \begin{cases} 0 & \text{if } n \in £ \\ n & \text{if } n \notin £, \end{cases}$

(c) $z_n = \begin{cases} \mathbb{R} & \text{if } n \in £ \\ 1 + \oslash & \text{if } n \notin £. \end{cases}$

Then $\{u_n\}$ is bounded since $|u_n| \leq 2$ for all $n \in \mathbb{N}$; $\{v_n\}$ is unbounded; and $\{z_n\}$ is eventually bounded, but not bounded.

Proposition 5.2.11. *Every N-convergent sequence is eventually bounded.*

Proof. Let N be a neutrix and $\{u_n\}$ be a sequence such that $u_n \underset{N}{\longrightarrow} \alpha$. Let $\epsilon > N$. Then there exists $n_0 \in \mathbb{N}$ such $|u_n - \alpha| < \epsilon$ that for $n \geq n_0$. It follows that $|u_n| - |\alpha| \leq |u_n - \alpha| < \epsilon$ for all $n \geq n_0$. This implies that $|u_n| \leq \epsilon + |\alpha|$ for all $n \geq n_0$. Hence $\{u_n\}$ is eventually bounded. □

Lemma 5.2.12. *Let $\alpha \in \mathbb{E}$ be an external number such that $\alpha \neq \mathbb{R}$. Then there exists a neutrix N such that $N \neq \mathbb{R}$ and $\alpha \subseteq N$.*

Proof. Let $N = \alpha£$. Then $N \neq \mathbb{R}$ and $\alpha \subseteq N$. □

We use the previous lemma to show that bounded sequences N-converge to N for some neutrix N.

Proposition 5.2.13. *Let $\{u_n\}$ be an eventually bounded sequence. Then there exists a neutrix N such that $u_n \underset{N}{\longrightarrow} N$.*

Proof. Because the sequence $\{u_n\}$ is eventually bounded, there exists an element $\alpha \in \mathbb{E}, \alpha \neq \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $|u_n| \leq \alpha$ for all $n \geq n_0$. By Lemma 5.2.12, there exists a neutrix N such that $|u_n| \leq \alpha \leq N$. Then for all $\epsilon > N$ we have $|u_n| < \epsilon$ for all $n \in \mathbb{N}, n \geq n_0$. Hence $u_n \underset{N}{\longrightarrow} N$. □

5.2.4 Monotonicity

Let $u\colon \mathbb{N} \longrightarrow \mathbb{R}$ and $n_0 \in \mathbb{N}$. In classical mathematics a real convergent sequence $u_n \geq 0$ $(n \geq n_0)$ has a non-negative limit. We give an adapted version for N-convergent sequences.

Proposition 5.2.14. *Let N be a neutrix and $n_0 \in \mathbb{N}$. Let $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be such that $N \leq u_n$ for $n \geq n_0$. Assume that $u_n \underset{N}{\longrightarrow} \alpha$ for some $\alpha = a + N \in \mathbb{E}$. Then $N \leq \alpha$.*

Proof. If $\alpha = N$, the conclusion is obvious. Assume that α is zeroless. Suppose on contrary that $\alpha = a + N < N$. Let $\epsilon = -\dfrac{a}{2} > N$. There exists $k_0 \in \mathbb{N}, k_0 \geq n_0$ such that for all $n \in \mathbb{N}, n \geq k_0$ it holds that $|u_n - a + N| < \epsilon$. So $a - \epsilon < u_n + N < a + \epsilon$ for all $n \geq k_0$. This means $\dfrac{3}{2}a < u_n < \dfrac{a}{2} < N$ for all $n \geq k_0$, which is a contradiction to the assumption. Hence $N < \alpha$. □

Proposition 5.2.15. *Let N be a neutrix and $u\colon \mathbb{N} \longrightarrow \mathbb{E}, v\colon \mathbb{N} \longrightarrow \mathbb{E}$ be two flexible sequences such that $u_n \underset{N}{\longrightarrow} \alpha$, $v_n \underset{N}{\longrightarrow} \beta$, for some $\alpha, \beta \in \mathbb{E}$. Assume that there is $n_0 \in \mathbb{N}$ such that $u_n \leq v_n$ for all $n \geq n_0$. Then $\alpha \leq \beta$.*

Proof. Write $\alpha = a + N, \beta = b + N$. Suppose that $\alpha \not\leq \beta$. By Corollary 2.2.34 we have $\beta < \alpha$. This implies that $\epsilon \equiv \frac{a-b}{2} > N$. So there exist $n_1, n_2 \in \mathbb{N}$ such that $|u_n - \alpha| < \epsilon$ for $n \geq n_1$ and $|v_n - \beta| < \epsilon$ for $n \geq n_2$. Let $n_3 = \max\{n_0, n_1, n_2\}$. Then for $n \geq n_3$ we have

$$v_n \leq v_n + N < b + \epsilon = b + \frac{a-b}{2} = a - \frac{a-b}{2} = a - \epsilon < u_n + N.$$

It follows that $v_n < u_n$ for all $n \geq n_3$, which is a contradiction. Hence $\alpha \leq \beta$. □

Remark 5.2.16. With similar arguments the conclusions in two propositions above are also true for the inequality $\geq$.

We prove next a version of the squeeze theorem for N-convergence. The squeeze theorem enables one to calculate the N-limit of a sequence (u_n) by comparison with two other sequences whose N-limits are equal and already known or easy to calculate.

Theorem 5.2.17 (Squeeze theorem)**.** *Let M, N be neutrices and $u\colon \mathbb{N} \longrightarrow \mathbb{E}, v\colon \mathbb{N} \longrightarrow \mathbb{E}, w\colon \mathbb{N} \longrightarrow \mathbb{E}$ be flexible sequences such that $u_n \underset{N}{\longrightarrow} \alpha$, $w_n \underset{M}{\longrightarrow} \alpha$, for some $\alpha \in \mathbb{E}$. Assume that there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $u_n \leq v_n \leq w_n$. Then $v_n \underset{N+M}{\longrightarrow} \alpha$. In particular, if $N = M$ then $v_n \underset{N}{\longrightarrow} \alpha$.*

Proof. Let $\epsilon \in \mathbb{R}$ and $\epsilon > N + M$. Then $\epsilon > N$ and $\epsilon > M$. So there exist $n_1, n_2 \in \mathbb{N}$ such that $|u_n - \alpha| < \epsilon$ for $n \geq n_1$ and $|w_n - \alpha| < \epsilon$ for $n \geq n_2$. Let $n_3 = \max\{n_0, n_1, n_2\}$. Then for $n \geq n_3$, by Proposition 2.2.39 and Proposition 2.2.40

$$-\epsilon < u_n - \alpha \leq v_n - \alpha \leq w_n - \alpha < \epsilon.$$

By Proposition 2.2.40 it holds that $|v_n - \alpha| < \epsilon$ for $n \geq n_3$. Hence, by Theorem 5.2.2, we conclude that $v_n \underset{N+M}{\longrightarrow} \alpha$. In particular, if $M = N$, the conclusion follows by the fact $M + N = N$. □

5.2.5 Operations on N-limits of sequences

Theorem 5.2.18. *Let N, M be two neutrices and $u\colon \mathbb{N} \longrightarrow \mathbb{E}, v\colon \mathbb{N} \longrightarrow \mathbb{E}$ be flexible sequences such that $u_n \underset{N}{\longrightarrow} \alpha$ and $v_n \underset{M}{\longrightarrow} \beta$, for some $\alpha, \beta \in \mathbb{E}$. Then*

$$u_n + v_n \underset{N+M}{\longrightarrow} \alpha + \beta.$$

In particular, if $M = N$ then $u_n + v_n \underset{N}{\longrightarrow} \alpha + \beta$.

Proof. Assume that $u_n \underset{N}{\longrightarrow} \alpha$ and $v_n \underset{M}{\longrightarrow} \beta$. Let $\epsilon > N + M$. Then $\epsilon > M$ and $\epsilon > N$. So there exists $n_0, n_1 \in \mathbb{N}$ such that $|u_n - \alpha| < \epsilon/2$ for all $n \geq n_0$ and $|v_n - \beta| < \epsilon/2$ for all $n \geq n_1$. Let $p = \max\{n_0, n_1\}$. Then for all $n \geq p$, by Lemma 2.2.43, we have

$$|(u_n + v_n) - (\alpha + \beta)| \leq |u_n - \alpha| + |v_n - \beta| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $u_n + v_n \underset{N+M}{\longrightarrow} \alpha + \beta$. In particular, if $N = M$ then $N + M = N$. So $u_n + v_n \underset{N}{\longrightarrow} \alpha + \beta$. □

In an analogous way one has the following result.

Theorem 5.2.19. *Let N, M be neutrixs and let $u\colon \mathbb{N} \longrightarrow \mathbb{E}, v\colon \mathbb{N} \longrightarrow \mathbb{E}$ be flexible sequences such that $u_n \underset{N}{\longrightarrow} \alpha$ and $v_n \underset{M}{\longrightarrow} \beta$, for some $\alpha, \beta \in \mathbb{E}$. Then*

$$(u_n - v_n) \underset{N+M}{\longrightarrow} \alpha - \beta.$$

In particular, if $N = M$ then $(u_n - v_n) \underset{N}{\longrightarrow} \alpha - \beta$.

In the following proposition we show that if $\{u_n\}$ is N-convergent and c is precise then $\{cu_n\}$ is cN-convergent.

Proposition 5.2.20. *Let N be a neutrix, $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a flexible sequence such that $u_n \underset{N}{\longrightarrow} \alpha$, for some $\alpha \in \mathbb{E}$. Let $c \in \mathbb{R}$. Then $cu_n \underset{cN}{\longrightarrow} c\alpha$.*

Proof. If $c = 0$, the conclusion is trivial. Assume $c \neq 0$. Let $\epsilon > cN = |c|N$. So $\epsilon / |c| > N$. Hence there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ one has $|u_n - \alpha| < \epsilon / |c|$. Because c is precise, the distributivity holds. It follows that

$$|cu_n - c\alpha| = |c(u_n - \alpha)| = |c|\,|u_n - \alpha| < |c|\frac{\epsilon}{|c|} = \epsilon.$$

We conclude that $cu_n \underset{cN}{\longrightarrow} c\alpha$. □

Example 5.2.21. Let ω be unlimited and consider the constant sequence $u_n = \omega + £$. Then $u_n \underset{£}{\longrightarrow} \omega$ and $u_n\omega \underset{\omega £}{\longrightarrow} \omega^2 + \omega £$. However, ωu_n is not £-convergent to $\omega^2 + \omega £$. Indeed, one has $\omega u_n = \omega(\omega + £) = \omega^2 + \omega £$. Suppose that $\omega u_n \underset{£}{\longrightarrow} \omega^2 + \omega £$. Let $\epsilon \in \omega £$ and $\epsilon > £$. For all $n \in \mathbb{N}$ one has $\epsilon \leq |\omega u_n - \omega^2| = \omega £$, which is a contradiction.

Next we present an adapted version of a classical result stating that the product of a bounded sequence with a sequence which converges to zero converges to zero.

Proposition 5.2.22. *Let $u, v \colon \mathbb{N} \longrightarrow \mathbb{E}$ be flexible sequences such that $u_n \underset{N}{\longrightarrow} 0$. If there exists $\alpha \in \mathbb{E}$ such that $\alpha \neq \mathbb{R}$ and $|v_n| < \alpha$, for all $n \in \mathbb{N}$, then $(u_n v_n) \underset{\alpha N}{\longrightarrow} 0$.*

Proof. Without restriction of generality we assume that the element α satisfying $|v_n| < \alpha$ is precise. Let $\epsilon > \alpha N$. Then there exists $n_0 \in \mathbb{N}$ such that $|u_n| < \epsilon / \alpha$ for all $n \geq n_0$. Then for all $n \geq n_0$ we have

$$|u_n v_n| = |u_n|\,|v_n| < \alpha \frac{\epsilon}{\alpha} = \epsilon.$$

Hence $(u_n v_n) \underset{\alpha N}{\longrightarrow} 0$. □

Now we turn to the problem of evaluating the N-limit of the product of two flexible sequences. This requires a somewhat delicate approach, as illustrated by the following example.

Example 5.2.23. Let ω be unlimited and $\{y_n\}$ be the constant sequence defined by $y_n = \omega^2 + \omega £$ for all $n \in \mathbb{N}$. Then $y_n \underset{\omega £}{\longrightarrow} \omega^2$. Then $y_n^2 = \omega^4 + 2\omega^3 £ + \omega^2 £ = \omega^4 + \omega^3 £$. Consequently $(y_n y_n) \underset{\omega^3 £}{\longrightarrow} \omega^4$. However $\{y_n y_n\}$ is $\omega £$-divergent.

Let M, N be neutrices. Let $\{u_n\}$ be a M-convergent sequence and (v_n) be a N-convergent sequence. As seen in Example 5.2.23, in some cases, the product sequence $\{u_n v_n\}$ may be neither N-convergent, nor M-convergent and not even (NM)-convergent. However, the sequence $\{u_n v_n\}$ is K-convergent, for a neutrix K.

Theorem 5.2.24. *Let M, N be neutrices and $\alpha = a + N, \beta = b + M$. Let $u \colon \mathbb{N} \longrightarrow \mathbb{E}, v \colon \mathbb{N} \longrightarrow \mathbb{E}$ be such that $u_n \underset{N}{\longrightarrow} \alpha, v_n \underset{M}{\longrightarrow} \beta$. Let $K = N + M + N^2 + M^2 + \alpha M + \beta N$. Then $u_n v_n \underset{K}{\longrightarrow} \alpha\beta$.*

Proof. Put $u_n = a_n + A_n$ and $v_n = b_n + B_n$. Let $\epsilon > K$. Then, by Lemma 2.2.43 and subdistributivity, we have

$$\begin{aligned}|u_nv_n - \alpha\beta| &= |a_nb_n + a_nB_n + b_nA_n + A_nB_n - \alpha\beta| \\ &\leq |a_nb_n - \alpha\beta| + a_nB_n + b_nA_n + A_nB_n \\ &\leq |a_n(b_n - \beta) + \beta a_n - \alpha\beta| + +a_nB_n + b_nA_n + A_nB_n \\ &\leq |a_n(b_n - \beta)| + |b||a_n - \alpha| + M|a_n| - M\alpha + a_nB_n + b_nA_n + A_nB_n.\end{aligned} \tag{5.3}$$

We show first that there exists $p_0 \in \mathbb{N}$ such that for $n \geq p_0$ one has

$$|a_n||b_n - \beta| < \epsilon/7. \tag{5.4}$$

We consider two cases, (i) $a \in N$ and (ii) $a \notin N$. In case (i) we can take $a = 0$. Because $\epsilon > N^2$ we have $\sqrt{\epsilon} > N$. So there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \sqrt{\epsilon}$ for $n > n_0$. Also $\epsilon > M^2$ implies $\sqrt{\epsilon} > M$ and hence $\sqrt{\epsilon}/7 > M$. It follows that there exists $n_1 \geq n_0$ such that $|b_n - \beta| < \sqrt{\epsilon}/7$ for all $n \geq n_1$. As a result, for $n \geq n_1$ it holds that

$$|a_n||b_n - \beta| < \sqrt{\epsilon}.\sqrt{\epsilon}/7 = \epsilon/7,$$

as required. In case (ii) we have $|a| > N$. So there exists n_2 such that $|a_n| < 2|a|$ for all $n \geq n_2$. On the other hand $\epsilon > |a|M = \alpha M$ implies $\dfrac{\epsilon}{14|a|} > M$. It follows that there exists $n_3 \geq n_2$ such that $|b_n - \beta| < \dfrac{\epsilon}{14|a|}$ for all $n \geq n_3$. Consequently, $|a_n||b_n - \beta| < 2|a||b_n - \beta| < \epsilon/7$ for all $n \geq n_3$, as required. Formula (5.4) follows by putting $p_0 = \max\{n_1, n_3\}$.

Secondly we show that there exists $p_1 \in \mathbb{N}$ such that for $n \geq p_1$ one has,

$$|a_n|M < \epsilon/7. \tag{5.5}$$

As above we distinguish the cases (i) $a \in N$ and (ii) $a \notin N$. In case (i) one has, as above, that for $n \geq n_0$ it holds that $|a_n| < \sqrt{\epsilon}$ and that $\sqrt{\epsilon}/7 > M$. So $|a_n|M < \sqrt{\epsilon} \cdot \sqrt{\epsilon}/7 = \epsilon/7$. In case (ii), as above, for $n \geq n_2$ it holds that $|a_n|M < 2|a|M < \epsilon/7$. Hence (5.5) holds by putting $p_1 = \max\{n_0, n_2\}$.

Thirdly we show that there exists $p_2 \in \mathbb{N}$ such that for $n \geq p_2$,

$$|b||a_n - \alpha| < \epsilon/7. \tag{5.6}$$

Note that if $b \in M$ then we may assume $b = 0$ and the result follows. Assume that $|b| > M$. Because $\epsilon > \beta N = bN$, we have $\dfrac{\epsilon}{7|b|} > N$. So there exists $p_2 \in \mathbb{N}$ such that $|a_n - \alpha| < \dfrac{\epsilon}{7|b|}$ for all $n \geq p_2$. Hence (5.6) holds for $n \geq p_2$.

Fourthly we prove that there exists $p_4 \in \mathbb{N}$ such that for $n \geq p_4$,

$$|b_n|A_n < \epsilon/7. \tag{5.7}$$

Again we consider two cases. (i) $b \notin M$ and (ii) $b \in M$. In case (i) we have $|b_n| < 2|b|$ for $n \geq n_4$. On the other hand, by Proposition 5.2.8 we obtain that $A_n \underset{N}{\longrightarrow} A$. Also $\epsilon > bN$ implies $N < \epsilon/|b|$. It follows that $N < \dfrac{\epsilon}{14|b|}$. So there exists k_0 such that $|A_n| < \dfrac{\epsilon}{14|b|}$ for all $n \geq k_0$. Put $p_4 = \max\{n_4, k_0\}$. Then for all $n \geq p_4$ it holds that $|b_n|A_n < 2|b|A_n < \epsilon/7$. In case (ii) we may assume that $b = 0$. Because $\epsilon > M^2$, on has $\epsilon/7 > M^2$. Hence $\sqrt{\dfrac{\epsilon}{7}} > M$. By Proposition 5.2.8 we have $b_n \underset{M}{\longrightarrow} 0$. So there exists $n_5 \in \mathbb{N}$ such that $|b_n| < \sqrt{\dfrac{\epsilon}{7}}$ for $n \geq n_5$. On the other hand, because $\sqrt{\dfrac{\epsilon}{7}} > N$ there exists $n_7 \in \mathbb{N}$ such that $A_n \leq \sqrt{\dfrac{\epsilon}{7}}$ for $n \geq n_7$. Put $n_8 = \max\{n_5, n_7\}$. Then $|b_n|A_n < \epsilon/7$ for all $n \geq n_8$, as required. Hence (5.7) holds by taking $p_4 = \max\{n_5, n_8\}$.

Similarly, there exists p_5 such that

$$a_n B_n < \epsilon/7 \quad \text{for all} \quad n \geq p_5. \tag{5.8}$$

As argument above, there exists k_1, k_2 such that $B_n < \sqrt{\epsilon}$ for all $n \geq k_1$ and $A_n < \sqrt{\epsilon}/7$ for all $n \geq k_2$. Put $p_6 = \max\{k_1, k_2\}$. Then for all $n \geq p_6$,

$$A_n B_n < \epsilon/7. \tag{5.9}$$

Clearly,

$$M\alpha < \epsilon/7. \tag{5.10}$$

Let $k = \max\{p_0, ..., p_6\}$. Then, from (5.3)-(5.10)we conclude that $|u_n v_n - \alpha\beta| < \epsilon$ for all $n \geq k$. Hence $u_n v_n \underset{K}{\longrightarrow} \alpha\beta$. □

In practice several neutrices occurring in the neutrix K can be neglected, according to circumstances. In fact, if $N, M \subseteq £$ we can neglect the terms $N^2 + M^2$. Also, if $£ \subset M, N$ we can neglect the terms $N + M$.

In classical mathematics, if $\epsilon^2 > 0$ then $\epsilon > 0$. In the case of a neutrix M, if $\epsilon^2 > M$ it is not always true that $\epsilon > M$. For example, let $\epsilon = \omega$ and $M = \omega£$. Then $\epsilon^2 > M$, but $\epsilon \not> M$.

In the proof above, as long as the terms of product sequences are outside the limit neutrix we can not always use ϵ-estimates, but sometimes we should resort to ϵ^2-estimates. We can illustrate this by considering the two following sequences u, v: $\mathbb{N} \longrightarrow \mathbb{E}$ given by

$$u_n = \omega£ \text{ for all } n \in \mathbb{N}$$

and

$$v_n = \begin{cases} n/2 & \text{if } n \leq \omega^4 \\ 0 & \text{if } n > \omega^4, \end{cases}$$

where $\omega \in \mathbb{N}$ is unlimited. Then $u_n \underset{\omega£}{\longrightarrow} \omega£$ and $v_n \longrightarrow 0$. Let $\epsilon = \omega^2 > \omega£$. Then $|v_n| < \epsilon$ for all $n \in \mathbb{N}, n \geq \omega^2$ and $|u_n| < \epsilon$ for all $n \in \mathbb{N}$. However, for $n = \omega^3$ we have $|u_n v_n| < \epsilon^2$ and $|u_n.v_n| \not< \epsilon$. As a consequence, if we neglect the terms $N^2 + M^2$ in the neutrix K, in general, it does not guarantee that these terms are less than ϵ for all $\epsilon > K$.

Finally we turn to the quotient of sequences. Let N be a neutrix and $\{u_n\}$ be a N-convergent sequence. In virtue of Theorem 5.2.24 it is enough to study the sequence $(1/u_n)$.

Definition 5.2.25. A flexible sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ is said to be *zeroless* if $u_n \neq N(u_n)$ for all $n \in \mathbb{N}$, i.e. if $0 \notin u_n$ for all $n \in \mathbb{N}$.

Lemma 5.2.26. *Let N be a neutrix and $\{a_n\}$ be a zeroless real sequence such that $a_n \underset{N}{\longrightarrow} a$, for some $a \in \mathbb{R}, |a| > N$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $|a|/2 < |a_n| < 2\,|a|$.*

Proof. Let $\epsilon = |a|\,/2$. Clearly $|a|\,/2 > N$. So there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $|a_n - a| < \epsilon = |a|/2$. It follows that $|a| - |a|/2 < |a_n| < |a| + |a|/2 = 3|a|/2$. Hence for all $n \geq n_0$, $|a|/2 < |a_n| < 3/2\,|a| \leq 2|a|$. □

Theorem 5.2.27. *Let N be a neutrix and $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a zeroless sequence such that $u_n \underset{N}{\longrightarrow} \alpha$, for some zeroless $\alpha = a + N \in \mathbb{E}$, where $|a| > N$. Then the sequence $(1/u_n)$ is N/a^2-convergent to $1/\alpha$.*

Proof. Write $u_n = a_n + A_n$ for all $n \in \mathbb{N}$. Let $\epsilon > N/a^2$. Then $a^2\epsilon/b > N$ for all $b \in @$. Because u_n is N-convergent to α, by Proposition 5.2.8 it holds that A_n N-converges to N and that a_n N-converges to a. So there exists n_0 such that for all $n \geq n_0$,

$$|A_n - N| < a^2\epsilon/3 \tag{5.11}$$

and

$$|a - a_n| < a^2\epsilon/6.$$

Formula (5.11) implies $|A_n| \leq a^2\epsilon/3$ for all $n \geq n_0$. By Lemma 2.2.20(i) and the fact there exists $n_1 \in \mathbb{N}$ such that $|a|/2 < |a_n| < 2|a|$ for all $n \geq n_1$ we obtain that for all $n \geq k = \max\{n_0, n_1\}$

$$\begin{aligned}
\left|\frac{1}{u_n} - \frac{1}{\alpha}\right| = \left|\frac{u_n}{a_n^2} - \frac{\alpha}{a^2}\right| = \frac{|a^2u_n - a_n^2\alpha|}{a_n^2a^2} &= \frac{|a^2a_n - a_n^2a|}{a_n^2a^2} + \frac{a^2A_n + a_n^2N}{a^2a_n^2} \\
&\leq \frac{|a - a_n|}{a_na} + \frac{a^2A_n + 4|a|^2N}{a^4/4} \\
&\leq \frac{|a - a_n|}{a^2/2} + \frac{A_n + N}{a^2} < \frac{a^2\epsilon/6}{a^2/2} + \frac{a^2\epsilon/3 + a^2\epsilon/3}{a^2} = \epsilon.
\end{aligned}$$

Hence $(1/u_n)$ is N/a^2-convergent to $1/\alpha$. □

Theorem 5.2.28. *Let $\{u_n\}, \{v_n\}$ be flexible sequences such that $\{v_n\}$ is zeroless and N be a neutrix. Assume that $u_n \underset{N}{\longrightarrow} \alpha$ and $v_n \underset{M}{\longrightarrow} \beta$. Then the sequence $\dfrac{u_n}{v_n}$ is K-convergent to $\dfrac{\alpha}{\beta}$, where $K = N + M/\beta^2 + N^2 + \left(\dfrac{M}{\beta^2}\right)^2 + \dfrac{N}{\beta} + \alpha\dfrac{M}{\beta^2}$.*

Proof. It follows from Theorem 5.2.24 and Theorem 5.2.27. □

5.2.6 Subsequences

In general, in non-standard analysis induction can not apply induction to an external formula, so we can not define subsequences of a flexible sequence like in classical mathematics. We will use the notion of *cofinal* *set* instead of subsequences of a flexible sequence. This is a generalization of classical definition because an internal cofinal set is a subsequence.

Definition 5.2.29. Let $X \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{R})$ be such that $X \neq \emptyset$. A set $J \subseteq X$ is called *cofinal* if

$$\forall k \in \mathbb{N}\, \exists n \geq k\, (n, x) \in J.$$

It is easy to see that the number of elements of a cofinal set can not be finite.

We denote by $P_{\mathbb{N}}(J)$ the projection of J on $\mathbb{N}$ and $\{P_{\mathbb{E}}(J)\}$ the projection of J on $\mathbb{E}$.

Definition 5.2.30. Let $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a flexible sequence. Each cofinal set J of $u = \{(n, u_n) : n \in \mathbb{N}\}$ is called a *subsequence* of $\{u_n\}$. We write $\{u_{n_m}\}_{P_{\mathbb{N}}(J)} \subseteq \{u_n\}$.

Next proposition gives a characterization of N-convergence in terms of the N-convergence of its subsequences.

Proposition 5.2.31. *Let N be a neutrix, $\alpha = a + N \in \mathbb{E}$ and $\{u_n\}$ be a flexible sequence. Then $u_n \underset{N}{\longrightarrow} \alpha$ if and only if every subsequence $\{u_{n_m}\}_{P_{\mathbb{N}}(J)} \subseteq \{u_n\}$, we have $u_{n_m} \underset{N}{\longrightarrow} \alpha$.*

Proof. Assume first that $\{u_n\}$ is N-convergent to α. Let J be a cofinal set of $\{u\}$ and $\{u_{n_m}\}_{P_{\mathbb{N}}(J)} \subseteq \{u_n\}$. Let $\epsilon > N$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|u_n - \alpha| < \epsilon$. Also J is a cofinal set, so there exists $k \in \mathbb{N}$ such that $k \geq n_0$. As a consequence, for all $n_m \in P_{\mathbb{N}}(J), n_m \geq k \geq n_0$ we have $|u_{n_m} - \alpha| < \epsilon$. We conclude that $u_{n_m} \underset{N}{\longrightarrow} \alpha$. The other implication is obvious because $\{u_n\}$ is a subsequence of itself. $\square$

Example 5.2.32. Consider the flexible sequence $u\colon \mathbb{N} \to \mathbb{E}$ defined by $u_n = (-1)^n + \frac{1}{n}\oslash$, $n \in \mathbb{N}$. Then $\{u_n\}$ is a divergent sequence. Indeed, we consider two subsequences $u_{2n} = 1 + \frac{1}{2n}\oslash$ and $u_{2n+1} = -1 + \frac{1}{2n+1}\oslash$. One has $u_{2n} \to 1$ and $u_{2n+1} \to -1$. Also $1 - (-1) = 2 > 0$. Applying Proposition 5.2.31 we conclude that the sequence $\{u_n\}$ is divergent.

An external subsequence may not satisfy all useful properties of conventional subsequences. The theorem below says that every real sequence has an internal subsequence.

Theorem 5.2.33. *Let $a\colon \mathbb{N} \longrightarrow \mathbb{R}$ be a real sequence. There exists an internal subsequence of $\{a_n\}$.*

Proof. If $\{a_n\}$ is an internal sequence, the conclusion is trivial. We now suppose that $\{a_n\}$ is an external sequence. Let $\{a\} = \{(n, a_n) : n \in \mathbb{N}\}$. The external set $\{a\}$ can be represent as $a = \bigcup_{\mathrm{st}(x) \in X} H_x$, where $H_x = \bigcap_{\mathrm{st}(y) \in Y} I_{xy}$ with X, Y standard sets and $I{:}X \times Y \rightrightarrows \mathcal{P}(\mathbb{N} \times \mathbb{R})$ is an internal set-valued mapping for all $x \in X, y \in Y$. There exists $x \in X$ such that H_x is cofinal set, otherwise, $\{a\}$ is included in a finite set, which is a contradiction. This implies that for all $\mathrm{st}(y) \in Y, I_{xy}$ is cofinal. So $\forall^{\mathrm{st} fin} Z \subseteq Y\, \exists\, J\, (\forall^{\mathrm{st}} y \in Y\, (J \subseteq I_{xy}))$ and J is cofinal and internal. Indeed, we can take $J = \bigcap_{\mathrm{st}(y) \in Z} I_{xy}$. By the Idealization principle, there exists an internal cofinal set J of $\{a\}$ such that for all $\mathrm{st}(y) \in Y$ one has $J \subseteq I_{xy}$. Hence $J \subseteq H_x$. So J is an internal subsequence of a. $\square$

5.2.7 N-Cauchy sequences

Intuitively, a sequence $\{u_n\}$ is a Cauchy sequence if the terms of the sequence become arbitrarily close to each other as the sequence progresses. In other words the difference between terms of the sequence converges to 0. In this section we use the notion of N-convergence defined above in order to generalize the notion of Cauchy sequence to sequences in which the difference between terms of the sequence is arbitrary close to a given neutrix.

Definition 5.2.34. Let N be a neutrix. A flexible sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ is said to be a N-*Cauchy sequence* if for all $\epsilon > N$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}, m, n \geq n_0$ one has $|u_n - u_m| < \epsilon$.

Remark 5.2.35. Similarly to Theorem 5.2.2, to verify whether a given flexible sequence is N-Cauchy or not, it is sufficient to do it with all $\epsilon \in \mathbb{R}$, instead of $\epsilon \in \mathbb{E}$.

A flexible sequence $\{u_n\}$ is a N-Cauchy sequence if and only if for all $\epsilon > N$, there exists $p \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ and $n \geq p$ one has $|u_{n+k} - u_n| < \epsilon$.

Example 5.2.36. Let N be a fixed neutrix and $u\colon \mathbb{N} \longrightarrow \mathbb{E}$ be a sequence defined by $u_n = s_n + N$, where $\{s_n\}$ is a Cauchy sequence. We will show that $\{u_n\}$ is a N-Cauchy sequence. Let $\epsilon > N$ arbitrary. Then there is $n_0 \in \mathbb{N}$ such that for $m, n > n_0$, $|s_m - s_n| < \delta$, for $\delta < N$. This implies that

$$\begin{aligned} |u_m - u_n| &= |s_m + N - (s_n + N)| \\ &= |s_m - s_n + N| \leq |s_m - s_n| + N \leq \delta + N \leq N + N < \epsilon. \end{aligned}$$

Hence $\{u_n\}$ is an N-Cauchy sequence.

Proposition 5.2.37. *Let $\{u_n\}$ be a flexible sequence with $u_n = a_n + A_n \in \mathbb{E}$ for all $n \in \mathbb{N}$. If the flexible sequence $\{u_n\}$ is N-Cauchy, then $\{a_n\}, \{A_n\}$ are two N-Cauchy sequences.*

Proof. Because $\{u_n\}$ is a N-Cauchy sequence, for each $\epsilon > N$ there exists n_0 such that for all $n > n_0$ and for all $p > 0$ one has $|u_{p+n} - u_n| < \epsilon$. That is $|a_{n+p} + A_{n+p} - (a_n + A_n)| = |a_{n+p} - a_n| + A_n + A_{n+p} < \epsilon$. It follows that $A_n + A_{n+p} < \epsilon$ and $|a_n - a_{n+p}| < \epsilon$. Hence $\{a_n\}, \{A_n\}$ are both N-Cauchy sequences. □

Proposition 5.2.38. *Let $\{A_n\}$ be a flexible sequence of neutrices. If $\{A_n\}$ is N-Cauchy then $A_n \underset{N}{\longrightarrow} N$.*

Proof. Let $\epsilon > N$. Because $\{A_n\}$ is N-Cauchy, there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$ and for all $p > 0$ one has $|A_{n+p} + A_n| < \epsilon/2$. Also $A_n \leq A_n + A_{n+p}$, so $A_n < \epsilon/2$ for all $n \geq n_0$. This implies that $|A_n - N| = A_n + N < \epsilon/2 + \epsilon/2 = \epsilon$; hence $A_n \underset{N}{\longrightarrow} N$. □

Lemma 5.2.39. *Let $\{a_n\}$ be an internal real sequence. If $\{a_n\}$ is N-Cauchy then $\{a_n\}$ is N-convergent.*

Proof. Because $\{a_n\}$ is a real N-Cauchy sequence, it is bounded. Also $\{a_n\}$ is internal, so there exists a subsequence $\{a_{n_m}\}$ of $\{a_n\}$ which convergent to a for some $a \in \mathbb{R}$. Let $\epsilon > N$. Then $\epsilon/2 > N$. So there exist $n_1, n_2 \in \mathbb{N}$ such that $|a_{n_m} - a| < \epsilon/2$ for all $n_m \geq n_1$ and $|a_m - a_n| < \epsilon/2$ for all $m, n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ it holds that $|a_n - a| = |a_n - a_{n_m} + a_{n_m} - a| \leq |a_n - a_{n_m}| + |a_{n_m} - a| < \epsilon$. □

Proposition 5.2.40. *Let $\{a_n\}$ be a real sequence and N be a neutrix. Let $\{a\} = \{(n, a_n) : n \in \mathbb{N}\}$. Assume that $\{a_n\}$ is N-Cauchy. If there exists an N-convergent subsequence $\{a_{n_m}\}_{P_{\mathbb{N}}(J)} \subseteq \{a\}$, where J is a cofinal set of $\{a\}$, then $\{a_n\}$ is N-convergent.*

Proof. Assume that $a_{n_m} \underset{N}{\longrightarrow} b$ for some $b \in \mathbb{R}$. Let $\epsilon > N$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $|a_{n_m} - b| \leq \epsilon/2$ for all $n_m \geq n_1$ and $|a_m - a_n| < \epsilon/2$ for all $m, n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ it holds that $n_m \geq n \geq n_0$. So $|a_n - b| = |a_n - a_{n_m} + a_{n_m} - b| \leq |a_{n_m} - a_n| + |a_{n_m} - b| < \epsilon$ for all $n \geq n_0$. Hence $\{a_n\}$ is N-convergent to b. □

Theorem 5.2.41. *Let $u_n = a_n + A_n$ for all $n \in \mathbb{N}$ be a flexible sequence. Then $\{u_n\}$ is N-convergent if and only if $\{u_n\}$ is N-Cauchy.*

Proof. We assume that $u_n \underset{N}{\longrightarrow} \alpha$ for some $\alpha \in \mathbb{E}$. Then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $|u_n - \alpha| < \epsilon/2$. So for $m, n > n_0$ it holds that

$$\begin{aligned} |u_m - u_n| &\leq |u_m - u_n + N| = |u_m - \alpha - u_n + \alpha| \\ &\leq |u_m - \alpha| + |u_n - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\{u_n\}$ is an N-Cauchy sequence.

Conversely, we assume that $\{u_n\}$ is N-Cauchy. By Theorem 5.2.33, there is an internal subsequence $\{a_{n_m}\} \subseteq \{a_n\}$. Since $\{a_n\}$ is N-Cauchy, the sequence $\{a_{n_m}\}$ is N-Cauchy. So $\{a_{n_m}\}$ is N-convergent by Lemma 5.2.39. It follows that $\{a_n\}$ is N-convergent by Proposition 5.2.40. Also, the sequence $\{A_n\}$ is N-Cauchy, so $A_n \underset{N}{\longrightarrow} N$ by Proposition 5.2.38. By Proposition 5.2.8 we conclude that $\{u_n\}$ is N-convergent. □

Let N be a neutrix. Next proposition states that if a sequence has two N-convergent subsequences whose N-limits are sufficiently far then it cannot be N-convergent.

Proposition 5.2.42. *Let N be a neutrix and $\{u_n\}$ be a flexible sequence. Let $\{u\} = \{(n, u_n) : n \in \mathbb{N}\}$. Assume that there exist subsequences $\{u_{n_m}\}_{P_{\mathbb{N}}(J)}, \{u_{n_k}\}_{P_{\mathbb{N}}(K)}$ of $\{u_n\}$, where J, K are two cofinal sets of $\{u\}$, such that $u_{n_m} \underset{N}{\longrightarrow} \alpha, u_{n_k} \underset{N}{\longrightarrow} \beta$ and $N < |\alpha - \beta|$. Then $\{u_n\}$ is N-divergent.*

Proof. Suppose that $\{u_n\}$ is N-convergent. Let ϵ be such that $N < \epsilon < |\alpha - \beta|$. By Proposition 5.2.41, there exists $n_0 \in \mathbb{N}$ such that $|u_m - u_n| < \epsilon/3$ for all $n, m \geq n_0$. On the other hand, because $u_{m_n} \underset{\rightarrow}{N} \alpha$, there exists $n_1 \in \mathbb{N}$ such that $|u_{m_n} - \alpha| < \epsilon/3$ for all $n \geq n_1$. Also $u_{k_n} \underset{N}{\longrightarrow} \beta$, so there exists $n_2 \in \mathbb{N}$ such that $|u_{k_n} - \beta| < \epsilon/3$ for all $n \geq n_2$. Let $k_0 = \max\{n_0, n_1, n_2\}$. Then for $n > k_0$ one has $|\alpha - \beta| \leq |\alpha - u_{m_n}| + |u_{m_n} - u_{k_n}| + |u_{k_n} - \beta| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon < |\alpha - \beta|$, a contradiction. Hence, $\{u_n\}$ is not N-convergent. □

5.3 Strong convergence

A well-known property of a real standard sequence a_n is that it converges to a if and only if $a_n \in a + \oslash$ for all n unlimited. Next we generalize this fact to a flexible sequence. We will show that a flexible sequence $\{u_n\}$ N-converges to $\alpha \in \mathbb{E}$ with $N \neq 0$, then there exists an index n_0 such that $u_n \subseteq \alpha$ for all $n \geq n_0$. We call it *strongly convergent*.

In this section we will also present properties and operations of strong limits. We will show that properties which hold for limits also hold for strong limits. In addition, we introduce the notion of strongly N-Cauchy sequence and demonstrate that flexible sequences are strong N-completely Cauchy, i.e. every strongly N-Cauchy sequence is N-convergent and vice versa.

5.3.1 Definition and example

Definition 5.3.1. Let $\{u_n\}$ be a flexible sequence and $\alpha = a + A$ be an external number. The sequence $\{u_n\}$ is said to be *strongly convergent to* α if there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $u_n \subseteq \alpha$. We write $\text{Lim} u_n = \alpha$ or $u_n \hookrightarrow \alpha$. The external number α is called a *strong limit* of $\{u_n\}$.

Example 5.3.2. Consider the flexible sequence $u_n = \dfrac{1}{n} + \oslash$. Then $\text{Lim} u_n = \oslash$.

Obviously, every flexible sequence is strongly convergent to $\mathbb{R}$. From now on, unless otherwise stated, when we say α is a strong limit we implicitly assume that $\alpha \neq \mathbb{R}$.

Also, if $\alpha \neq \mathbb{R}$ is a strong limit of $\{u_n\}$, for every neutrix M, $N(\alpha) \subseteq M$ it holds that $\alpha + M$ is a strong limit of $\{u_n\}$. So in practice we prefer to find neutrices which are as small as possible.

Observe that if $u_n \hookrightarrow \alpha$ then $N(u_n) \subseteq N(\alpha)$ for all $n \geq n_0$. Moreover, it is easy to see that if $u_n \hookrightarrow \alpha$ then $u_n \underset{N(\alpha)}{\longrightarrow} \alpha$. However, in general, the converse is not true. For example, consider the flexible sequence $u_n = \dfrac{1}{n} + \dfrac{1}{n}\oslash$. Then $u_n \rightarrow 0$ but u_n is not strongly convergent to 0.

5.3.2 Operations on strong convergence

The behaviour of the strong limit under operations is as expected, but the proofs are easier than the case of the ordinary N-limit.

Proposition 5.3.3. *Let* $\{u_n\}, \{v_n\}$ *be two flexible sequences and* $\gamma \in \mathbb{E}$. *Assume that* $\text{Lim} u_n = \alpha$ *and* $\text{Lim} v_n = \beta$*, where* $\alpha \neq \mathbb{R}, \beta \neq \mathbb{R}$. *Then*

(i) $\text{Lim}(\gamma u_n) = \gamma \text{Lim} u_n = \gamma\alpha$.

(ii) $\text{Lim}(u_n \pm v_n) = \text{Lim} u_n \pm \text{Lim} v_n = \alpha \pm \beta$.

Proof. The properties follow from the definition of strong convergence. □

Proposition 5.3.4. *Let $\{u_n\}, \{v_n\}$ be two flexible sequences and N, M be two neutrices. Assume that $u_n \hookrightarrow \alpha$ and $v_n \hookrightarrow \beta$ for some $\alpha, \beta \in \mathbb{E}$. Then $\{u_n v_n\}$ is strongly convergent to $\alpha\beta$.*

Proof. By assumptions there exists n_0 such that for all $n \geq n_0$, $u_n \subseteq \alpha$ and $v_n \subseteq \beta$. This implies that $u_n v_n \subseteq \alpha\beta$ for all $n \geq n_0$, and hence $u_n v_n \hookrightarrow \alpha\beta$. □

Proposition 5.3.5. *Let $\{u_n\}$ be a zeroless sequence. Assume that $u_n \hookrightarrow \alpha$, where $\alpha \in \mathbb{E}$ is zeroless. Then $\frac{1}{u_n} \hookrightarrow \frac{1}{\alpha}$.*

Proof. Because $u_n \hookrightarrow \alpha$, there exists $n_0 \in \mathbb{N}$ such that $u_n \subseteq \alpha$ for all $n \geq n_0$. It follows that $\frac{1}{u_n} \subseteq \frac{1}{\alpha}$ for all $n \geq n_0$. Hence $\frac{1}{u_n} \hookrightarrow \frac{1}{\alpha}$. □

Proposition 5.3.6. *Let $\{u_n\}$ be a flexible sequence. If $\{u_n\}$ is strongly convergent to α then $|u_n|$ is strongly convergent to $|\alpha|$.*

Proof. Assume that $u_n \hookrightarrow \alpha$. Then there exists $n_0 \in \mathbb{N}$ such that $u_n \subseteq \alpha$ for all $n \geq n_0$. This means $u_n - \alpha \subseteq N(\alpha)$. So $||u_n| - |\alpha|| \leq |u_n - \alpha| \subseteq N(\alpha)$. It follows that $|u_n| \subseteq |\alpha|$ for all $n \geq n_0$. One concludes that $|u_n| \hookrightarrow |\alpha.|$ □

Next we present a version of squeeze theorem for strong limits.

Proposition 5.3.7. *Let $\{u_n\}, \{v_n\}, \{w_n\}$ be flexible sequences. Assume that $u_n \leq w_n \leq v_n$ for all $n \in \mathbb{N}$ and $u_n \hookrightarrow \alpha, v_n \hookrightarrow \alpha$. Then $w_n \hookrightarrow \alpha$.*

Proof. For $n \in \mathbb{N}$, let $u_n = a_n + A_n, v_n = b_n + B_n, w_n = d_n + D_n$ and $\alpha = a + A$. We may assume that always $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Because $u_n \hookrightarrow \alpha$, there exists $n_1 \in \mathbb{N}, n_1 \geq n_0$ such that $u_n \subseteq \alpha$ for all $n \geq n_1$. Similarly, since $v_n \hookrightarrow \alpha$, there exists $n_2 \geq n_0$ such that $v_n \subseteq \alpha$ for all $n \geq n_2$. Let $p = \max\{n_1, n_2\}$. We prove that $w_n \subseteq \alpha$ for all $n \geq p$. Let $n \geq p$. Then we have the following cases:

Case 1: $w_n \subseteq v_n$. Then $w_n \subseteq \alpha$.

Case 2: $w_n < v_n$ and $u_n < w_n$. Then $w_n \subseteq [a_n, b_n] \subseteq \alpha$.

Case 3: $w_n < v_n$ and $u_n \subseteq w_n$. Then $a_n + D_n^+ \subseteq [a_n, b_n] \subseteq \alpha$. In particular, $D_n^+ \subseteq N(\alpha)$. Because $N(\alpha)$ is symmetric, one concludes that $D_n \subseteq \alpha$. So $w_n \subseteq \alpha$.

Hence $w_n \subseteq \alpha$ for all $n \geq k$ and we conclude that $w_n \hookrightarrow \alpha$. □

5.3.3 Some properties of strongly convergent flexible sequences

We below obtain similar results on properties of strong limits as in the case of N-limits.

Proposition 5.3.8. *Let $\{u_n\}$ be a flexible sequence and $\alpha \in \mathbb{E}$. Then $\{u_n\}$ is strongly convergent to α if and only if every subsequence $\{u_n\}_{P_{\mathbb{N}}(J)}$ of $\{u_n\}$, one has $\{u_{P_{\mathbb{N}}(J)}\}$ is strongly convergent to α.*

Proof. Assume that $u_n \hookrightarrow \alpha$. So $u_n \subseteq \alpha$ for all $n \geq n_0$. For a subsequence $\{u_n\}_{P_{\mathbb{N}}(J)}$ there exists $k \in P_{\mathbb{N}}(J) \geq n_0$. This implies that $u_m \subseteq \alpha$ for all $m \in P_{\mathbb{N}}(J)$, $m \geq k \geq n_0$. We conclude that $\{u_n\}_{P_{\mathbb{N}}(J)}$ is strongly convergent to α.

The converse is trivial because $\{u_n\}$ is a subsequence of itself. □

Proposition 5.3.9. *Let $\{u_n\}$ be a flexible sequence and $\alpha \in \mathbb{E}$. If $\{u_n\}$ is strongly convergent to α, then $\{u_n\}$ is eventually bounded.*

Proof. We have $u_n \subseteq \alpha$ for all $n \geq n_0$. Because $\alpha \neq \mathbb{R}$, it holds that $\{u_n\}$ is eventually bounded. □

Proposition 5.3.10. *Let $\{u_n\}$ be a flexible sequence, where $u_n = a_n + A_n$ for all $n \in \mathbb{N}$. Then $u_n \hookrightarrow \alpha$ if and only if $a_n \hookrightarrow \alpha$ and $A_n \hookrightarrow N(\alpha)$.*

Proof. It follows by the fact that $u_n \subseteq \alpha$ if and only if $a_n \in \alpha$ and $A_n \subseteq N(\alpha)$. □

5.3.4 The relationship between N-limits and strong limits

Next we investigate the relationship between N-convergence and strong convergence. We first consider it for a real sequence and then for a flexible sequence. We will show that the two notions are equivalent when $N \neq 0$.

Proposition 5.3.11. *Let $\{a_n\} \subset \mathbb{R}$ be an internal sequence and $N \neq 0$ be a neutrix such that $a_n \notin N$ for all n. Then $\{a_n\}$ is not N-convergent to N.*

Proof. Firstly we assume that for all $n \in \mathbb{N}$, $a_n > N$. We consider two cases.

Case 1: The sequence $\{a_n\}$ is convergent to a for some $a \in \mathbb{R}$. Then $a \notin N$. Indeed, suppose that $a \in N$. Let $\epsilon \in N, \epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$. So $|a_n| < |a| + \epsilon$ for all $n \geq n_0$. Because $|a|+\epsilon \in N$, it follows that $a_n \in N$ for all $n \geq n_0$, which is a contradiction to the assumption. Suppose that $a < N$. Let $\eta_0 = |a|/2 > N$. Then there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < \eta_0$ for all $n \in \mathbb{N}, n \geq n_0$. This implies that $a_n < a + \eta_0 = a/2 < 0$ for all $n \geq n_0$, a contradiction. Hence $a > N$.

Suppose that $a_n \underset{N}{\longrightarrow} N$. Let $N < \epsilon = a/2$. Then $a/2 = a - \epsilon < a_n < a + \epsilon$ for all $n \geq n_0$. It follows that $|a_n - N| = a_n + N > a/2 + N > a/4 > N$ for all $n \geq n_0$, which is a contradiction. Hence N is not a N-limit of (a_n).

Case 2: The sequence $\{a_n\}$ is divergent. Suppose on contrary that $a_n \underset{N}{\longrightarrow} N$. Then $\{a_n\}$ is bounded. So there exists a subsequence $\{a_{m_n}\} \subset \{a_n\}$ such that $\{a_{m_n}\}$ has a limit $b \in \mathbb{R}$. By Case 1, the subsequence $\{a_{m_n}\}$ does not N-converge to N, a contradiction to Proposition 5.2.31. Hence N is not a N-limit of $\{a_n\}$.

Secondly, let $\{a_n\}$ be an arbitrary sequence. Suppose that $a_n \underset{N}{\longrightarrow} N$. Then $|a_n| \underset{N}{\longrightarrow} N$ with $|a_n| > N$ for all $n \in \mathbb{N}$, which is a contradiction. Hence $\{a_n\}$ is not N-convergent to N. □

Theorem 5.3.12. *Let $\{a_n\}$ be a real sequence and $N \neq 0$ be a neutrix. Assume that $a_n \underset{N}{\longrightarrow} N$. Then $a_n \hookrightarrow N$.*

Proof. Because $a_n \underset{N}{\longrightarrow} N$, we may assume that $\{a_n\}$ is bounded. Suppose that for each $n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $p \geq n$ and $a_p \notin N$. Let $J = \{(n, a_n) : a_n \notin N\}$. Then J is cofinal. By Theorem 5.2.33, there is an internal subsequence K of J. Let $b_n = a_n, n \in P_{\mathbb{N}}(K)$. Because $\{a_n\}$ is N-convergent to N, by Proposition 5.2.31 the sequence $\{b_n\}$ is N-convergent to N with $b_n \notin \mathbb{N}$ for all $n \in P_{\mathbb{N}}(K)$, which is a contradiction to Proposition 5.3.11. Hence there exists $n_0 \in \mathbb{N}$ such that $u_n \in N$ for all $n \geq n_0$. We conclude that $a_n \hookrightarrow N$. □

Corollary 5.3.13. *Let $\{a_n\}$ be a real sequence and $N \neq 0$ be a neutrix. Assume that $a_n \underset{N}{\longrightarrow} \alpha$ for some $\alpha \in \mathbb{E}$. Then $a_n \hookrightarrow \alpha$.*

Proof. Let $\alpha = a + N$. Because $a_n \underset{N}{\longrightarrow} \alpha$, we have $a_n - a \underset{N}{\longrightarrow} N$. By Proposition above, it follows that $a_n - a \hookrightarrow N$ and hence $a_n \hookrightarrow a + N = \alpha$. □

Lemma 5.3.14. *Let $A = \{(n, A_n)\} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{R})$ be a cofinal external set, where A_n is a non-empty for all $n \in P_{\mathbb{N}}(A)$. Then there exists a cofinal internal set $J = \{(k, a_k)\} \subseteq \mathbb{N} \times \mathbb{R}$ such that $a_k \in A_k$ for all $k \in P(J)$.*

Proof. We have $A = \bigcup_{\mathrm{st}(x) \in X} \bigcap_{\mathrm{st}(y) \in Y} I_{xy}$, where X, Y are standard and $I \colon X \times Y \rightrightarrows \mathcal{P}(\mathbb{N} \times \mathbb{R})$ is an internal set-valued mapping for all $x \in X, y \in Y$. Let $H_x = \bigcap_{\mathrm{st}(y) \in Y} I_{xy}$. Because A is cofinal, there exists $\mathrm{st}(x) \in X$ such that H_x is cofinal. It follows that for all $\mathrm{st}(y) \in Y$, I_{xy} is cofinal. Hence $\forall^{stfin} Z \subseteq Y\, \exists J\, \forall^{\mathrm{st}} y \in Z,\, J \subseteq I_{xy}$. In fact, we can take $J = \bigcap_{\mathrm{st}(y) \in Z} I_{xy}$. By the idealization principle we have

$$\exists J\, \forall^{\mathrm{st}} y \in Y\, J \subseteq I_{xy},$$

where J is internal and cofinal. Because J is internal, applying the axiom of choice we have that for each $k \in P_{\mathbb{N}}(J)$ there exists $a_k : (k, a_k) \in J$, that is, $a_k \in A_k$ for all $k \in P_{\mathbb{N}}(J)$. □

Proposition 5.3.15. *Let $\{A_n\}$ be a neutrix sequence and $N \neq 0$ be a neutrix. If $A_n \underset{N}{\longrightarrow} N$, there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $A_n \subseteq N$.*

Proof. Suppose on contrary that for all $n \in \mathbb{N}$ there exist $p_n \geq n$ such that $N \subset A_{p_n}$. Let

$$D = \{(n, A_n \setminus N) : N \subset A_n\}.$$

Then D is cofinal. By Lemma 5.3.14, there exists an internal subsequence $\{b_k\}$ such that $b_k \in A_k \setminus N$ for all $k \in P(J)$. Because $A_n \underset{N}{\longrightarrow} N$, we have $b_k \underset{N}{\longrightarrow} N$, a contradiction to Lemma 5.3.11. Hence there exists n_0 such that for all $n \geq n_0$ we have $A_n \subseteq N$. □

Lemma 5.3.16. *Let $\{u_n\}$ with $u_n = a_n + A_n$ be a flexible sequence and $N \neq 0$ be a neutrix. If $u_n \underset{N}{\longrightarrow} N$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $u_n \subseteq N$.*

Proof. Because $u_n \underset{N}{\longrightarrow} N$, by Proposition 5.2.8 we have $a_n \underset{N}{\longrightarrow} N$ and $A_n \underset{N}{\longrightarrow} N$. By Proposition 5.3.15 and Theorem 5.3.12, there exist $n_1, n_2 \in \mathbb{N}$ such that $a_n \in N$ for all $n \geq n_1$ and $A_n \subseteq N$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ we have $u_n \subseteq N$. □

Theorem 5.3.17. *Let u_n be a flexible sequence and $\alpha \in \mathbb{E}$ be an external number with $N(\alpha) \neq 0$. Then $u_n \underset{N(\alpha)}{\longrightarrow} \alpha$ if and only if $u_n \hookrightarrow \alpha$.*

Proof. Assume that $u_n \hookrightarrow \alpha$ with $N(\alpha) \neq 0$. Then there exists $n_0 \in \mathbb{N}$ such that $u_n \subseteq \alpha$ for all $n \geq n_0$. It follows that $u_n \underset{N(\alpha)}{\longrightarrow} \alpha$.

Conversely, assume that $u_n \underset{N(\alpha)}{\longrightarrow} \alpha$. Then $u_n - \alpha \underset{N(\alpha)}{\longrightarrow} N(\alpha)$. By Lemma 5.3.16, there exists $n_0 \in \mathbb{N}$ such that $u_n - \alpha \subseteq N(\alpha)$ for all $n \geq n_0$. This implies that $u_n \subseteq \alpha$ for all $n \geq n_0$. □

5.3.5 Strong Cauchy sequences

Definition 5.3.18. Let N be a neutrix. A flexible sequence $\{u_n\}$ is said to be a *strongly N-Cauchy sequence* if there exists $n_0 \in \mathbb{N}$ such that $u_n - u_m \subseteq N$ for all $n, m \geq n_0$.

Theorem 5.3.19. *Let N be a neutrix. Let $\{u_n\}$ be a flexible sequence with $u_n = a_n + A_n$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ is a strongly N-Cauchy sequence if and only $\{a_n\}, \{A_n\}$ are strongly N-Cauchy sequences.*

Proof. It follows by the fact that $u_m - u_n \subseteq N$ if and only $a_m - a_n \subseteq N$ and $A_m - A_n \subseteq N$. □

Lemma 5.3.20. *Let $\{a_n\}$ be a real internal sequence and N be a neutrix. If $\{a_n\}$ is strongly N-Cauchy then $\{a_n\}$ is strongly convergent to $\alpha = a + N$ for some $a \in \mathbb{R}$.*

Proof. If $N = 0$ then there exists $n_0 \in \mathbb{N}$ and a constant c such that $a_n = c$ for all $n \geq n_0$. So a_n is strongly convergent to $c + N = c$. If $N \neq 0$, because $\{a_n\}$ is strongly N-Cauchy, it is bounded. It follows from Theorem 5.2.33 that there exists an internal subsequence $\{a_{m_n}\}$ of $\{a_n\}$ such that $\{a_{m_n}\}$ is convergent to a for some $a \in \mathbb{R}$. Also, $N \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $a_{m_n} \in a + N$ for all $m_n \geq n_0$. Because the sequence $\{a_n\}$ is strongly N-Cauchy, there exists $p_0 \in \mathbb{N}$ such that $a_m - a_n \in N$ for all $m, n \geq p_0$. Let $k_0 = \max\{n_0, p_0\}$. Then for all $n \geq k_0$ we have $a_n - a = a_n - a_{m_n} + a_{m_n} - a \in N + N = N$. It follows that $a_n \in a + N$ for all $n \geq k_0$. Hence $\{a_n\}$ is strongly convergent to $a + N$. □

Theorem 5.3.21. *Let $\{u_n\}$ be a flexible sequence with $u_n = a_n + A_n$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ is strongly convergent to α if and only if $\{u_n\}$ is a strongly $N(\alpha)$-Cauchy sequence.*

Proof. Assume that $\{u_n\}$ is strongly convergent to α for some $\alpha = a + N(\alpha) \in \mathbb{E}$. Then there exists n_0 such that for all $n \geq n_0$ we have $u_n \subseteq \alpha$. It follows that $u_n - a \subseteq N(\alpha)$ for all $n \geq n_0$. So for all $n, m \geq n_0$ we have $u_n - u_m = u_n - a + a - u_m \subseteq N(\alpha) + N(\alpha) = N(\alpha)$. Hence $\{u_n\}$ is strongly $N(\alpha)$-Cauchy.

Conversely, assume that N is a neutrix and $\{u_n\}$ is strongly N-Cauchy. Then there exists n_0 such that for all $n \geq n_0$ we have $u_n - u_m \subseteq N$. It follows that $A_n \subseteq N$ for all $n \geq n_0$. By Theorem 5.2.33, there exists an internal subsequence $\{a_{m_n}\}$ of $\{a_n\}$. Since $\{u_n\}$ is strongly N-Cauchy, by Theorem 5.3.19 we have that $\{a_n\}$ is strongly N-Cauchy which implies that $\{a_{m_n}\}$ is also strongly N-Cauchy. Then $\{a_{m_n}\}$ is strongly convergent to $a+N$ for some $a \in \mathbb{R}$ by Lemma 5.3.20. So there exists p_0 such that for all $n \geq p_0$ it holds that $a_{m_n} - a \in N$. Also, because $\{a_n\}$ is strongly N-Cauchy, there exists p_1 such that for all $n, m \geq p_1$, we have $a_m - a_n \in N$. Let $k_0 = \max\{p_0, p_1\}$. Then for all $n \geq k_0$, it holds that $a_n - a = a_n - a_{m_n} + a_{m_n} - a \in N + N = N$. Hence $\{a_n\}$ is strongly convergent to $a + N$. By Proposition 5.3.19 we conclude that $\{u_n\}$ is strongly convergent to $a + N$. $\square$

5.4 Flexible sequences in $\mathbb{E}^p$

In this section we expand results on flexible sequences in $\mathbb{E}$ to a vector flexible sequence in $\mathbb{E}^p$ in which we always assume that $p \in \mathbb{N}$ is standard. Vector flexible sequences are used when we study the convergence of a flexible function of several variables in the next chapter. We will use the norm $\|\alpha\| = \max\limits_{1\leq i\leq p} |\alpha_i|$ with $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{E}^p$.

Definition 5.4.1. A mapping $u\colon \mathbb{N} \longrightarrow \mathbb{E}^p$ is called a flexible sequence in $\mathbb{E}^p$.

Definition 5.4.2. Let N be neutrices and $\{u_n\}$ be a flexible sequence in E^p. The sequence $\{u_n\}$ is said to be *N-convergent* to a vector $\alpha = (\alpha_1, \ldots, \alpha_p)$ if for all $\epsilon > N$ there exists $n_0 \in \mathbb{N}$ such that for al $n \geq n_0$ we have $d(u_n, \alpha) < \epsilon$. Then we also say that α is an *N-limit* of the sequence $\{u_n\}$ and we write N-$\lim u_n = \alpha$.

Similar to sequence in $\mathbb{E}$, we assume that $N(\alpha_i) = N$ for all $1 \leq i \leq p$.

Example 5.4.3. Let $u\colon \mathbb{N} \longrightarrow \mathbb{E}^2$ be given by $u_n = \left(1 + \oslash, \dfrac{1}{n} + \dfrac{1}{n}\epsilon\oslash\right)$. Then $u_n \underset{\oslash}{\longrightarrow} (1 + \oslash, \oslash)$.

Theorem 5.4.4. *Let $\{u_n\}$ be a flexible sequence in $\mathbb{E}^p$ with $u_n = (u_{1n}, \ldots, u_{pn})$ for all $n \in \mathbb{N}$, N be a neutrix and $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{E}^p$. Then $u_n \underset{N}{\longrightarrow} \alpha$ if and only if $u_{in} \underset{N}{\longrightarrow} \alpha_i$ for all $1 \leq i \leq p$.*

Proof. Assume that $u_n \underset{N}{\longrightarrow} \alpha$. Then for $\epsilon > N$ there exists n_0 such that for all $n \geq n_0$ we have $d(u_n, \alpha) < \epsilon$. It follows that for all $n \geq n_0$ we have $|u_{in} - \alpha_i| \leq d(u_n, \alpha) < \epsilon$, with $1 \leq i \leq p$. So N-$\lim u_{in} = \alpha_i$.

Conversely, assume that $u_{in} \underset{N}{\longrightarrow} \alpha_i$ for all $1 \leq i \leq p$. For $\epsilon > N$, and $1 \leq i \leq p$ there exists $n_{i0} \in \mathbb{N}$ such that for all $n \geq n_{i0}$ we have $|u_{in} - \alpha_i| < \epsilon$. Let $n_0 = \max\{n_{10}, \ldots, n_{p0}\}$. Then for all $n \geq n_0$ we have $d(u_n, \alpha) = \max\limits_{1\leq i\leq p} |u_{in} - \alpha_i| = |u_{kn} - \alpha_k| < \epsilon$. We conclude that $u_n \underset{N}{\longrightarrow} \alpha$. $\square$

Definition 5.4.5. A flexible sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}^p$ with $u_n = (u_{1n}, \ldots, u_{pn}), n \in \mathbb{N}$ is said to be strongly convergent to $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{E}^n$ if the flexible sequence u_{in} is strongly convergent to α_i for $1 \leq i \leq p$. Then we write $u_n \hookrightarrow \alpha$ or $\mathrm{Lim} u_n = \alpha$. The vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ is also called a *strong limit* of $\{u_n\}$.

Example 5.4.6. Let $\epsilon > 0$ be infinitesimal. Consider the flexible sequence $u\colon \mathbb{N} \longrightarrow \mathbb{E}^2$ given by $u_n = \left(\dfrac{n}{n+1} + \epsilon\oslash, \dfrac{1}{n} + \left(1 + \dfrac{1}{n}\right)^n \epsilon£\right)$ for all $n \in \mathbb{N}$. Then $u_n \hookrightarrow (1 + \epsilon\oslash, \epsilon£)$.

6

Flexible functions

This chapter is devoted to the study of functions with uncertainties. We only consider functions with precise variables and imprecise values. The imprecisions are modelled by external numbers. A function such that its values are external numbers is called a *flexible function.*

The structure of the chapter as follows.

In Section 6.1, we introduce the notion of flexible function and give some examples.

In Section 6.2 we generalize some topological notions. By using neutrices instead of zero, we introduce the notions of M*-neighbourhood*, an M*-interior point*, an M*-ball*, where M is a neutrix

In Section 6.3 the convergence of flexible functions is considered. Like Chapter 5 we will develop an adapted version of traditional convergence of function for a flexible function in terms of neutrices. Properties and arithmetic operations of this kind of limit as well. We also present the relationship between the convergence of a flexible sequence and of a flexible function. The Cauchy criterion for the convergence of a flexible function is obtained as in conventional analysis.

In Section 6.4 We introduce one-sided convergence for flexible functions and study the relationship to both-sided convergence.

In Section 6.5 we define a notion of continuity for flexible functions. Properties of and arithmetical operations on continuous flexible functions are investigated.

Recall that a standard function f is uniformly continuous if and only if $f(x+\oslash) \subseteq f(x)+\oslash$ for all x. Generalizing this properties we introduce in section 6.6 a notion of *inner convergence* and of *inner continuity*. Some properties and arithmetic operations are considered.

In order to construct the derivative of a flexible function in Section 6.7 we introduce another notion of convergence, which is called *outer convergent*. Using this notion we define so-called $M \times N$-derivatives of first and higher order of flexible functions. Monotonicity of these functions is studied in Section 6.8.

In Section 6.9 we introduce the $M \times N$-partial derivative and the $M \times N$-total derivative for flexible functions of several variables. The relationship between the two notions is studied. Also, we will provide conditions such that an implicit function or an inverse function is $M \times N$-totally differentiable in Sections 6.10 and 6.11. The study of the various types of differentiation is motivated by Chapter 8 on approximate optimal solutions of optimization problems with flexible objective functions.

6.1 Definitions and example

Definition 6.1.1. Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$. A mapping $F\colon X \longrightarrow \mathbb{E}$, of the form $\bigcup\limits_{st(u)\in U} \bigcap\limits_{st(v)\in V} I_{uv}$ where U, V are standard sets and $I\colon U \times V \rightrightarrows X \times \mathcal{P}(\mathbb{R})$ is an internal set-valued mapping, is said to be a *flexible function*. A flexible real function f defined on X such that $f(x) \in F(x)$ for all $x \in X$ is called a *representative* of F. The mapping $N_F\colon X \longrightarrow \mathbb{E}$ defined by $N_F(x) = N(F(x))$ for $x \in X$ is called the *neutrix part* of F; observe that the neutrix-part is also a flexible function. In general, we call a flexible mapping $N\colon X \to \mathcal{N}$ a *neutrix-function*. Then for each flexible function F defined on X and for all $x \in X$ we have $F(x) = f(x) + N_F(x)$.

We recall that as a consequence of Nelson's Reduction Algorithm [14] every external set with internal elements can be expressed in the form $\bigcup\limits_{st(u)\in U} \bigcap\limits_{st(v)\in V} I_{uv}$ where U, V are standard sets and $I\colon U \times V \rightrightarrows X \times \mathcal{P}(\mathbb{R})$ is an internal set-valued mapping.

Convention 6.1.2. In the whole chapter whenever we mention $X \subseteq \mathbb{R}^n$ we implicitly assume that $n \in \mathbb{N}$ is standard and $X \neq \emptyset$.

Example 6.1.3. a. A mapping $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ given by $F(x) = \sin x + \cos x \cdot \oslash$ for $x \in \mathbb{R}$ is a flexible function.

Indeed, let $V = \mathbb{N}$ and for $n \in \mathbb{N}$ we define $I\colon \mathbb{N} \rightrightarrows \mathbb{R} \times \mathcal{P}(\mathbb{R})$ given by

$$I_n = \left\{ \{x\} \times \left\{ \sin x + \left[-\frac{\cos}{n}, \frac{\cos x}{n} \right] \right\}, \, x \in \mathbb{R} \right\}.$$

Then $F = \bigcap\limits_{st(n)\in\mathbb{N}} I_n$.

The function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \sin x$ for $x \in \mathbb{R}$ is a representative of F and $N_F\colon \mathbb{R} \longrightarrow \mathcal{N}$ given by $N_F(x) = \cos x \cdot \oslash$, $x \in \mathbb{R}$ is the neutrix part of F.

b. Let $\epsilon > 0$ be infinitesimal and $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by

$$F(x) = \begin{cases} e^x + \epsilon\pounds & \text{if } x \in \pounds \\ \ln(x) + e^x\epsilon\oslash & \text{if } x \notin \pounds. \end{cases}$$

Observe that the function F can be expressed by

$$F = \left(\bigcup_{st(m)\in\mathbb{N}} \bigcup_{st(n)\in\mathbb{N}} \{\{x\} \times \{e^x + [-\epsilon n, \epsilon n]\}, -m \le x \le m\} \right)$$
$$\bigcup \left(\bigcap_{st(m)\in\mathbb{N}} \bigcap_{st(k)\in\mathbb{N}} \left\{\{x\} \times \left\{\ln x + e^x \left(-\frac{\epsilon}{k}, \frac{\epsilon}{k}\right)\right\}, x \notin [-m, m]\right\} \right).$$

By the Reduction Algorithm the mapping F can be expressed only by one intersection and one union relation. Hence F is a flexible function.

c. For example, the mapping $F\colon \mathbb{R} \longrightarrow \mathcal{N}$ given by $F(x) = e^x \cdot \epsilon\oslash$ for all $x \in \mathbb{R}$, is a neutrix-function.

6.2 Some topological notions

Treating a neutrix as a kind of generalized zero we will expand classical notions of topology in which conditions related to zero are replaced by neutrices, in such a way that if the neutrix is zero we obtain the classical notions. These notions allow us to study properties of convergence and continuity of a flexible function. Also, it enables us to measure the order of magnitude of uncertainties of local optimal solutions in Chapter 8.

Recall that in the classical mathematics, x_0 is an accumulation point of X if every ball of radius $r > 0$, centered at x_0 contains points of X which are different from x_0. In this context, we generalize this notion with a condition that $r > M$ instead of $r > 0$.

Let d be a metric on $\mathbb{R}^n$ and M be a neutrix.

The open ball centered at x_0 of radius $r > M$

$$B(x_0, r) = \{x \in \mathbb{R}^n | d(x, x_0) < r\}$$

is called the *open M-ball centered at x_0 of radius r* .

The closed ball centered x_0 of radius $r > M$

$$B[x_0; r] = \left\{x \in \mathbb{R}^n \middle| d(x, x_0) \le r\right\}$$

is called the *closed M-ball centered at x_0 of radius r*.

In case $M = 0$ we use usual notations.

Later on we also use the following notion: the *outer M-ball centered at x_0 of radius $r > M$* by

$$B_M(x_0, r) = \{x \in \mathbb{R}^n | M < d(x, x_0) < r\}.$$

Definition 6.2.1. Let $S \subseteq \mathbb{R}^n$ be not empty and $x_0 \in \mathbb{R}^n$. Let M be a neutrix. We say that x_0 is *M-close* to S if there exists $x \in S$ such that $d(x, x_0) \in M$.

Let S be a non-empty subset of $\mathbb{R}^n$ and M be a neutrix. Then every point of S is M-close to S. Another example, for instance, every infinitesimal is $\oslash$-close to $\{0\}$.

Every appreciable point is not $\oslash$-close to $\oslash$.

Definition 6.2.2. Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $x_0 \in \mathbb{R}^n$. We say that x_0 is an *M-accumulation point* of X if for all $\delta > M$ one has $B(x_0; \delta) \cap (X \setminus \{x_0 + M\}) \neq \emptyset$.

In case $M = 0$ we have the usual notion of accumulation point so, we may use the terminology *accumulation point* instead of 0-accumulation point.

Example 6.2.3. a. 0 is a $\oslash$-accumulation point of $X = (-1, 1)$. In this example, 0 is a point which belongs to X. However, like the classical definition, an M-accumulation point of a set may not belong to this set. For example, 0 is a $\oslash$-accumulation point of $X = @$ but it does not belong to X.

b. Let $\epsilon_0 > 0$ be infinitesimal. Then ϵ_0 is an accumulation point of $\oslash$.

Definition 6.2.4. Let M be a neutrix. Let $x_0 \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ be a non-empty subset. We say that x_0 an *M-interior point* of U if there is $r > M$ such that $B(x_0; r) \subseteq U$. Then the subset U is said to be an *M-neighbourhood* of x_0. Similarly, if $M = 0$, we use the usual terminology.

For example, $\oslash$ is a neighbourhood of 0, but not a $\oslash$-neighbourhood of 0 and @ is a $\oslash$-neighbourhood of all its members. Let $\epsilon > 0$ be infinitesimal. Then $B(0; \epsilon) \subset \mathbb{R}^2$ is a 0-neighbourhood of $x_0 = (0, \epsilon/2)$.

Also, we have that 1 is a $\oslash$-interior point of £ and ω is an £-interior point of ω£, here ω is unlimited.

Definition 6.2.5. Let $\alpha \in E$ and $U \subseteq \mathbb{R}$. The set U is said to be a *real neighbourhood* of α if and only if there exists an open interval $V \equiv (a, b) \subseteq \mathbb{R}$ such that $\alpha \subset V \subseteq U$.

For example, the subset $U =$ £ is a real neighbourhood of $1 + \oslash$.

6.3 Both-sided $M \times N$-limits

In general, the classical notion of convergence can not apply to flexible functions. Using neutrices as a kind of generalized zero we generalize the notion of convergence of real function in traditional mathematics to flexible functions. We also investigate properties and operations of these convergences.

6.3.1 Definition and example

Definition 6.3.1. Let M, N be neutrices, $X \subseteq \mathbb{R}^n$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function. Let x_0 be an M-accumulation point of X and $\alpha = a + A$ be an external number. We say that α is a $M \times N$*-limit* of F at x_0, written ${}^{N}_{M}\lim\limits_{x \to x_0} F(x) = \alpha$, if for all $\epsilon \in \mathbb{E}, \epsilon > N$, there exists $\delta > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta$ we have $|F(x) - \alpha| < \epsilon$.

In this case we also say that $F(x)$ is $M \times N$-convergent to α when x approaches x_0.

If $F(x)$ is not $M \times N$-convergent to any element in $\mathbb{E}$, we say that it is $M \times N$*-divergent*.

In case $M = N \neq 0$ we use the notation $M\text{-}\lim\limits_{x \to x_0} F(x)$, in case $M = 0, N \neq 0$ we use the notation ${}^{N}\lim\limits_{x \to x_0} F(x)$ instead of ${}^{N}_{M}\lim\limits_{x \to x_0} F(x)$. In particular, if $M = N = 0$, the notion reduces to conventional one, so we use the usual notation as the classical one.

Similarly to the definition of N-limit of a flexible sequence, in the definition above we also can replace the condition $\epsilon \in \mathbb{E}$ by the condition $\epsilon \in \mathbb{R}$.

Example 6.3.2. Consider the flexible function given by $F(x) = x + x \cdot \oslash$ for all $x \in \mathbb{R}$. One has

$$\oslash\text{-}\lim_{x \to 1}(x + x \cdot \oslash) = 1 + \oslash.$$

Indeed, let $\epsilon > \oslash$, taking $\delta = \epsilon/2 > \oslash$. Then for all $x \in \mathbb{R}$, $|x - 1| < \delta$ one has $|F(x) - (1 + \oslash)| = |x - 1| + \oslash < \epsilon/2 + \epsilon/2 = \epsilon$.

Remark 6.3.3. Using the notion of a neighbourhood of an external number we can rewrite the definition of $M \times N$- limits at one point. In fact, ${}^{N}_{M}\lim\limits_{x \to x_0} F(x) = \alpha$ if and only if for each real neighbourhood V of α, there exists $\delta > M$ such that for all $x \in X,\ 0 < d(x, x_0) < \delta$ one has $F(x) \subseteq V$.

6.3.2 Properties and operations

The neutrix part of an $M \times N$-limit must be included in N.

Proposition 6.3.4. *Assume that* ${}^{N}_{M}\lim\limits_{x \to x_0} F(x) = \alpha$. *Then* $N(\alpha) \subseteq N$.

Proof. We write $\alpha = a + A$. Suppose on the contrary that $N \subset A$. Let $\epsilon \in A$ be a real number such that $N < \epsilon$. Then there exists $\delta > M$ such that $|F(x) - \alpha| = |f(x) + N_F(x) - a + A| < \epsilon$. It follows that $|f(x) - a| < \epsilon$. So $|f(x) - a| \in N(\alpha) = A$. As a result, $\epsilon \leq |f(x) - a + N_F(x) + A| = N_F(x) + A < \epsilon$, which is a contradiction. We conclude that $A \equiv N(\alpha) \subseteq N$. $\square$

In contrast to a conventional convergence, $M \times N$-limits of a given flexible function at a point are not unique. In fact, if α is an $M \times N$-limit of F at x_0, then every element $\beta \subseteq \alpha + N$ is also an $M \times N$-limit of F at x_0. We will show that the $M \times N$-limit is unique up to the neutrix N.

Proposition 6.3.5. *Assume that* $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} F(x)=\alpha$ *and* $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} F(x)=\beta$. *Then* $\alpha-\beta\subseteq N$.

Proof. Suppose that $\alpha-\beta\nsubseteq N$. This implies that $N<|\alpha-\beta|$ or $N\subset\alpha-\beta$. Pick $\epsilon\in\mathbb{R}$ such that $N<\epsilon\leq|\alpha-\beta|$. So there are $\delta_1,\delta_2>M$ such that for all $x\in X, 0<d(x,x_0)<\delta_1$ it holds that $|F(x)-\alpha|<\epsilon/2$ and for all $x\in X, 0<d(x,x_0)<\delta_2$ we have $|F(x)-\beta|<\epsilon/2$. It follows that

$$\epsilon\leq|\alpha-\beta|\leq|\alpha-F(x)+F(x)-\beta|\leq|F(x)-\alpha|+|F(x)-\beta|<\epsilon/2+\epsilon/2=\epsilon,$$

which is a contradiction. Hence $\alpha-\beta\subseteq N$. □

In fact, if α is an $M\times N$-limit of F at x_0, every external number $\beta\subseteq\alpha+N$ is also an $M\times N$-limit of F at x_0.

Proposition 6.3.6. *Let* M,N *be neutrices and* $F\colon X\longrightarrow\mathbb{E}$ *be a flexible function. Assume that* $\alpha\in\mathbb{E}$ *is an* $M\times N$*-limit of* F *at* x_0. *Then* $\alpha+N$ *is also an* $M\times N$*-limit of* F *at* x_0. *In addition, every* $\beta\in\mathbb{E}$ *such that* $\beta-\alpha\subseteq N$ *is an* $M\times N$*-limit of* F *at* x_0.

Proof. For $\epsilon>N$ one has $\epsilon/2>N$. Because $\alpha\in\mathbb{E}$ is an $M\times N$-limit of F at x_0, there exists $\delta>M$ such that for all $x\in X, 0<d(x,x_0)<\delta$ one has $|F(x)-\alpha|<\epsilon/2$. So $|F(x)-(\alpha+N)|=|F(x)-\alpha|+N<\epsilon/2+\epsilon/2=\epsilon$. Hence $\alpha+N$ is an $M\times N$-limit of F at x_0.

In addition, by Proposition 6.3.4 we have $N(\alpha)\subseteq N$. So $\beta-\alpha\subseteq N$ implies $\beta\subseteq\alpha+N$. As above, we obtain that β is an $M\times N$-limit of F at x_0. □

Convention 6.3.7. Because of Proposition 6.3.6, unless stated otherwise, we always assume that if $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} F(x)=\alpha$, then $N(\alpha)=N$.

Proposition 6.3.8. *Let* M_1,N_1,M_2,N_2 *be neutrices such that* $M_2\subseteq M_1$ *and* $N_1\subseteq N_2$. *Let* $F\colon X\longrightarrow\mathbb{E}$ *be a flexible function and* $x_0\in\mathbb{R}^n$ *be a* M_1*-accumulation point of* X. *Assume that* $\overset{N_1}{\underset{M_1}{}}\lim\limits_{x\to x_0} F(x)=\alpha$. *Then*

$$\overset{N_2}{\underset{M_2}{}}\lim_{x\to x_0} F(x)=\alpha.$$

Proof. Let $\epsilon>N_2$. Because $N_1\subseteq N_2$, it holds that $\epsilon>N_1$. Also, by assumption that $\overset{N_1}{\underset{M_1}{}}\lim\limits_{x\to x_0} F(x)=\alpha$, there exists $\delta>M_1$ such that

$$\text{for all } x\in X \text{ with } 0<d(x,x_0)<\delta \text{ we have } |F(x)-\alpha|<\epsilon. \tag{6.1}$$

On the other hand, $M_2\subseteq M_1$, it holds that $\delta>M_2$. From (6.1) we conclude that $\overset{N_2}{\underset{M_2}{}}\lim\limits_{x\to x_0} F(x)=\alpha$. □

Because of this result, when we consider $M\times N$-limits we are implicit in working with the largest M and the smallest N possible.

Each flexible function is expressed by a sum of two components: a real part and the neutrix part. The result below indicates that if there exists $M \times N$-limit of a flexible function at one point, there also exist $M \times N$-limits of its components at this point and vice versa.

Theorem 6.3.9. *Let M, N be two neutrices and F be a flexible function defined on $X \subseteq \mathbb{R}^n$. Let f be a representative of F and N_F be the neutrix part of F. Let x_0 be an M-accumulation point of X. Then ${}_M^N\lim\limits_{x \to x_0} F(x) = \alpha \equiv a + N$ if and only if ${}_M^N\lim\limits_{x \to x_0} f(x) = a$ and ${}_M^N\lim\limits_{x \to x_0} N_F(x) = N$.*

Proof. Assume that ${}_M^N\lim\limits_{x \to x_0} F(x) = a + N$. Let $\epsilon > N$. There exists $\delta > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta$ one has $|F(x) - \alpha| = |f(x) - a + N_F(x) + N| < \epsilon$. It follows that $|f(x) - a| < \epsilon$ and $\big(N_F(x) + N\big) < \epsilon$. Hence ${}_M^N\lim\limits_{x \to x_0} f(x) = a$ and ${}_M^N\lim\limits_{x \to x_0} N_F(x) = N$.

Conversely, we assume that ${}_M^N\lim\limits_{x \to x_0} f(x) = a$ and ${}_M^N\lim\limits_{x \to x_0} N_F(x) = N$. Let $\epsilon > N$. It holds that $\epsilon/2 > N$ and hence there exists $\delta_1 > N$ such that for all $x \in X, 0 < d(x, x_0) < \delta_1$ one has $|f(x) - a| < \epsilon/2$ and $\delta_2 > N$ such that for all $x \in X, 0 < d(x, x_0) < \delta_2$, it holds that $\big(N_F(x) + N\big) < \epsilon/2$. Put $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x \in X, 0 < d(x, x_0) < \delta$ we have $|F(x) - \alpha| = |f(x) - a| + \big(N_F(x) + N\big) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence ${}_M^N\lim\limits_{x \to x_0} F(x) = \alpha$. $\square$

Next propositions state how the limit of a flexible function behaves under algebraic operations, and to what extent the involved neutrices need to be adjusted.

Theorem 6.3.10. *Let M, N_1, N_2 be neutices and $\alpha, \beta \in \mathbb{E}$. Let F, G be flexible functions defined on $X \subseteq \mathbb{R}^n$ and x_0 is an M-accumulation point of X. Assume that ${}_M^{N_1}\lim\limits_{x \to x_0} F(x) = \alpha$ and ${}_M^{N_2}\lim\limits_{x \to x_0} G(x) = \beta$. Then*

(i) ${}_M^N\lim\limits_{x \to x_0} (F + G)(x) = \alpha + \beta$, where $N = N_1 + N_2$.

(ii) ${}_M^N\lim\limits_{x \to x_0} (F - G)(x) = \alpha - \beta$, where $N = N_1 + N_2$..

(iii) Let $k \in \mathbb{R}$. Then ${}_M^{kN_1}\lim\limits_{x \to x_0} (kF)(x) = k\alpha$.

(iv) ${}_M^{N_1}\lim\limits_{x \to x_0} |F(x)| = |\alpha|$.

Proof. (i) Let $\epsilon > N$. Then $\epsilon > N_1$ and $\epsilon > N_2$. Because ${}_M^{N_1}\lim\limits_{x \to x_0} F(x) = \alpha$, there exists $\delta_1 > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta_1$ one has $F(x) - \alpha| < \epsilon/2$. Similarly, because ${}_M^{N_2}\lim\limits_{x \to x_0} G(x) = \beta$, there exists $\delta_2 > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta_2$ one has $|G(x) - \beta| < \epsilon/2$. Put $\delta_0 = \min\{\delta_1, \delta_2\}$. Then $\delta_0 > M$ and for all $x \in X, 0 < d(x, x_0) < \delta_0$ one has $|F(x) + G(x) - \alpha - \beta| \leq |F(x) - \alpha| + |G(x) - \beta| < \epsilon/2 + \epsilon/2 = \epsilon$.

(ii) The proof is similar to the proof of Part (i).

(iii) If $k = 0$, the conclusion is trivial. We assume that $k \neq 0$. Let $\epsilon > kN_1 = |k|N_1$. This implies that $\dfrac{\epsilon}{|k|} > N_1$. Because ${}_M^{N_1}\lim\limits_{x \to x_0} F(x) = \alpha$, there exists $\delta_0 > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta_0$ one has

$|F(x) - \alpha| < \epsilon/|k|$. It follows that $|kF(x) - k\alpha| = |k||F(x) - \alpha| < |k|\dfrac{\epsilon}{|k|} = \epsilon$. Hence ${}^{kN_1}_{M}\lim\limits_{x \to x_0} (kF)(x) = k\alpha$.

(iv) The result follows from the fact that $||F(x)| - |\alpha|| \le |F(x) - \alpha|$. $\square$

Theorem 6.3.11. *Let M, N_1, N_2 be neutrices and $\alpha, \beta, \gamma \in \mathbb{E}$. Let F, G be flexible functions defined on $X \subseteq \mathbb{R}^n$ and x_0 be an M-accumulation point of X. Assume that ${}^{N_1}_{M}\lim\limits_{x \to x_0} F(x) = \alpha$ and ${}^{N_2}_{M}\lim\limits_{x \to x_0} G(x) = \beta$. Then*

$${}^{N}_{M}\lim_{x \to x_0} (FG)(x) = \alpha\beta,$$

where $N = N_1 + N_2 + N_1^2 + N_2^2 + \alpha N_2 + \beta N_1$.

Proof. Write $F(x) = f(x) + A(x)$, $G(x) = g(x) + B(x)$ for all $x \in X$ and $\alpha = a + N_1, \beta = b + N_2$. Then

$$\begin{aligned} |F(x)G(x) - \alpha\beta| =& |f(x)G(x) - \alpha\beta + A(x)G(x)| \le |f(x)G(x) - f(x)\beta + f(x)\beta - \alpha\beta + A(x)G(x)| \\ \le& |f(x)||G(x) - \beta| + |b||f(x) - \alpha| + N_2 f(x) + N_2\alpha + A(x)g(x) + A(x)B(x) \end{aligned} \tag{6.2}$$

Let $\epsilon > N$. We first show that there exists $\delta_1 > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta_1$ one has

$$|f(x)||G(x) - \beta| < \frac{\epsilon}{6}. \tag{6.3}$$

We consider two cases, (i) $a \in N_1$ and (ii) $a \notin N_1$. For the case (i) we can take $a = 0$. The inequality $\epsilon > N_1^2$ implies $\sqrt{\epsilon} > N_1$. Hence there exists $k_1 > N_1$ such that for all $x \in X, 0 < d(x, x_0) < k_1$ we have $|f(x)| < \sqrt{\epsilon}$. Similarly, one has $\sqrt{\epsilon} > N_2$. So $\sqrt{\epsilon}/6 > N_2$ and hence there exists $k_2 > M$ such that for all $x \in X, 0 < d(x, x_0) < k_2$ one has $|G(x) - \beta| < \sqrt{\epsilon}/6$. Let $\delta_1 = \min\{k_1, k_2\} > M$. Then that for all $x \in X, 0 < d(x, x_0) < \delta_1$, we obtain

$$|f(x)||G(x) - \beta| < \frac{\epsilon}{6},$$

as required. For the case (ii) we have $|a| > N_1$ and hence there exists p_1 such that for $x \in X, 0 < d(x, x_0) < p_1$, $|f(x) - a| < |a|$, which implies that $|f(x)| < 2|a|$. Also $\epsilon > |a|N_2$, so $\dfrac{\epsilon}{12|a|} > N_2$. There exists $p_2 > M$ such that for $x \in X, 0 < d(x, x_0) < p_2$ it holds that $|G(x) - \beta| < \dfrac{\epsilon}{12|a|}$. Let $\delta_1 = \min\{p_1, p_2\}$. Then for all $x \in X, 0 < d(x, x_0) < \delta_1$ we obtain $|f(x)||G(x) - \beta| \le 2|a||G(x) - \beta| < \dfrac{\epsilon}{6}$, as required.

Secondly, we indicate that there exists $\delta_2 > M$ such that for $x \in X, 0 < d(x, x_0) < \delta_2$ it holds that

$$|b||f(x) - \alpha| < \epsilon/6. \tag{6.4}$$

Note that if $b \in N_2$, we can choose $b = 0$ and the result follows. Assume that $|b| > N_2$. The inequality $\epsilon > \beta N_1 = bN_1$ implies $\dfrac{\epsilon}{6|b|} > N_1$. Then there exists δ_2 such that for all $x \in X, 0 < d(x, x_0) < \delta_2$ one has $|f(x) - \alpha| < \dfrac{\epsilon}{6|b|}$. Hence (6.4) holds.

Thirdly we show that there exists $\delta_3 > 0$ such that for $x \in X, 0 < d(x, x_0) < \delta_3$ one has,

$$|f(x)|N_2 < \epsilon/6. \tag{6.5}$$

As above we distinguish the cases (i) $a \in N$ and (ii) $a \notin N$. For the case (i), similarly to the first part, there exists $k_1 > M$ such that for all $x \in X, 0 < d(x, x_0) < k_1$ one has $|f(x)| < \sqrt{\epsilon}$ and that $\sqrt{\epsilon}/6 > N_2$. Let $\delta_3 = k_1$. Then for all $x \in X, 0 < d(x, x_0) < \delta_3$ it holds that $|f(x)|N_2 < \sqrt{\epsilon}.\sqrt{\epsilon}/6 = \epsilon/6$. For the case (ii), taking $\delta_3 = p_1$, for all $x \in X,\ 0 < d(x, x_0) < \delta_3$ it holds that $|f(x)|N_2 < 2|a|N_2 < \epsilon/6$. Hence (6.5) holds.

Fourthly, we prove that there exists $\delta_4 > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta_4$ one has

$$|g(x)|A(x) < \epsilon/6. \tag{6.6}$$

Again we distinguish two cases. (i) $b \in N_2$ and (ii) $b \notin N_2$. (i) Again we can choose $b = 0$. Since $\epsilon > N_2^2$ one has $\epsilon/6 > N_2^2$. Hence $\sqrt{\frac{\epsilon}{6}} > N_2$. Also ${}^{N_2}_{M}\lim\limits_{x\to x_0} g(x) = 0$, so there exists $p_3 > M$ such that for all $x \in X, 0 < d(x, x_0) < p_3$ it holds that $|g(x)| < \sqrt{\frac{\epsilon}{6}}$. Moreover, since $\sqrt{\frac{\epsilon}{6}} > N_1$ there exists $p_4 > M$ such that for all $x \in X, 0 < d(x, x_0) < p_4$ it holds that $A(x) \leq \sqrt{\frac{\epsilon}{6}}$. Let $\delta_4 = \min\{p_3, p_4\}$. Then, for all $x \in X, 0 < d(x, x_0) < \delta_4$ we have $|g(x)|A(x) < \epsilon/6$, as required (ii) Let $\epsilon = |b| > N_2$. Then there exists $p_5 > M$ such that for all $x \in X, 0 < d(x, x_0) < p_5$ it holds that $|g(x) - b| < |b|$, which implies that $|g(x)| < 2|b|$. Furthermore $\epsilon > b.N_1$, so $N_1 < \epsilon/|b|$. It follows that $N_1 < \frac{\epsilon}{12|b|}$. Then there exists $p_6 > M$ such that for all $x \in X, 0 < d(x, x_0) < p_6$ one has $|A(x)| < \frac{\epsilon}{12|b|}$. Put $\delta_4 = \min\{p_5, p_6\}$. Then for all $x \in X, 0 < d(x, x_0) < \delta_4$ one has , $|g(x)|A(x) \leq 2|b|A(x) < \epsilon/6$. Hence (6.6) holds.

Finally, we saw that for all $x \in X, 0 < d(x, x_0) < p_4$ it holds that $A(x) < \sqrt{\epsilon/6}$. Similarly, there exists p_7 such that for all $x \in X, 0 < d(x, x_0) < p_7$ one has $B(x) < \sqrt{\epsilon/6}$. Put $\delta_5 = \min\{p_4, p_7\}$. Then for all $x \in X, 0 < d(x, x_0) < \delta_5$ one has

$$A(x)B(x) < \epsilon/6. \tag{6.7}$$

Clearly,

$$N_2\alpha < \epsilon/6. \tag{6.8}$$

Let $\delta_0 := \min\{\delta_1, ..., \delta_5\}$. Then, from (6.2)-(6.8), we conclude that, for all $x \in X, 0 < d(x, x_0) < \delta_0$ one has $|(FG)(x) - \alpha\beta| < \epsilon$. Hence ${}^{N}_{M}\lim\limits_{x\to x_0} = \alpha\beta$. □

Remark 6.3.12. Note that in case $N_1, N_2 \subseteq £$, we can neglect the term $N_1^2 + N_2^2$. So $N \equiv N_1 + N_2 + N_1^2 + N_2^2 + \alpha N_2 + \beta N_1 = N_1 + N_2 + \alpha N_2 + \beta N_1$. In case α, β are zeroless, N reduces to $N = \alpha N_2 + \beta N_1$.

Example 6.3.13. Let ω be unlimited and $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible function given by $F(x) = e^x + \omega £$.

We have ${}^{\omega £}_{\oslash}\lim\limits_{x\to 0}(e^x + \omega £) = \omega £$. However, $F^2 = F \cdot F$ is not $\oslash \times \omega £$-convergent to $\omega £$. In fact, it is $\oslash \times \omega^2 £$-convergent to $\omega^2 £$.

Example 6.3.14. Let $\epsilon > 0$ be infinitesimal F, G: $\mathbb{R} \longrightarrow \mathbb{E}$ be flexible functions given by

$$F(x) = \sin x + (x^2+1)\epsilon\oslash$$

and

$$G(x) = e^{x+\frac{1}{\epsilon^2}} + \cos x \cdot \epsilon\pounds.$$

Then we have

$${}^{\epsilon\oslash}\lim_{x\to 0} F(x) = \epsilon\oslash$$

and

$${}^{\epsilon\pounds}\lim_{x\to 0} G(x) = e^{\frac{1}{\epsilon^2}} + \epsilon\pounds.$$

Also, $\epsilon\oslash + \epsilon\pounds = \epsilon\pounds$. However, $F \cdot G$ is not $\epsilon\pounds$-convergent to $(\epsilon\oslash)\left(e^{\frac{1}{\epsilon^2}} + \epsilon\pounds\right)$ because $N\left(\epsilon\oslash(e^{\frac{1}{\epsilon^2}} + \epsilon\pounds)\right) = e^{\frac{1}{\epsilon^2}} \cdot \epsilon\oslash > \epsilon\pounds$. In fact, we have

$${}^{(e^{\frac{1}{\epsilon^2}}\cdot\epsilon\oslash)}\lim_{x\to 0}(F\cdot G)(x) = (\epsilon\oslash)\left(e^{\frac{1}{\epsilon^2}} + \epsilon\pounds\right) = e^{\frac{1}{\epsilon^2}} \cdot \epsilon\oslash$$

by Theorem 6.3.11 with $N_1 = \epsilon\oslash$, $\beta = e^{\frac{1}{\epsilon^2}} + \epsilon\pounds$ and $N = N_1\beta$.

Theorem 6.3.15. *Let M, N be neutrices. Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and x_0 be an M-accumulation point of X. Assume that there exists $\delta > M$ such that $F(x)$ is zeroless for all $x \in X, 0 < d(x, x_0) < \delta$ and that ${}^N_M\lim_{x\to x_0} F(x) = \alpha$, where $|\alpha| > N$. Then*

$${}^K_M\lim_{x\to x_0} \frac{1}{F(x)} = \frac{1}{\alpha}$$

where $K = N/\alpha^2$.

Proof. Let $\alpha = a + N, F(x) = f(x) + A(x)$ for all $x \in X$. Let $\epsilon > K$. Then $\epsilon > N/a^2$. So there exists $M < p_1 \leq \delta$ such that for all $x \in X, 0 < d(x, x_0) < p_1$ one has $|f(x) - a| < a^2\epsilon/4$ and by Theorem 6.3.9, there also exists $M < p_2 \leq \delta$ such that for all $x \in X, 0 < d(x, x_0) < p_2$ one has $(A(x) + N) < a^2\epsilon/2$. Moreover $|a| > N$, so there exists $M < p_3 \leq \delta$ such that for all $x \in X, 0 < d(x, x_0) < p_3$ one has $|f(x) - a| \leq |F(x) - \alpha| < |a|$, which implies that $|f(x)| < 2|a|$. Similarly, there exists $M < p_4 \leq \delta$ such that for all $x \in X, 0 < d(x, x_0) < p_4$ one has $|a|/2 < |f(x)|$. Let $p_5 = \min\{p_3, p_4\}$. Then for all $x \in X, 0 < d(x, x_0) < p_5$ one has $a^2/4 < f^2(x) < 4a^2$. Let $p = \min\{p_1, p_2, p_5\}$. Then for all $x \in X, 0 < d(x, x_0) < p$ one has

$$\begin{aligned}\Big|\frac{1}{F(x)} - \frac{1}{\alpha}\Big| &= \Big|\frac{F(x)}{f^2(x)} - \frac{\alpha}{a^2}\Big| = \Big|\frac{a^2F(x) - f^2(x)\alpha}{f^2(x)a^2}\Big| \leq \Big|\frac{a - f(x)}{f(x)a}\Big| + \frac{a^2A(x) - f^2(x)N}{f^2(x)a^2}\\ &\leq \Big|\frac{a - f(x)}{a^2/2}\Big| + \frac{a^2A(x) - 4a^2N}{a^4/4} \leq \Big|\frac{a^2\epsilon/4}{a^2/2}\Big| + \frac{(A(x)+N)}{a^2} < \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence ${}^K_M\lim_{x\to x_0} \frac{1}{F(x)} = \frac{1}{\alpha}$. □

Corollary 6.3.16. *Let M, N_1, N_2 be neutices and $\alpha, \beta \in \mathbb{E}$. Let F, G be flexible functions defined on X and x_0 is an M-accumulation point of X. Assume that $G(x)$ is zeroless in an M-neighbourhood of x_0 and ${}^{N_1}_{M}\lim\limits_{x\to x_0} F(x) = \alpha$ and ${}^{N_2}_{M}\lim\limits_{x\to x_0} G(x) = \beta$. If $|\beta| > N_2$ then*

$$ {}^{N}_{M}\lim_{x\to x_0} \frac{F(x)}{G(x)} = \frac{\alpha}{\beta}, $$

where $N = N_1 + N_2/\beta^2 + N_1^2 + (N_2/\beta^2)^2 + \alpha\big(N_2/\beta^2\big) + \beta N_1$.

Proof. Using Theorem 6.3.15, one has ${}^{K}_{M}\lim\limits_{x\to x_0} \dfrac{1}{G(x)} = \dfrac{1}{\beta}$, where $K = N_2/\beta^2$. The Corollary follows by applying Theorem 6.3.11 to the flexible functions $F(x)$ and $\dfrac{1}{G(x)}$, where N_2 is replaced by K. □

Example 6.3.17. Let $\epsilon > 0$ be infinitesimal, $\omega > 0$ be unlimited. Consider flexible functions F, G, $H\colon \mathbb{R} \longrightarrow \mathbb{E}$ given by $F(x) = e^{x+\omega} + \oslash x$, $G(x) = 1 + x + \big(\epsilon \oslash\big)e^x$ and $H(x) = \epsilon + \epsilon \oslash x$ for $x \in \mathbb{R}$.

(a) We have $\oslash$-$\lim\limits_{x\to 0} F(x) = e^{\omega}(1+\oslash)$. So, by Theorem 6.3.15 the function $\dfrac{1}{F}$ is not only $\oslash \times \oslash$-convergent but also $\oslash \times \dfrac{\oslash}{e^{2\omega}}$-convergent, and ${}^{\frac{\oslash}{e^{2\omega}}}_{\oslash}\lim\limits_{x\to 0} \dfrac{1}{F(x)} = \dfrac{1}{e^{\omega}} + \dfrac{\oslash}{e^{2\omega}}$.

(b) Also, ${}^{\epsilon\oslash}\lim\limits_{x\to 0} G(x) = 1 + \epsilon\oslash$ and by Convention 6.3.7 we have ${}^{\epsilon\oslash}\lim\limits_{x\to 0} H(x) = \epsilon + \epsilon\oslash$. Note that $\dfrac{G(x)}{H(x)}$ is not $0 \times \epsilon\oslash$-convergent to $\dfrac{1+\epsilon\oslash}{\epsilon+\epsilon\oslash}$ when x approaches 0, because $\epsilon\oslash \subset N\left(\dfrac{1+\epsilon\oslash}{\epsilon+\epsilon\oslash}\right) = \oslash$. However, by Corollary 6.3.16 the function $\dfrac{G(x)}{H(x)}$ is $0 \times \oslash$-convergent to $\dfrac{1+\epsilon\oslash}{\epsilon+\epsilon\oslash} = \dfrac{1}{\epsilon} + \oslash$ at $x = 0$.

In the next result we state one version of the squeeze theorem for $M \times N$-limits of flexible functions.

Theorem 6.3.18. *Let M, N be neutrices and F, G, H be flexible functions defined on $X \subseteq \mathbb{R}^n$. Assume that $F(x) \leq G(x) \leq H(x)$ for all $x \in V \subseteq X$, where V is an M-neighbourhood of x_0 and ${}^{N}_{M}\lim\limits_{x\to x_0} F(x) = {}^{N}_{M}\lim\limits_{x\to x_0} H(x) = \alpha$. Then ${}^{N}_{M}\lim\limits_{x\to x_0} G(x) = \alpha$.*

Proof. Let $\epsilon > N$. Because ${}^{N}_{M}\lim\limits_{x\to x_0} F(x) = {}^{N}_{M}\lim\limits_{x\to x_0} H(x) = \alpha$, there exists $\delta > M$ such that $|F(x) - \alpha| < \epsilon$ and $|G(x) - \alpha| < \epsilon$ for all $x \in V, 0 < d(x, x_0) < \delta$. Moreover, for all $x \in V$ one has $F(x) - \alpha \leq G(x) - \alpha \leq H(x) - \alpha$. It follows that $|G(x) - \alpha| \leq \max\{|H(x) - \alpha|, |F(x) - \alpha|\} < \epsilon$. Hence ${}^{N}_{M}\lim\limits_{x\to x_0} G(x) = \alpha$. □

We know that in classical analysis, if a function is bounded from above by a constant c, and has a limit, the limit is less than or equal to this constant. We state below a version for $M \times N$-limits of flexible functions.

Theorem 6.3.19. *Let M, N be neutrices and F be a flexible function defined on $X \subseteq \mathbb{R}^n$ such that $F(x) \leq \beta + N$ for all $x \in X$. Let x_0 be an M-accumulation point of X. Assume that ${}^{N}_{M}\lim\limits_{x\to x_0} F(x) = \alpha$. Then $\alpha \leq \beta + N$.*

Proof. If $\alpha \subseteq \beta + N$, the conclusion is trivial. We assume that $\alpha \not\subseteq \beta + N$. Because $N(\alpha) = N$ by Convention 6.3.7, it follows that $\alpha \cap (\beta + N) = \emptyset$. Suppose that $\alpha > \beta + N$. This implies that $\alpha - \beta > N$ by Proposition 2.2.38. Let ϵ be a real number such that $N < \epsilon < \alpha - \beta$. There exists $\delta > M$ such that for all $x \in X, 0 < d(x, x_0) < \delta$ one has $|F(x) - \alpha| < \epsilon$. Consequently, $\alpha - \epsilon < F(x) + N$. It follows that $\alpha - (\alpha - \beta) \leq \alpha - \epsilon < F(x) + N$ and hence $N + \beta < F(x) + N$. One obtains that $F(x) > \beta + N$, which is a contradiction. We conclude that $\alpha \leq \beta + N$. □

Let f be a real function defiend on X. In classical mathematics it is well-known that a function f has the limit l at a point x_0 if and only if for every sequence $\{x_n\} \subseteq X$ converges to x_0, the sequence $f(x_n)$ converges to l. We show below that if a flexible function has $M \times N$-limits at x_0, for every sequence u_n M-converges to $x_0 + M$, the sequence $F(u_n)$ N-converges to the $M \times N$- limit of F at x_0. Also, if $M = 0$, the converse is true.

Theorem 6.3.20. *Let $p \in \mathbb{N}$ be standard, F be a flexible function defined on $X \subseteq \mathbb{R}^p$ and x_0 be an M-accumulation point of X. The following statements hold:*

If ${}^N_M\lim\limits_{x \to x_0} F(x) = \beta$, one has N-$\lim\limits_{n \to \infty} F(x_n) = \beta$ for every sequence $\{x_n\} \subseteq X \setminus \{x_0\}$ which M-converges to x_0.

Proof. Assume that ${}^N_M\lim\limits_{x \to x_0} F(x) = \beta$ and $\{x_n\} \subset X \setminus \{x_0\}$, M-$\lim\limits_{n \to \infty} x_n = x_0$. We will prove that N-$\lim\limits_{n \to \infty} F(x_n) = \beta$.

Let $\epsilon > N$. Because ${}^N_M\lim\limits_{x \to x_0} F(x) = \beta$, there exists $\delta > M$ such that for all $x \in X$ with $0 < d(x, x_0) < \delta$ one has $|F(x) - \beta| < \epsilon$. Also N-$\lim\limits_{n \to \infty} x_n = x_0$, so there exists n_0 such that for all $n \geq n_0$ one has $0 < d(x_n, x_0) < \delta$. It implies that $|F(x_n) - \beta| < \epsilon$ for all $n \geq n_0$. Hence N-$\lim\limits_{n \to \infty} F(x_n) = \beta$. □

We present below the Cauchy criterion for convergence of flexible functions. In some situations we can not calculate $M \times N$-limits of flexible function but we want to know if the function is $M \times N$-convergent or not. The Cauchy criterion is a useful tool to do this.

Theorem 6.3.21 (Cauchy criterion)**.** *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and M, N be neutrices. Let x_0 be an M-accumulation point of X. Then F is $M \times N$-convergent at x_0 if and only if for all $\epsilon > N$, there exists $\delta > M$ such that for all $x, x' \in X$, $0 < d(x, x_0) < \delta$, $0 < d(x', x_0) < \delta$ it holds that $|F(x) - F(x')| < \epsilon$.*

Proof. Assume that F is $M \times N$-convergent at x_0. Then there is $\alpha \in E$ such that ${}^N_M\lim\limits_{x \to x_0} F(x) = \alpha$. Let $\epsilon > N$. There exists $\delta > M$ such that for all $x, x' \in X$, $0 < d(x, x_0) < \delta$, $0 < d(x', x_0) < \delta$ we have $|F(x) - \alpha| < \epsilon/2$ and $|F(x') - \alpha| < \epsilon/2$. It follows that $|F(x) - F(x')| \leq |F(x) - \alpha| + |F(x') - \alpha| < \epsilon/2 + \epsilon/2 = \epsilon$.

Conversely, let $\{x_n\} \subset X$ such that $x_n \underset{M}{\longrightarrow} x_0$. By the assumption, the sequence $F(x_n)$ is N-Cauchy. Because of Theorem 5.2.41 it holds that $F(x_n) \underset{N}{\longrightarrow} a + N$. Let $\epsilon > N$. There exists $\delta > M$ such that for all $x, x' \in X$, $0 < d(x, x_0) < \delta$, $0 < d(x', x_0) < \delta$ we have $|F(x) - F(x')| \leq \epsilon/2$. Also $\lim x_n = x_0$, so there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $0 < d(x_n, x_0) < \delta$. Because $F(x_n) \underset{N}{\longrightarrow} a + N$, there exists m_0 such that for all $n \geq n_0$, $|F(x_n) - a + N| < \epsilon/2$. Let $y \in X, 0 < d(y, x_0) < \delta$ and $p \geq \max\{m_0, n_0\}$. Then

$|F(y)-a+N| \leq |F(y)-F(x_p)|+|F(x_p)-a+N| \leq \epsilon/2+\epsilon/2 = \epsilon$. We conclude that F is $M\times N$-convergent to $a+N$ at x_0. □

6.4 One-sided $M \times N$-limits

In this section we only consider flexible functions of one variable. This means that the domain of function X is a subset of $\mathbb{R}$.

Sometimes it is not necessary or impossible to consider two-sided convergence. For instance, let $F\colon @ \longrightarrow \mathbb{E}$ be a flexible function defined by $F(x) = x + \oslash \cdot x,\ x \in @$. Then 0 is a $\oslash$-accumulation point of $X = @$. In this case when we investigate the $\oslash$-convergence of F at 0, we can only consider argument $x > 0 + \oslash$. This means that $x \in @$ can only approach 0 from above.

Also behaviours of some flexible functions are quite different when x approaches x_0 from one side to another. For example, consider the flexible function

$$F(x) = \begin{cases} x^2 + \oslash x & \text{if } x \geq 0 \\ \sin x + \oslash & \text{if } x < 0. \end{cases}$$

Then $F(0) = 0$, while $F(x) = \oslash$ for $x \in \oslash, x < 0$.

Limits of flexible functions when x approaches a point from one side are called *one-sided limits*.

Definition 6.4.1. Let M, N be two neutrices and α_1, α_2 be external numbers. Let $F\colon X \subseteq \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible function and x_0 be an M-accumulation of X.

(i) The external number α_1 is called a *left $M \times N$- limit* of F at x_0, written as

$$\mathop{{}_{M}^{N}\lim}_{x\to x_0^-} F(x) = \alpha_1,$$

if for all $\epsilon > N$ there exists $\delta > M$ such that for all $x \in X,\ 0 < x_0 - x < \delta$ it holds that $|F(x) - \alpha_1| < \epsilon$.

(ii) The external number α_2 is called a *right $M \times N$-limit* of F at x_0, written as

$$\mathop{{}_{M}^{N}\lim}_{x\to x_0^+} F(x) = \alpha_2,$$

if for all $\epsilon > N$ there exists $\delta > M$ such that for all $x \in X,\ 0 < x - x_0 < \delta$ it holds that $|F(x) - \alpha_2| < \epsilon$.

Remark 6.4.2. It is easy to verify that the results stated above for both-sided limits also hold for one-sided limits.

As for relationships between both-sided and one-sided limits, there exists an $M \times N$-limit of F at x_0 if and only if there exist a left $M \times N$- limit and a right $M \times N$-limit, and both of them are equal.

Theorem 6.4.3. *There exists* $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} F(x) = \alpha$ *if and only if there exist left and right* $M \times N$*-limits of* F *at* x_0*, and*

$$\overset{N}{\underset{M}{}}\lim_{x\to x_0^-} F(x) = \overset{N}{\underset{M}{}}\lim_{x\to x_0^+} F(x) = \alpha.$$

Proof. Assume that $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} = \alpha$. Let $\epsilon > N$. Then there exists $\delta > M$ such that for all $x \in X, 0 < |x-x_0| < \delta$ we have $|F(x) - \alpha| < \epsilon$. Hence for all $x \in X$, $0 < x - x_0 < \delta$ it holds that $0 < |x - x_0| < \delta$ and hence $|F(x)-\alpha| < \epsilon$. So $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0^+} F(x) = \alpha$. Similarly, for all $x \in X, 0 < x_0 - x < \delta$ it holds that $0 < |x - x_0| < \delta$ and hence $|F(x) - \alpha| < \epsilon$. So $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0^-} F(x) = \alpha$.

Conversely, we assume that $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0^-} F(x) = \overset{N}{\underset{M}{}}\lim\limits_{x\to x_0^+} F(x) = \alpha$. Let $\epsilon > N$. Then there exists $\delta_1, \delta_2 > M$ such that for all $x \in X$, $0 < x_0 - x < \delta_1$ one has $|F(x) - \alpha| < \epsilon$ and for all $x \in X$, $0 < x - x_0 < \delta_2$ one has $|F(x) - \alpha| < \epsilon$. Put $\delta = \min\{\delta_1, \delta_2\} > M$. Then for all $x \in X, 0 < |x - x_0| < \delta$, it follows that $0 < x - x_0 < \delta \le \delta_2$ and $0 < x_0 - x < \delta \le \delta_1$. Hence $|F(x) - \alpha| < \epsilon$. □

6.5 Continuity

Using $M \times N$-limits we develop continuity for flexible functions. For sake of simplicity, we denote by $N_0 = N_F(x_0)$ the neutrix part of $F(x_0)$.

6.5.1 Both-sided continuity

Definition 6.5.1. Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and M, N be neutrices. Let $x_0 \in X$ be an M-accumulation point of X. The flexible function F is said to be $M \times N$*-continuous* at x_0 if $\overset{N}{\underset{M}{}}\lim\limits_{x\to x_0} F(x) = F(x_0)$. In particular, if $N = N(F(x_0))$, the function F is said to be M*-continuous* at x_0. Furthermore, if $M = 0$ then F is said to be *continuous* at x_0.

From the definition of $M\times N$-limit, a flexible function F is M- continuous at x_0 if and only if for every $\epsilon > N_0$, there exists $\delta > M$ such that for all $x \in X, d(x, x_0) < \delta$ we have $|F(x) - F(x_0)| < \epsilon$.

The proposition below give us one characterization of continuity of flexible functions.

Proposition 6.5.2. *A flexible function* F *is* $M \times N$*-continuous at* x_0 *if and only if for every neighbourhood* V *of* $F(x_0) + N$ *there exists* $\delta > M$*, such that for all* $x \in X$ *with* $d(x, x_0) < \delta$ *one has* $F(x) \in V$*.*

Proof. Let f be a representative of F. Assume that F is M-continuous at x_0. Let V be an arbitrary neighbourhood of $F(x_0) + N$. By the definition of neighbourhood V, there exists a real number $\epsilon > N$ such that $F(x_0) + N \subset (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq V$. Because F is $M \times N$-continuous at x_0, there exists $\delta > M$ such that $|F(x) - F(x_0)| < \epsilon$. This means that $f(x_0) - \epsilon < F(x) + N_F(x_0) < f(x_0) + \epsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Hence $F(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq V$ for all $x \in X, d(x, x_0) < \delta$.

Conversely, assume that for every V a neighbourhood of $F(x_0) + N$ there exists $\delta > M$ such that for all $x \in X, d(x, x_0) < \delta$, one has $F(x) \in V$. We need to prove that F is $M \times N$-continuous at x_0. We take $V = (F(x_0) - \epsilon, F(x_0) + \epsilon)$ with $\epsilon > N$. Then V is a neighbourhood of $F(x_0) + N$. By the assumption, there exists $\delta > M$ such that for all $x \in X, d(x, x_0) < \delta$ implies $F(x) \in V = (F(x_0) - \epsilon, F(x_0) + \epsilon)$. That is $F(x_0) - \epsilon < F(x) < F(x_0) + \epsilon$. Then $|F(x) - F(x_0)| < \epsilon + N_F(x_0)$. Also, $N_F(x_0) \leq N < \epsilon$ it follows that $|F(x) - F(x_0)| < \epsilon$. Hence F is $M \times N$-continuous at x_0. □

Example 6.5.3. Let ϵ be infinitesimal. Consider the internal functions defined by

$$f(x) = \arctan(x/\epsilon), \; g(x) = \begin{cases} -\epsilon, & x < 0 \\ \epsilon, & x \geq 0, \end{cases} \quad j(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

The first function is a well-known example of a continuous, not S-continuous function, in our terminology, it is neither $\oslash$-continuous nor $\oslash \times \oslash$-continuous. It is not difficult to see that g is not continuous at $x_0 = 0$ but is $0 \times \oslash$-continuous. The function j is clearly neither continuous nor $0 \times \oslash$-continuous at $x_0 = 0$. However j is $0 \times$ £-continuous at that point.

Example 6.5.4. Let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by

$$F(x) = \begin{cases} \oslash & \text{if } x \in \oslash \\ 0 & \text{if } x \notin \oslash. \end{cases}$$

Then F is $\oslash$-continuous at 0.

Examples (6.5.3) and (6.5.4) show that in our definition of continuity, the continuity of a flexible function depends on the order of magnitude considered. Due to Proposition 6.3.8 a flexible function F is $M \times N$-continuous, then it is $M' \times N'$-continuous with $M' \leq M$ and $N \leq N'$.

Let $F{:}X \rightrightarrows \mathcal{P}(\mathbb{R})$ be a set-valued map. Recall that F is called upper semi-continuous at x_0 if for every neighbourhood U of $F(x_0)$, there is a $\delta > 0$ such that for all $x \in X, 0 < d(x, x_0) < \delta$ we have $F(x) \subseteq U$. Hence the notion of M-continuity of a flexible function becomes the notion of upper semi-continuity of a set-valued map. However, consider the function

$$F(x) = \begin{cases} 0 & \text{if } x = 0 \\ \oslash & \text{if } x \neq 0. \end{cases}$$

If F is seen as a set-valued mapping, it is not upper semi continuous at 0. In fact, it is lower semi-continuous at 0. In our definition it is $0 \times \oslash$-continuous at 0. We see that in some cases our definition implies upper semi-continuity and some other cases it implies lower semi-continuity.

Example 6.5.5. Let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible function given by $F(x) = x^2 + \oslash \cdot x$ for $x \in \mathbb{R}$. Then F is continuous at $x_0 = 0$, but not $\oslash$-continuous at $x_0 = 0$.

6.5.2 One-sided continuity

Definition 6.5.6. Let F be a flexible function defined on $X \subseteq \mathbb{R}$ and M, N be neutrices. The flexible function F is said to be *left $M \times N$-continuous* at x_0 if there exists $\delta > M$ such that $[x_0, x_0 + \delta) \subset X$ and $\overset{N}{\underset{M}{}} \lim\limits_{x \to x_0^-} F(x) = F(x_0)$.

Definition 6.5.7. *Let F be a flexible function defined on $X \subseteq \mathbb{R}$ and M, N be a neutrices. A flexible function F is said to be right $M \times N$-continuous at x_0 if there exists $\delta > M$ such that $(x_0 - \delta, x_0] \subset X$ and $\overset{N}{\underset{M}{}} \lim\limits_{x \to x_0^+} F(x) = F(x_0)$.*

As consequence of Theorem 6.4.3, one has:

Theorem 6.5.8. *A flexible function F is $M \times N$-continuous at x_0 if and only if it is left and right $M \times N$-continuous at x_0.*

We next define the continuity of a flexible function on a closed interval.

Definition 6.5.9. Let F be a flexible function defined on $[a, b]$ and M, N be neutrices. Assume that there exist $\delta_1, \delta_2 > M$ such that $[a, a + \delta_1) \subset [a, b]$ and $(b - \delta_2, b] \subset [a, b]$. The flexible function $F(x)$ is said to be *$M \times N$-continuous on $[a, b]$* if it is $M \times N$-continuous on (a, b) and left $M \times N$-continuous at a and right $M \times N$-continuous at b.

6.5.3 Operations on continuous flexible functions

We consider how the continuity of flexible functions behaves under algebra operations. As a result of Theorem 6.3.10 and Theorem 6.3.11 one obtains the following.

Theorem 6.5.10. *Let N_1, N_2, M be neutrices and $x_0 \in \mathbb{R}^n$. Let F be $M \times N_1$-continuous function and G be an $M \times N_2$-continuous function at $x = x_0$. Then*

(i) The flexible function $F + G$ is $M \times K$-continuous at x_0 with $K = N_1 + N_2$.

(ii) The flexible function $F - G$ is $M \times K$-continuous at x_0 with $K = N_1 + N_2$.

(iii) Let $k \in \mathbb{R}$. Then (kF) is $M \times K$-continuous at x_0, where $K = k.N_1$.

(iv) The flexible function $F \cdot G$ is $M \times K$-continuous at x_0, where $K = N_1 + N_2 + N_1^2 + N_2^2 + F(x_0) \cdot N_2 + G(x_0) \cdot N_1$.

We now turn to continuity of a composition of function between a real function and a flexible function.

Theorem 6.5.11. *Let N_1, N_2 be neutrices and $I \subseteq \mathbb{R}^m$, $J \subseteq \mathbb{R}^n$. Let $f \colon I \longrightarrow J$ be a real function, $G \colon J \longrightarrow \mathbb{E}$ be a flexible function. Let $H \colon I \longrightarrow \mathbb{E}$ be the composition of f and G defined by $H(x) = G(f(x))$ for all $x \in I$. If f is $K \times M$-continuous at the point x_0 and G is $M \times N$-continuous at $f(x_0)$ then H is $K \times N$-continuous at x_0.*

Proof. Let $\epsilon > N$. Because G is $M \times N$-continuous at $y_0 = f(x_0)$, there exists $\eta > M$ such that for all $y \in J, d\big(y, f(x_0)\big) < \eta$ it holds that $|G(y) - G(f(x_0))| < \epsilon$. Moreover f is $K \times M$-continuous at x_0, there exists $\delta > K$ such that for all $x \in I$, $d(x, x_0) < \delta$ we have $d\big(f(x), f(x_0)\big) < \eta$. It follows that for all $x \in I, d(x, x_0) < \delta$ one has $|H(x) - H(x_0)| = |G(f(x)) - G(f(x_0))| < \epsilon$. We conclude that H is $K \times N$-continuous at x_0. □

6.6 Inner convergence and inner continuity

It is well-known that a standard function f defined on standard $X \subseteq \mathbb{R}$ is continuous if and only if f is S-continuous at every limited real number $x \in X$, meaning that $f(x + \oslash) \subseteq f(x) + \oslash$. The function f is uniformly continuous if and only if $f(x + \oslash) \subseteq f(x) + \oslash$ for all $x \in \mathbb{R}$. Taking this guide we define another notion of convergence and continuity for flexible functions. We replace the neutrix $\oslash$ on the left side by an arbitrary neutrix and the term on the right side by an external number. We call it an *inner convergence*. Also, this notion is corresponding to the notion of strong convergence of a flexible sequence.

The word "inner" implies that we only consider real points in $x_0 + M$ and also the values of f at these points are inside the limit.

Definition 6.6.1. Let $M = (M_1, \ldots, M_n) \neq 0$ be a neutrix vector and $\alpha = a + A \in \mathbb{E}$. Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ such that $x_0 + M \subseteq X$. We call α an *inner limit* of F at $x_0 + M$ if $F(x) \subseteq \alpha$ for all $x \in x_0 + M \setminus \{x_0\}$. Then we write

$$\lim_{x \to x_0 + M} F(x) = \alpha.$$

We also say that $F(x)$ is *inner convergent* to α when x approaches $x_0 + M$.

Example 6.6.2. Consider the flexible function F given by $F(x) = \dfrac{x}{x^2 + y^2 + 1} + \dfrac{y}{x^2 + y^2 + 1} + \oslash \cdot (x + y)$ for all $x, y \in \mathbb{R}$. Then $\lim\limits_{(x,y) \to (0,0) + (\oslash, \oslash)} F(x) = \oslash$.

By this definition of limit it is easy to see that the following operations hold.

Theorem 6.6.3. *Let M be a neutrix and F, G be flexible functions defined on X. Assume that $\lim\limits_{x \to x_0 + M} F(x) = \alpha$, $\lim\limits_{x \to x_0 + M} G(x) = \beta$. Then*

(i) $\lim\limits_{x \to x_0 + M} (F + G)(x) = \alpha + \beta$.

(ii) $\lim\limits_{x \to x_0 + M} (F - G)(x) = \alpha - \beta$.

(iii) $\lim\limits_{x \to x_0 + M} kF(x) = k\alpha$ *for* $k \in \mathbb{R}$.

(iv) $\lim\limits_{x \to x_0 + M} (FG)(x) = \alpha \cdot \beta$.

(v) If $G(x)$ is zeroless for all $x \in x_0 + M \setminus \{x_0\}$ and β is zeroless then $\lim\limits_{x \to x_0 + M} \dfrac{F(x)}{G(x)} = \dfrac{\alpha}{\beta}$.

As for strong convergence we also obtain a squeeze theorem.

Theorem 6.6.4. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$, $M = (M_1, \dots, M_n) \neq 0$ be a neutrix vector and x_0 be point such that $x_0 + M \subseteq X$. Assume that $F(x) \leq G(x) \leq H(x)$ for all $x \in x_0 + M \setminus \{x_0\}$. Assume also that $\lim\limits_{x \to x_0 + M} F(x) = \lim\limits_{x \to x_0 + M} H(x) = \alpha$. Then $\lim\limits_{x \to x_0 + M} G(x) = \alpha$.*

Proof. Let $x \in x_0 + M \setminus \{x_0\}$ and $u \in G(x)$. Since $F(x) \leq G(x) \leq H(x)$, there are $v_1 \in F(x)$ and $v_2 \in H(x)$ such that $v_1 \leq u \leq v_2$. Also $F(x) \subseteq \alpha$, $H(x) \subseteq \alpha$. It follows that $u \in \alpha$. So $G(x) \subseteq \alpha$ for all $x \in x_0 + M \setminus \{x_0\}$. Hence $\lim\limits_{x \to x_0 + M} G(x) = \alpha$. □

As for relationship between limits of flexible functions and limits of flexible sequences we obtain the same result as classical mathematics.

Theorem 6.6.5. *Let $\alpha \in \mathbb{E}$, $M = (M_1, \dots, M_n) \neq 0$ be a neutrix vector and F be a flexible function defined on $X \subseteq \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ such that $x_0 + M \setminus \{x_0\} \subseteq X$. Then $\lim\limits_{x \to x_0 + M} F(x) = \alpha$ if and only if for every sequence $\{x_n\} \subseteq X \setminus \{x_0\}, x_n \hookrightarrow x_0 + M$, it holds that $F(x_n) \hookrightarrow \alpha$.*

Proof. Assume that $\lim\limits_{x \to x_0 + M} F(x) = \alpha$, with $\alpha = a + A$. Let $\{x_n\} \subseteq X \setminus \{x_0\}, x_n \hookrightarrow x_0 + M$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $x_n \in x_0 + M$. It follows that $F(x_n) \subseteq \alpha$ for all $n \geq n_0$. This means that $\text{Lim} F(x_n) = \alpha$.

Conversely, we assume that $F(x_n) \hookrightarrow \alpha$ for every sequence $\{x_n\} \subseteq X, x_n \hookrightarrow x_0 + M$. We need to prove that $\lim\limits_{x \to x_0 + M} F(x) = \alpha$. We suppose that it is not true. Then there is $x' \in x_0 + M \setminus \{x_0\}$ such that $F(x') \nsubseteq \alpha$. Let $x_n = x'$ for all $n \in \mathbb{N}$. Then $x_n \hookrightarrow x_0 + M$. This implies that $F(x_n) = F(x') \subseteq \alpha$, a contradiction. □

Next theorem shows that a flexible function is inner convergent if and only if it satisfies the Cauchy criterion.

Theorem 6.6.6. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$, $M = (M_1, \dots, M_n) \neq 0$ be a neutrix vector and x_0 be point such that $x_0 + M \subseteq X$. Then $\lim\limits_{x \to x_0 + M} F(x) = a + N$ if and only if for all $x, x' \in x_0 + M \setminus \{x_0\}$ we have $F(x) - F(x') \subseteq N$.*

Proof. Assume that $\lim\limits_{x \to x_0 + M} F(x) = a + N$. Let $x, x' \in x_0 + M \setminus \{x_0\}$. One has $F(x) - F(x') \subseteq F(x) - a + a - F(x') \subseteq N + N = N$.

Conversely, let $x_n = x_0 + \dfrac{1}{n} \hookrightarrow x_0 + M$. There exists k_0 such that for all $n \geq k_0$ we have $x_n \in x_0 + M$. Also $F(x_n)$ is N-strongly Cauchy. By Theorem 5.3.21, it follows that $\text{Lim} F(x_n) = a + N$. So, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $F(x_n) \subseteq a + N$. This means that $F(x_n) - a \subseteq N$. Let $m_0 = \max\{k_0, n_0\}$. Let $y \in x_0 + M \setminus \{x_0\}$ and $p \geq m_0$. Then $x_p \in x_0 + M \setminus \{x_0\}$. It follows that $F(y) - F(x_p) \subseteq N$. As a result, $F(y) - a + a - F(x_p) \subseteq N$. Since $a - F(x_p) \subseteq N$, one has $F(y) - a \subseteq N$. Hence $\lim\limits_{x \to x_0 + M} F(x) = a + N$.

□

Corresponding to the notion of inner limit, we define here another notion of continuity of a flexible function. As argued at the beginning of this section, this notion can be seen as a generalization of the notion of uniform continuity of a standard function.

Definition 6.6.7. $X \subseteq \mathbb{R}^n$, $M = (M_1, \ldots, M_n) \neq 0$ be a neutrix vector and N be a neutrix. Let $F\colon X \longrightarrow \mathbb{E}$ be a flexible function and $x_0 + M \subseteq X$. The function F is said to be $M \times N$-*inner continuous* at x_0 if $\lim\limits_{x \to x_0 + M} F(x) = F(x_0) + N$. In case $N = N_F(x_0)$ we say that F is M-inner continuous at x_0.

Example 6.6.8. (a) Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{E}$ be a flexible function defined by $F(x, y) = \sin(\pi x)\cos y + (x^2\epsilon \oslash +y^2\epsilon £) + \oslash$ and $x_0 = (1, 0)$. Let $M = (\oslash, \epsilon £)$. Then F is M-inner continuous at x_0.

(b) Let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by $F(x) = e^{x+\oslash} = e^x(1+\oslash)$. Then F is $\oslash$-inner continuous at $x \in £$. However, it is not $\oslash$-inner continuous at $x \notin £$.

As a sequence of Theorem 6.6.3, we obtain the followings.

Theorem 6.6.9. *Let $M = (M_1, \ldots, M_n) \neq 0$ be a neutrix vector and N_1, N_2 be neutrices. Let $N = N_1 + N_2$ and $X \subseteq \mathbb{R}^n$. Let F, G be flexible functions defined on X, with $F(x) = f(x) + N_F(x)$ and $G(x) = g(x) + N_G(x)$ for $x \in X$ and $x_0 \in \mathbb{R}^n$ such that $x_0 + M \subseteq X$. Assume that F is $M \times N_1$-inner continuous, G is $M \times N_2$-inner continuous at x_0. Then*

(i) $F + G$ is $M \times N$-inner continuous at x_0.

(ii) $F - G$ is $M \times N$-inner continuous at x_0.

(iii) kF is $M \times kN_1$-inner continuous at x_0 for all $k \in \mathbb{R}$.

(iv) FG is $M \times K$-inner continuous at x_0, where $K = N_1G(x_0) + N_2F(x_0) + N_1N_2$.

(v) If $G(x) + N_2$ is zeroless for all $x \in x_0 + M$, the function (F/G) is $M \times K$-inner continuous at x_0, where $K = \dfrac{F(x_0) + N_1}{g(x_0)} + \dfrac{F(x_0) + N_1}{g^2(x_0)}\big(N_G(x_0) + N_2\big)$.

6.7 The $M \times N$-derivative of a flexible function

When x approaches x_0 the neutrix part of the expression $\dfrac{F(x) - F(x_0)}{x - x_0}$, in general, approaches $\mathbb{R}$. For example $\dfrac{\oslash}{x}$ tends to $\mathbb{R}$ when x approaches 0. So we can not use the classical technique to build the notion of derivative for flexible functions. However, for a neutrix and a point x, which x is not an absorber of N, we have that $\dfrac{N}{x} \subseteq N$. Using this fact we will construct the notion of derivative for a flexible function. To do it we will introduce another notion of limit, called an *outer limit*. In chapter 8 we will use the notion of derivative to construct necessary conditions for the existence of an approximate local optimal solution of optimization with flexible objective functions.

6.7.1 Outer limit

Let F be a flexible function and M be a neutrix. In this section we consider behaviour of $F(x)$ when x approaches x_0, but x always stays outside of $x_0 + M$.

Definition 6.7.1. Let $X \subseteq \mathbb{R}^n$, F: $X \longrightarrow \mathbb{E}$ be a flexible function and $\alpha = a + A$ be an external number. Assume that x_0 is an M-accumulation point of X . We say that α is an *$M \times N$-outer limit* of F at x_0, if for all $\epsilon > N$, there exists $\delta > M$ such that for all $x \in X,\ M < |x - x_0| < \delta$ one has $|F(x) - \alpha| < \epsilon$. We write

$$N\text{-}\lim_{x \to x_0 + M} F(x) = \alpha.$$

Then we also say that $F(x)$ is $M \times N$- outer convergent to α when x approaches x_0.

The "outer" here hints that x approaches x_0 but always stays the outside of $x_0 + M$.

Remark 6.7.2. Similarly to Proposition 6.3.6 it is true that if α is an $M \times N$-outer limit of F at x_0 then $N(\alpha) \subseteq N$ and $\alpha + N$ is an $M \times N$-outer limit of F at x_0.

When $M = 0$, the notion of $M \times N$-outer limit coincides exactly with the notion of $M \times N$ limit.

Example 6.7.3. One has $\oslash\text{-}\lim_{x \to 0 + \oslash} (x + x\oslash) = \oslash$. Indeed, let $\epsilon > \oslash$. Let $\delta = \epsilon/2 > \oslash$. Then for all $x \in \mathbb{R}$, $\oslash < |x| < \delta$ we have $|F(x) - F(0)| = |x + x \oslash | < \epsilon/2 + \epsilon/2 = \epsilon$.

The difference between an $M \times N$-limit and an $M \times N$-outer limit is that an $M \times N$-limit considers values of a given function at points $x \in \mathbb{R}$ with $|x - x_0| \in M$ while an $M \times N$-outer limit does not. As a consequence it is easy to see that if α is an $M \times N$-limit of F at x_0, it is an $M \times N$-outer limit of F at x_0. That is

$$\mathop{{}_{M}^{N}\lim}_{x \to x_0} F(x) = \alpha \Longrightarrow N\text{-}\lim_{x \to x_0 + M} F(x) = \alpha. \tag{6.9}$$

Remark 6.7.4. Because of (6.9), the results which hold for $M \times N$-limits also hold for $M \times N$-outer limits.

Similarly to $M \times N$-limits we also have notions of one-sided $M \times N$-outer limits.

Definition 6.7.5. Let F: $X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and $x_0 \in \mathbb{R}^n$ be an M-accumulation point of X. An external number α is called a *left $M \times N$- outer limit* of F at x_0 if for all $\epsilon > N$ there exists $\delta > M$ such that for all $M < x_0 - x < \delta$ we have $|F(x) - F(x_0)| < \epsilon$. We write

$$N\text{-}\lim_{x \to x_0 + M^-} F(x) = \alpha.$$

An external number $\beta \in \mathbb{E}$ is called a *right $M \times N$-outer limit* of F at x_0 if for all $\epsilon > N$ there exists $\delta > M$ such that for all $x \in X, M < x - x_0 < \delta$ we have $|F(x) - F(x_0)| < \epsilon$. We write

$$N\text{-}\lim_{x \to x_0 + M^+} F(x) = \beta.$$

Remark 6.7.6. It is easy to see that $N\text{-}\lim\limits_{x\to x_0+M} F(x)=\alpha$ if and only if

$$N\text{-}\lim_{x\to x_0+M^+} F(x)=N\text{-}\lim_{x\to x_0+M^-} F(x)=\alpha.$$

6.7.2 The notion of $M \times N$-derivative

Using the notion of outer limit we define the $M\times N$-derivative of a flexible function. It will be used to determine conditions for the existence of (approximate) optimal solutions in Chapter 8. This notion of derivative is devoted to a function of one variable.

Definition 6.7.7. Let M, N be neutrices, $F\colon X \subseteq \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible function and $x_0 \in X$ be an M-accumulation point of X. The flexible function F is called $M \times N$*-differentiable* at x_0 if the $M \times N$-outer limit of the fraction $\dfrac{F(x)-F(x_0)}{x-x_0}$ exists. Then this $M \times N$-outer limit is called the $M \times N$*-derivative* of F at x_0 and denoted by $\dfrac{d_N F}{d_M x}(x_0)$ or ${}^{N}_{M}DF(x_0)$. So

$$\frac{d_N F}{d_M x}(x_0)=N\text{-}\lim_{x\to x_0+M}\frac{F(x)-F(x_0)}{x-x_0}.$$

In particular, in case $N=N_F(x_0)$ we call it M-derivative and write $\dfrac{dF}{d_M x}(x_0)=\lim\limits_{x\to x_0}\dfrac{F(x)-F(x_0)}{x-x_0}$. We also say that F is M-differentiable at x_0.

Convention 6.7.8. Because of Remark 6.7.4, from now on, we always assume that the neutrix part of the $M \times N$-derivative of F at x_0 is N.

Example 6.7.9. Let F be a flexible function given by $F(x)=x^2+\oslash$ for all $x\in\mathbb{R}$. One has

$$\frac{dF}{d_\oslash x}(x_0)=\lim_{x\to x_0+\oslash}\frac{x^2+\oslash-x_0^2+\oslash}{x-x_0}=2x_0+\oslash.$$

Indeed, we have $N_F(x)=\oslash$ for all $x\in\mathbb{R}$. Let $\epsilon>\oslash$. Let $\delta=\epsilon/2>\oslash$. Then for all $x\in\mathbb{R}, \oslash<d(x,x_0)<\delta$ one has

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0}-2x_0+\oslash\right| &=\left|\frac{x^2+\oslash-x_0^2+\oslash}{x-x_0}-2x_0+\oslash\right|\\ &=\left|x+x_0+\frac{\oslash}{x-x_0}-2x_0+\oslash\right|<d(x,x_0)+\oslash<\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Example 6.7.10. Let F be a flexible function given by $F(x)=x+x\cdot\oslash$ for all $x\in\mathbb{R}$ and $x_0\in\mathbb{R}$. One has $\dfrac{d_\oslash F}{d_\oslash x}(x_0)=1+\oslash$. Note that $\dfrac{dF}{d_\oslash x}(0)$ does not exist. Indeed, we have $N_F(0)=0$. Let $\epsilon_0>0$ be a fixed infinitesimal. Then $\dfrac{F(x)-F(0)}{x-0}=1+\oslash>\epsilon_0$ for all $x\in\mathbb{R}$ which implies that $\lim\limits_{x\to 0}\dfrac{F(x)-F(0)}{x-0}$ does not exist.

Definition 6.7.11. Let M, N be neutrices, $F\colon X\longrightarrow\mathbb{E}$ be a flexible function and $x_0\in X\subseteq\mathbb{R}$ be M-

accumulation point of X. The *left $M \times N$-derivative* of F at x_0 is defined and written as follows:

$$\frac{d_N F_-}{d_M x}(x_0) = N\text{-}\lim_{x \to x_0 + M^-} \frac{F(x) - F(x_0)}{x - x_0}.$$

The *right $M \times N$-derivative* of F at x_0 is defined and written as follows:

$$\frac{d_N F_+}{d_M x}(x_0) = N\text{-}\lim_{x \to x_0 + M^+} \frac{F(x) - F(x_0)}{x - x_0}.$$

Theorem 6.7.12. *The $M \times N$-derivative of F at x_0 exists if and only if the left and right $M \times N$-derivatives of F at x_0 exist and they are equal to each other.*

The theorem is a consequence of Remark 6.7.6.

Theorem 6.7.13. *Let M, N_1, N_2 be neutrices. Let F, G be flexible functions defined on $X \subseteq \mathbb{R}$ and $x_0 \in X$ is an M-accumulation point of X. Assume that F is $M \times N_1$-differentiable at x_0 and G is $M \times N_2$-differentiable at x_0. Let $K = (N_1 + N_2)$. Then*

(i) The flexible function $(F \pm G)$ is $M \times K$-differentiable at x_0 and

$$\frac{d_N(F \pm G)}{d_M x}(x_0) = \frac{d_{N_1} F}{d_M x}(x_0) \pm \frac{d_{N_2} G}{d_M x}(x_0).$$

(ii) For $k \in \mathbb{R}$, the flexible function kF is $M \times kN_1$-differentiable at x_0 and

$$\frac{d_{kN_1} F}{d_M x}(x_0) = k \cdot \frac{d_{N_1} F}{d_M x}(x_0).$$

The theorem is a consequence of Remark 6.7.4 and Theorem 6.3.10.

6.7.3 Higher order derivatives

Definition 6.7.14. Let $F\colon X \longrightarrow \mathbb{E}$ be a flexible function and $N\colon X \longrightarrow \mathcal{N}$ be a neutrix-function. Let M be a neutrix. We say that the flexible function F is *$M \times N$-differentiable on* X if F is $M \times N(x)$-differentiable at x for all $x \in X$. In case $N(x) = N_F(x)$ for all $x \in X$ we say that F is M-differentiable on X.

Example 6.7.15. Let F be a flexible function given by $F(x) = x^2 + \oslash$. Then F is $\oslash$-differentiable on $\mathbb{R}$.

Example 6.7.16. The function $F(x) = x^2 + \oslash x$ is not $\oslash$-differentiable on $\mathbb{R}$ because it is not $\oslash$-differentiable at $x = \omega \simeq \infty$. Indeed, we have $\dfrac{F(x) - F(\omega)}{x - \omega} = \dfrac{x^2 + x \oslash - \omega^2 + \oslash\omega}{x - \omega} = x + \omega + \dfrac{x \oslash + \oslash \omega}{x - \omega}$. Since $|x - \omega| \in @$ one has $\oslash \subset \omega\oslash = \dfrac{x \oslash + \oslash \omega}{x - \omega}$. Hence $\oslash\text{-}\lim\limits_{x \to \omega + \oslash} \dfrac{F(x) - F(\omega)}{x - \omega}$ does not exist. However, by the above, F is $\oslash \times \omega\oslash$-differentiable at ω.

Definition 6.7.17. Let $N_1\colon X \to \mathcal{N}$ be a neutrix-function and F be a flexible function. Let M_1, M_2 be neutrices. Assume that F is $M \times N_1$-differentiable on X. Then ${}^{N_1(x)}_{M}DF$ is a flexible function defined on X with the neutrix part $N(x)$ and a representative $g(x)$. That is for all $x \in X$, the value of $M \times N_1(x)$-derivative of F has the form

$$ {}^{N_1(x)}_{M}DF(x) = g(x) + N_1(x). \tag{6.10}$$

If this flexible function is N_2-differentiable at $x_0 \in X$, we say that the function F is $(M_1, M_2) \times (N_1, N_2)$-*differentiable of degree 2* at x_0, where $N_1 = N_1(x_0)$, and its derivative is called the *second* $(M_1, M_2) \times (N_1, N_2)$-*derivative* or the $(M_1, M_2) \times (N_1, N_2)$-*derivative of degree 2*, denoted by ${}^{N_2 N_1}_{M_2 M_1}D^2F(x_0)$. So

$$ {}^{N_2 N_1}_{M_2 M_1}D^2F(x_0) = {}^{N_2}_{M_2}D\left({}^{N_1}_{M_1}DF\right)(x_0).$$

By external inductive, for every standard $n \in \mathbb{N}$ we define the $(M_1, \ldots, M_n) \times (N_1, \ldots, N_n)$-*derivative of degree* n by

$$ {}^{N_n,\ldots,N_1}_{M_n,\ldots,M_1}D^{(n)}F(x_0) = {}^{N_n}_{M_n}D\left({}^{N_{n-1},\ldots N_1}_{M_{n-1},\ldots,M_1}D^{(n-1)}F\right)(x_0),$$

where M_i, N_i are neutrices for $1 \le i \le n$.

6.8 Monotonicity

Many flexible functions are locally constant at each point. For example, consider the flexible function $F(x) = x + \oslash$ for $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ then for all $x \in x_0 + \oslash$ one has $F(x) = F(x_0)$. Because of this fact, we should consider monotonicity of flexible functions with certain order steps of variables.

Note that $\alpha \le \beta$ is not equivalent to $\beta \ge \alpha$ where α, β are external numbers. Then the monotonicity of a flexible function depends on a relationship considered.

Definition 6.8.1. Let $F \colon X \longrightarrow \mathbb{E}$ be a flexible function. The function F is said to be

(i) *increasing with order step* M on X if $F(x) \ge F(y)$ for all $x, y \in X, x - y > M$,

(ii) *strictly increasing with order step* M on X if $F(x) > F(y)$ for all $x, y \in X, x - y > M$,

(iii) *decreasing with order step* M if $F(x) \le F(y)$ for all $x, y \in X, x - y > M$,

(iv) *strictly decreasing with order step* M if $F(x) < F(y)$ for all $x, y \in X, x - y > M$.

The function F which is decreasing with order step M or increasing with order step M is called *monotone with order step* M.

In case $M = 0$ we call it *increasing, decreasing, monotone*, respectively.

Example 6.8.2. The flexible function $F(x) = x + \oslash$ is strictly $\oslash$-increasing on $\mathbb{R}$. Indeed, for all $x, y \in \mathbb{R}, y - x > \oslash$, one has $F(y) - F(x) = y - x + \oslash > \oslash$.

In classical mathematics if $f'(x_0) > 0$, there is $\delta > 0$ such that $f(x)$ is increasing on $(x_0 - \delta, x_0 + \delta)$ and if $f'(x_0) < 0$, there is $\eta > 0$ such that $f(x)$ is decreasing on $(x_0 - \eta, x_0 + \eta)$. Using neutrices N instead of zero, we develop a version for flexible functions.

Proposition 6.8.3. *Let M, N be neutrices such that $N_A \subseteq M$ and F be a flexible function defined on $X \subseteq \mathbb{R}$. Assume that $F(x)$ is $M \times N$-differentiable at $x_0 \in X$ and $\frac{d_N F}{d_M x}(x_0) > N$. Then there exists $\delta > M$ such that*

(i) $F(x) - F(x_0) < N$ for all $x \in X, M < x_0 - x < \delta$,

(ii) $F(x) - F(x_0) > N$ for all $x \in X, M < x - x_0 < \delta$.

In particular, if $N_F(x_0) \subseteq N$ we have

(i') $F(x) < F(x_0)$ for all $x \in X, M < x_0 - x < \delta$,

(ii') $F(x) > F(x_0)$ for all $x \in X, M < x - x_0 < \delta$.

Proof. Since $\frac{d_N F}{d_M x}(x_0) > N$, let $\epsilon \in \frac{\frac{d_N F}{d_M x}(x_0)}{2} > N$ be a representative of $\frac{\frac{d_N F}{d_M x}(x_0)}{2}$. By the definition of the $M \times N$-derivative, there exists $\delta > M$ such that $|\frac{F(x) - F(x_0)}{x - x_0} - \frac{d_N F}{d_M x}(x_0)| < \epsilon$. It follows by Proposition 2.2.41 that

$$\frac{d_N F}{d_M x}(x_0) - \epsilon < \frac{F(x) - F(x_0)}{x - x_0} \text{ for all } x \in X, M < |x - x_0| < \delta. \tag{6.11}$$

Note that $\frac{d_N F}{d_M x}(x_0) - \epsilon = \frac{\frac{d_N F}{d_M x}(x_0)}{2} > N$. On the other hand, $N_A \subseteq M$, so $x - x_0 \notin M$ implies that $x - x_0$ is not an absorber of N. Hence, for all $x \in X,\ M < x - x_0 < \delta$, formula (6.11) implies $F(x) - F(x_0) > N$.

In particular, if $N_F(x_0) \subseteq N$, this implies $F(x) > F(x_0)$.

For all $x \in X, M < x_0 - x < \delta$, formula (6.11) implies $F(x) - F(x_0) < N$. In particular, if $N_F(x_0) \subseteq N$, it follows that $F(x) < F(x_0)$ for all $x \in X,\ M < x_0 - x < \delta$. □

With similar arguments we obtain the following.

Proposition 6.8.4. *Let M, N be neutrices such that $N_A \subseteq M$ and F be a flexible function defined on $X \subseteq \mathbb{R}$. Assume that F is $M \times N$-differentiable at $x_0 \in X$ and $\frac{d_N F}{d_M x}(x_0) < N$. Then there exists $\delta > M$ such that*

(i) $F(x) - F(x_0) > N$ for all $x \in X, M < x_0 - x < \delta$,

(ii) $F(x) - F(x_0) < N$ for all $x \in X, M < x - x_0 < \delta$.

In particular, if $N_F(x_0) \subseteq N$, we have

(i') $F(x) > F(x_0)$ for all $x \in X, M < x_0 - x < \delta$,

(ii') $F(x) < F(x_0)$ for all $x \in X, M < x - x_0 < \delta$.

6.9 The $M \times N$-differentiability of a vector flexible function

6.9.1 The $M \times N$-partial derivatives of a flexible function of several variables

This section is devoted to studying differential a flexible function of several variables. We will apply these results to investigate an optimization problem.

We known that partial derivatives of a function of several variables at a point are defined as the derivative of a function of one variable by fixing other variables as constants. Analogously, we define $M \times N$-partial derivatives of flexible functions of several variables.

Definition 6.9.1. Let $n \in \mathbb{N}$ be standard, $X \subseteq \mathbb{R}^n$ and M_i, N_i be neutrices for $1 \leq i \leq n$. Let $F\colon X \longrightarrow \mathbb{E}$ is a flexible function defined on X. Let $x_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in \mathbb{R}^n$ be such that

$$\left\{x = (x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \ldots, x_n^{(0)}) \in \mathbb{R}^n \middle| M_i < d(x, x_0) < \delta\right\} \cap X \setminus \{x_0\} \neq \emptyset.$$

We say that the flexible function F has an $M_i \times N_i$*-partial derivative* corresponding to the i-th variable, x_i, at the point x_0 if the following $M_i \times N_i$-outer limit exists:

$$N_i\text{-}\lim_{x \to x_i^{(0)} + M_i} \frac{F(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \ldots, x_n^{(0)}) - F(x_1^{(0)}, \ldots, x_n^{(0)})}{x_i - x_i^{(0)}}.$$

The $M_i \times N_i$-partial derivative corresponding to the variable x_i at x_0 is denoted by $\dfrac{\partial_{N_i} F}{\partial_{M_i} x_i}(x_0)$ or $\dfrac{d_N F}{d_M x}_{x_i}(x_0)$.

Hence

$$\frac{\partial_{N_i} F}{\partial_{M_i} x_i}(x_0) = N_i\text{-}\lim_{x \to x_i^{(0)} + M_i} \frac{F(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \ldots, x_n^{(0)}) - F(x_1^{(0)}, \ldots, x_n^{(0)})}{x_i - x_i^{(0)}}.$$

6.9.2 The $M \times N$-total derivative of a vector flexible function of several variables

We now define the *total* $M \times N$*-derivative* of a vector flexible function of several variables.

Definition 6.9.2. Let M, N be neutrices and $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}^m$ be a function with $F = (F_1, \ldots, F_m)$. We say that F is $M \times N$*-totally differentiable* if there exists an $m \times n$ matrix $\mathcal{A} = [\alpha_{ij}]_{m \times n} \in \mathcal{M}_{m,n}(\mathbb{E})$ such that

$$0 \in \left(N\text{-}\lim_{\|h\| \to 0 + M} \frac{\|F(x_0 + h) - F(x_0) - \mathcal{A}h\|}{\|h\|}\right).$$

Then the matrix $\mathcal{A}$ is called the $M \times N$*-total derivative* of F at x_0 and we write $\mathcal{A} = {}_M^N DF(x_0)$.

Similarly to the derivative of a flexible function of one variable, the neutrix parts of all entries of ${}_M^N DF(x_0)$ are included in N. So we always take the neutrix parts of all entries of ${}_M^N DF(x_0)$ are N.

Theorem 6.9.3. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}^m$ be an $M \times N$-totally differentiable at a, where a is an M-interior point of X. Then for each $i \in \{1,\dots,m\}$ the function $F_i\colon X \longrightarrow \mathbb{E}$ is $M \times N$-totally differentiable at a.*

Proof. Let $\mathcal{A} = {}^N_M DF(a)$ and $\mathcal{A}_i$ be a row vector of $\mathcal{A}$ for $1 \le i \le m$. One has

$$|F_i(a+h) - F_i(a) - \langle \mathcal{A}_i, h\rangle| \le \max_{k\in\{1,\dots,m\}} |F_k(a+h) - F_k(a) - \langle \mathcal{A}_k, h\rangle| = \|F(a+h) - F(a) - \mathcal{A}h\|\,.$$

Then

$$0 \le \frac{|F_i(a+h) - F_i(a) - \langle \mathcal{A}_i, h\rangle|}{\|h\|} \le \frac{\|F(a+h) - F(a) - \mathcal{A}h\|}{\|h\|} \underset{N}{\longrightarrow} 0$$

when $\|h\| \to M$, $\|h\| \notin M$. This implies that

$$0 \in \left(N\text{-}\lim_{\|h\|\to 0+M} \frac{|F_i(a+h) - F_i(a) - \langle \mathcal{A}_i, h\rangle|}{\|h\|}\right).$$

So F_i is $M \times N$-totally differentiable at a. □

Theorem 6.9.4. *Let $X \subseteq \mathbb{R}^n$ and $F\colon X \longrightarrow \mathbb{E}$ be N-totally differentiable at a, where a is an M-interior point of X. Then F has $M \times N$-partial derivatives with respect to the variable x_i for every $i \in \{1,\dots,n\}$ at a.*

Proof. Let $\mathcal{A} = [\alpha_i] \in \mathcal{M}_{1,n}(\mathbb{E})$ be the $M \times N$-total derivative of F at x_0. Since F is $M \times N$-totally differentiable at a, we have

$$0 \in \left(N\text{-}\lim_{\|h\|\to 0+M} \frac{\|F(a+h) - F(a) - \mathcal{A}h\|}{\|h\|}\right).$$

In particular $h = h_i e_i$ with $h_i \in \mathbb{R}$, $1 \le i \le n$ we obtain

$$F(a+h) - F(a) - \mathcal{A}h = F(a_1,\dots,a_{i-1}, a_i + h_i, a_{i+1},\dots,a_n) - F(a) - \alpha_i h_i.$$

It follows that

$$\begin{aligned} 0 &\in \left(N\text{-}\lim_{|h_i|\to 0+M} \frac{|F(a_1,\dots,a_{i-1}, a_i + h_i, a_{i+1},\dots,a_n) - F(a) - \alpha_i h_i|}{|h_i|}\right) \\ &= N\text{-}\lim_{\|h\|\to 0+M} \frac{\|F(a+h) - F(a) - \mathcal{A}h\|}{\|h\|}. \end{aligned}$$

Hence $\mathcal{A}_i = \dfrac{\partial_N F}{\partial_M x_i}(a)$, $\forall i = 1,\dots,n$. □

6.9.3 The $M \times N$-partial derivatives of a composite function

Recall that in classical mathematics if f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$ then $h = g \circ f$ is differentiable at x_0. We now investigate conditions to guarantee that h is $M \times N$-differentiable.

Definition 6.9.5. Let $f\colon \mathbb{R}^m \longrightarrow \mathbb{E}$ be a flexible function and N, M be neutrices. Let $b = (b_1,\dots,b_m) \in \mathbb{R}^m$. We say that f is *M-outer N-inner continuous* at b if there exists $r > M$ such that for all $y = (y_1,\dots,y_m) \in$

$\mathbb{R}^m, M < |y-b| < r$, for each $0 \le k \le m$, and $i_1, \dots, i_k \in \{1, \dots, m\}$ for all $u = (u_1, \dots, u_m)$ with

$$u_p = \begin{cases} b_p \text{ if } p \in \{i_1, \dots, i_k\} \\ y_p \text{ if } p \notin \{i_1, \dots, i_k\} \end{cases}$$

one has

$$f(y) - f(u) \subseteq N.$$

Example 6.9.6. Consider the flexible function $F\colon \mathbb{R}^2 \longrightarrow \mathbb{E}$ given by $F(x,y) = x+y+x^2\epsilon\oslash+\epsilon\pounds y$. Then F is $\epsilon\pounds$-outer $\oslash$-inner continuous at $w_0 = (x_0, y_0) \in \pounds\times\pounds$. Indeed, let $r > \epsilon\pounds, r \in \oslash$. For all $w = (x,y) \in B(w_0, r)$ and $u = (u_1, u_2)$ where $u_1 = x_0 \vee u_1 = x$ and $u_2 = y_0 \vee u_2 = y$, we have

$$|F(w) - F(u)| = |x + y + x^2\epsilon \oslash +\epsilon\pounds y - \big(u_1 + u_2 + u_1^2\epsilon \oslash + u_2^2\epsilon\pounds\big)| \le |x - x_0| + |y - y_0| + \oslash \subseteq \oslash.$$

Theorem 6.9.7. *Let N_1, N_2, M_1, M_2 be neutrices. Let $f\colon R^m \longrightarrow \mathbb{E}$ be a flexible function, $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector function and $b = \varphi(a_1, \dots, a_n) = (b_1, \dots, b_m)$, with $b_j = \varphi_j(a)$ for all $j \in \{1, \dots, m\}$. Let $g = f \circ \varphi\colon \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function defined by $g(x) = f\big(\varphi_1(x), \dots, \varphi_m(x)\big)$ for every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Assume that*

(i) φ is $M_1 \times N_1$-totally differentiable at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$,

(ii) for $i \in \{1, \dots, m\}$, let $z_i = (a_1, \dots, a_{i-1}, a_i+h, a_{i+1}, a_n)$. Then we have $\varphi_i(z_i)-\varphi_i(a) \to M_2, \varphi_i(z_i)-\varphi_i(a) \notin M_2$ when $h \to M_1, h \notin M_1$,

(iii) f is M_2-outer N_2-inner continuous and $M_2 \times N_2$-totally differentiable at b.

Then

$$\frac{\partial_K g}{\partial_{M_1} x_i}(a) = \sum_{k=1}^{m} \frac{\partial_{N_2} f}{\partial_{M_2} y_k}(b) \frac{\partial_{N_1} \varphi_k}{\partial_{M_1} x_i}(a), \tag{6.12}$$

where $K = \sum\limits_{q=1}^{m} K_q$ with $K_q = N_1 + N_2 + N_1^2 + N_2^2 + \dfrac{\partial_{N_1}\varphi_q}{\partial_{M_1} x_i}(a) \cdot N_2 + \dfrac{\partial_{N_2} f}{\partial_{M_2} y_q}(b_1, \dots, b_m) \cdot N_1.$

Proof. Let $i \in \{1, \dots, n\}$ be arbitrary. Put $\delta = g(z) - g(a) = f(\varphi(z_i)) - f(\varphi(a)); y_j = \varphi_j(z_i), 1 \le j \le m$. One has

$$\begin{aligned}\frac{\delta}{h} =&\frac{f(y_1, \dots, y_m) - f(b_1, \dots, b_m)}{h} \subseteq \frac{f\Big(y_1, \dots, y_m\Big) - f\big(y_1, \dots, y_{m-1}, b_m\big)}{h} \\ &+\frac{f\big(y_1, \dots, y_{m-1}, b_m\big) - f\big(y_1, \dots, y_{m-2}, b_{m-1}, b_m\big)}{h} + \dots + \frac{f\big(y_1, b_2, \dots, b_m\big) - f\big(b_1, \dots, b_m\big)}{h} \equiv D.\end{aligned}$$

We prove only that the first term is K_m-partial differentiable with respect to the variable x_i. The other terms are treated similarly. Then the conclusion follows from the sum rule.

One has

$$\frac{\delta_m}{h} \equiv \frac{f\big(y_1,\dots,y_m\big)-f\big(y_1,\dots,y_{m-1},b_m\big)}{h} = \frac{f\big(y_1,\dots,y_m\big)-f\big(y_1,\dots,y_{m-1},b_m\big)}{y_m-b_m}\cdot\frac{y_m-b_m}{h}$$

By assumption (ii), if $h \longrightarrow M_1, h \notin M_1$ we have $y_j - b_j \longrightarrow M_2$, $y_j - b_j \notin M_2$ $j = 1,\dots,m$. By the assumption that f is M_2-outer N_2-inner continuous, it follows

$$\begin{aligned} f\big(y_1,\dots,y_{m-1},y_m\big) &\subseteq f(b_1,\dots,b_{m-1},y_m)+N_2 \\ f\big(y_1,\dots,y_{m-1},b_m\big) &\subseteq f(b_1,\dots,b_{m-1},b_m)+N_2, \end{aligned}$$

for all $(y_1,\dots,y_{m-1}) \in B_{M_2}\big((b_1,\dots,b_{m-1}),r\big)$. Hence

$$\begin{aligned} \frac{f\Big(y_1,\dots,y_m\Big)-f\big(y_1(z),\dots,y_{m-1},b_m\big)}{y_m-b_m} &\subseteq \frac{f(b_1,\dots,b_{m-1},y_m)-f(b_1,\dots,b_{m-1},b_m)+N_2}{y_m-b_m} \\ &\underset{N_2}{\longrightarrow} \frac{\partial_{N_2} f}{\partial_{M_2} y_m}(b_1,\dots,b_m). \end{aligned}$$

Also, $\dfrac{y_m-b_m}{h} = \dfrac{\varphi_m(z_i)-\varphi_m(a)}{h} \underset{N_1}{\longrightarrow} \dfrac{\partial_{N_1}\varphi_m}{\partial_{M_1} x_i}(a)$ when $h \longrightarrow M_1, h \notin M_1$. Hence, by Theorem 6.3.11, it holds that

$$\frac{\delta_m}{h} \underset{K_m}{\longrightarrow} \frac{\partial_{N_2} f}{\partial_{M_2} y_m}(b_1,\dots,b_m)\cdot\frac{\partial_{N_1}\varphi_m}{\partial_{M_1} x_i}(a),$$

where $K_m = N_1 + N_2 + N_1^2 + N_2^2 + \dfrac{\partial_{N_1}\varphi_m}{\partial_{M_1} x_i}(a)\cdot N_2 + \dfrac{\partial_{N_2} f}{\partial_{M_2} y_m}(b_1,\dots,b_m)\cdot N_1$.

Similarly, for $q \in \{2,\dots,m\}$ one has

$$\begin{aligned} \frac{\delta_q}{h} =& \frac{f\big(y_1,\dots,y_q,b_{q+1},\dots,b_m\big)-f\big(y_1,\dots,y_{q-1},b_q,\dots,b_m\big)}{h} \\ \underset{K_q}{\longrightarrow}& \frac{\partial_{N_2} f}{\partial_{M_2} y_q}(b_1,\dots,b_m)\cdot\frac{\partial_{N_1}\varphi_q}{\partial_{M_1} x_i}(a), \end{aligned}$$

where $K_q = N_1 + N_2 + N_1^2 + N_2^2 + \dfrac{\partial_{N_1}\varphi_q}{\partial_{M_1} x_i}(a)\cdot N_2 + \dfrac{\partial_{N_2} f}{\partial_{M_2} y_q}(b_1,\dots,b_m)\cdot N_1$. By Theorem 6.3.10 one has

$$D \underset{K}{\longrightarrow} \sum_{k=1}^{m}\frac{\partial_{N_2} f}{\partial_{M_2} y_k}(b)\cdot\frac{\partial_{N_1}\varphi_k}{\partial_{M_1} x_i}(a).$$

This implies that $\dfrac{\delta}{h} \underset{K}{\longrightarrow} \sum_{k=1}^{m}\dfrac{\partial_{N_2} f}{\partial_{M_2} y_k}(b)\cdot\dfrac{\partial_{N_1}\varphi_k}{\partial_{M_1} x_i}(a)$. One concludes that $\dfrac{\partial_K g}{\partial x_i}(a) = \sum_{k=1}^{m}\dfrac{\partial_{N_2} f}{\partial_{M_2} y_k}(b)\cdot\dfrac{\partial_{N_1}\varphi_k}{\partial_{M_1} x_i}(a)$.
□

6.10 The inverse flexible function theorem

In conventional mathematics the inverse function theorem and the implicit function theorem play a key role in the Lagrange multiplier method. Below we investigate under which conditions inverse functions or implicit functions are N-differentiable.

Theorem 6.10.1. *Let N, M be neutrices, $X \subseteq \mathbb{R}^n$ be an open set and a be an M-interior point of X. Let Z be an M-neighbourhood of a, $Z \subseteq X$ and $f\colon X \longrightarrow \mathbb{R}^n$ be an internal function satisfying*

(a) f is $M \times N$-differentiable on X,

(b) f is continuously differentiable on X,

(c) f' is $M \times N$-continuous and invertible at a, and

$$\left\| \left(f'(a)\right)^{-1} \right\|^{-1} > N, \tag{6.13}$$

(d) $\left\| \left(f'(x)\right)^{-1} \right\|$ is not an absorber of N for all $x \in Z$,

(e) $f'(x) \in {}_M^N DF(x)$ for all $x \in Z$,

(f) $\|f(x+h) - f(x)\| \leq r\|h\|$ for all $x, x+h \in Z$, and for some $r \in \mathbb{R}$, where r^{-1} is not an absorber of M,

(g) f maps an open M-neighbourhood of a to an open M-neighbourhood of b, that is for every open M-neighbourhood U of a and $f(U) = V$ then V is an open M-neighbourhood of b with $b = f(a)$.

Then

(i) There exists an open M-neighbourhoods U and V of a and b, respectively, such that f is one-to-one mapping on U and $f(U) = V$.

(ii) if g is the inverse function of f defined in V by $g(f(x)) = x$, $(x \in U)$ then g is $M \times N$-totally differentiable at b and ${}_M^N Dg(y) = \Big(f'\big(g(y)\big)\Big)^{-1} + N$.

To prove this theorem we recall the following result.

Theorem 6.10.2 ([31, p. 209])**.** *Let Ω be the set of all invertible linear operations on $\mathbb{R}^n$.*

(i) If $A \in \Omega, B \in L(\mathbb{R}^n)$, and

$$\|A - B\| \cdot \left\|A^{-1}\right\| < 1,$$

then $B \in \Omega$.

(ii) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \longrightarrow A^{-1}$ is continuous.

Proof of Theorem 6.10.1. (i) Put $f'(a) = A$, and put

$$\lambda = \frac{1}{2\|A^{-1}\|}. \tag{6.14}$$

Note that $\lambda > N$ by condition (6.13). Moreover, because f' is $M \times N$-continuous at a, there is an open M-ball U of radius $r > M$, centered at a such that for all $x \in U$ it holds that

$$\|f'(x) - A\| < \lambda. \tag{6.15}$$

We associate to each $y \in \mathbb{R}^n$ a function φ, defined by

$$\varphi(x) = x + A^{-1}(y - f(x)), (\forall x \in X). \tag{6.16}$$

Note that $f(x) = y$ if and only if x is a fixed point of φ. Because $\varphi'(x) = I - A^{-1}f'(x) = A^{-1}(A - f'(x))$, (6.14) and (6.15) imply that for all $x \in U$,

$$\|\varphi'(x)\| < \frac{1}{2}. \tag{6.17}$$

Hence

$$\|\varphi(x_1) - \varphi(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|, \ \ (x_1, x_2 \in U) \tag{6.18}$$

by the mean value theorem in several variables. By the contraction principle [31], it follows that φ has the unique fixed point x in U, so that $f(x) = y$ for exactly one $x \in U$. Hence f is 1-1 in U.

Next, put $V = f(U)$. By assumption (g), V is an open M-neighbourhood of $b = f(a)$.

(ii) Let $y \in V, y \in V$ and $k \in \mathbb{R}^n$ such that $y + k \in V$. Then there exists $x \in U, x + h \in U$ such that

$$y = f(x), y + k = f(x + h). \tag{6.19}$$

With φ as in (6.16) we obtain

$$\varphi(x + h) - \varphi(x) = h + A^{-1}[f(x) - f(x + h)] = h - A^{-1}k.$$

By (6.18) it holds that $\left\|h - A^{-1}k\right\| \leq \frac{1}{2}\left\|h\right\|$. Hence $\|A^{-1}k\| \geq \frac{1}{2}|h|$. Also $\|A^{-1}k\| \leq \|A^{-1}\| \cdot \|k\|$, so

$$\|h\| \leq 2\|A^{-1}\| \cdot \|k\| = \lambda^{-1}\|k\|. \tag{6.20}$$

By (6.14), (6.15) and Theorem 6.10.2, $f'(x)$ has an inverse, say T. Note that from (6.19) and $g = f^{-1}$ we have $g(y) = x, g(y + k) = x + h$. Also, from equality $k = f(x + h) - f(x)$ and $T = \left(f(x)\right)^{-1}$ we obtain

$$g(y + k) - g(y) - Tk = h - Tk = -T[f(x + h) - f(x) - f'(x)h].$$

Combining with the inequality (6.20) one has

$$\left\|\frac{g(y + k) - g(y) - Tk}{k}\right\| \leq \frac{\|T\|}{\lambda} \cdot \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|}$$

Because $\|k\| \to M, \|k\| \notin M$, formula (6.20) and assumption (f) imply that $\|h\| \to M, \|h\| \notin M$. Because of assumptions (d) and (e), the right side of the last inequality $M \times N$-converges to 0. It follows that the left side is $M \times N$-converges to 0. So $g'(y) = \Big(f'\big(g(y)\big)\Big)^{-1} \in {}^N_M Dg(y)$. We conclude that ${}^N_M Dg(y) = g'(y) + N$. $\square$

6.11 The implicit flexible function theorem

Using the inverse function theorem we prove $M \times N$-totally differentiability of implicit functions.

In the result below, a point in $\mathbb{R}^{n+m}$ is written as $(x, y) \in \mathbb{R}^{n+m}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

Theorem 6.11.1. *Let M, N be neutrices such that $M \subseteq N$ and $g\colon X \longrightarrow \mathbb{R}^m$ with $X \subseteq \mathbb{R}^{n+m}$ be an internal vector function. Let (a, b) with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n, b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ be an M-interior point of X such that $g(a, b) = 0$. Let Z be an M-neighbourhood of (a, b). Suppose that*

(a) g is $M \times N$-differentiable at (a, b),

(b) g is continuously differentiable on X,

(c) g' is $M \times N$-continuous at (a, b), and $g'(x, y) \in {}^N_M Dg(x, y)$ for all $(x, y) \in Z$,

(d) Let $\mathsf{A}(x, y) = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & a_{1n} & \cdots & a_{1n+m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & \cdots & a_{mn+m} \end{pmatrix}$*, where*

$a_{ij} = \dfrac{\partial g_i}{\partial x_j}(x, y), i = 1, \ldots, m; j = 1, \ldots, n+m.$

Assume that $\mathsf{A}(x, y)$ is invertible for all $(x, y) \in Z$ and $\left\|\big(\mathsf{A}(a, b)\big)^{-1}\right\|^{-1} > N$ and $\left\|\big(\mathsf{A}(x, y)\big)^{-1}\right\|$ is not an absorber of N for $(x, y) \in Z$, where Z is an M-neighbourhood of (a, b),

(e) $\|g(x+h) - g(x)\| \le r\|h\|$ for all $x, x+h \in Z$ and $r \in \mathbb{R}$ such that r^{-1} is not an absorber of M.

Then there exists an open M-neighbourhood U of (a, b) and an open M-neighbourhood $W \subseteq \mathbb{R}^n$ of b such that the following property holds.

For every $x \in W$ corresponds a unique y such that

$$(x, y) \in U \text{ and } g(x, y) = 0. \tag{6.21}$$

If this y is defined to be $h(x)$ then h is N-differentiable at a and $h(a) = b$, and for all $x \in W$

$$g(x, h(x)) = 0. \tag{6.22}$$

Proof. We first change variable by putting $y_j = x_{n+j}$ for all $1 \leq j \leq m$. Note that from assumption (d) we have the Jacobi determinant

$$|J| \equiv \begin{vmatrix} \frac{\partial g_1}{\partial y_1}(a,b) & \frac{\partial g_1}{\partial y_2}(a,b) & \cdots & \frac{\partial g_1}{\partial y_m}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial y_1}(a,b) & \frac{\partial g_m}{\partial y_2}(a,b) & \cdots & \frac{\partial g_m}{\partial y_m}(a,b) \end{vmatrix} \neq 0.$$

The existence of h is implied by the implicit theorem [31], so it is sufficient to prove that h is $M \times N$-differentiable.

Define F by

$$F(x,y) = \big(x, g(x,y)\big). \tag{6.23}$$

Note that $F'(x,y) = \mathsf{A}(x,y)$. So F satisfies all conditions of Theorem 6.10.1. Then F has the inverse function, say G. By Theorem 6.10.1, the flexible function G is $M \times N$-differentiable. On the other hand $\big(x, h(x)\big) = G(x,0)$. Hence h is $M \times N$-differentiable at a. $\square$

7

Linear programming with flexible objectives and constraints

7.1 Introduction

In optimization problems, input and/or output data are normally not precise. As mentioned before we can model these imprecise amounts by neutrices. This chapter is devoted to studying linear programming problems with uncertainties by using neutrices to model imprecise quantities. In this model, coefficients are not real numbers but external numbers. In fact, we investigate problems of the form

$$F(x) = \sum_{j=1}^{n} \alpha_j x_j = \langle \alpha, x \rangle \to \min(\max) \tag{7.1a}$$

subject to the constraints

$$D = \begin{cases} \sum\limits_{i=1}^{n} \beta_{ij} x_j \geq \gamma_i, \ i \in M_1, \\ \sum\limits_{i=1}^{n} \beta_{ij} x_j \leq \gamma_i, \ i \in M_2 \\ \sum\limits_{i=1}^{n} \beta_{ij} x_j \subseteq \gamma_i, \ i \in M_3, \end{cases} \tag{7.1b}$$

where $n \in \mathbb{N}$ is standard, M_1, M_2, M_3 are disjointed subsets of $\mathbb{N}^\sigma$ and $\alpha_j, \beta_{ij}, \gamma_i$ are external numbers for all $j \in \{1, \dots, n\}, i \in M$ with $M = M_1 \cup M_2 \cup M_3$.

The functions in this model are not linear but nearly linear in sense that $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$. This is caused by the fact that the addition operation on external numbers is not distributive but sub-distributive. We call the problem (7.1a)-(7.1b) a *nearly linear programming problem with flexible objective function and constraints* or simply a *nearly linear programming problem*.

In this chapter we will consider two cases. In the first case we deal with the problem in which the objective function is flexible and the values of variables are precise. A sufficient condition for the existence of optimal solutions are given. In the second case we study the problem in which the coefficients of the objective function, the constraints and the values of variables are external numbers. In this case, the domain is not precise. Conditions are constructed to guarantee that an optimal solution of a nearly linear programming problem may be determined through solving an associated ordinary linear programming problem.

We start by introducing some notions regarding solutions of the problem, some remarks about expressions of the domain, and about the relationship between a maximization problem and a minimization problem.

Definition 7.1.1. A point $x_0 \in \mathbb{E}$ is called a *feasible solution* of the problem (7.1a)-(7.1b) if x_0 satisfies all the constraints of D.

Definition 7.1.2. A feasible solution $x_0 \in D$ is called an *optimal solution* of problem (7.1a)-(7.1b) if for all $x \in D$, $F(x) \geq F(x_0)$ $\Big(F(x) \leq F(x_0)$, respectively$\Big)$ and written x_{opt}. In particular, if x_0 is an optimal solution of a minimization problem then we call it a *minimizer* , if it is an optimal solution of a maximization problem we call it a *maximizer*.

Note that $F(x) \leq F(x_0)$ is not equivalent to $F(x_0) \geq F(x)$ (see the definition of order relations in Subsection 2.2.2). Also, from the definition of order relations on external numbers a point x_0 is a minimizer of a nearly linear programming problem if and only if for all $x \in D$, $F(x_0) < F(x)$ or $F(x) \subseteq F(x_0)$. Similarly, x_0 is a maximizer if and only if for all $x \in D$, $F(x) < F(x_0)$ or $F(x) \subseteq F(x_0)$. However, it is sufficient to study minimization problems. Indeed, $\alpha \leq \beta$ if and only if $-\alpha \geq -\beta$ and hence a point $x_0 \in D$ is a maximizer of a nearly linear programming problem with a objective function F if and only if x_0 is a minimizer of a nearly linear programming problem with the objective function $G = -F$. Moreover, the inequality $\sum\limits_{i=1}^{n} \beta_{ij} x_j \leq \gamma_i$ is equivalent to $\sum\limits_{i=1}^{n} -\beta_{ij} x_j \geq -\gamma_i$ and the constraint $\sum\limits_{i=1}^{n} \beta_{ij} x_j \subseteq \gamma_i$ can be transformed into two inequalities $\sum\limits_{i=1}^{n} \beta_{ij} x_j \geq \gamma_i$ and $\sum\limits_{i=1}^{n} \beta_{ij} x_j \leq \gamma_i$. Consequently, every nearly linear programming problem can be transformed to the form

$$F(x) = \sum_{i=1}^{n} \alpha_i x_i = \langle \alpha, x \rangle \to \min \tag{7.2a}$$

subject to the constraints

$$D = \sum_{i=1}^{n} \beta_{ij} x_j \geq \gamma_i,\ i \in \{1, \dots, m\}. \tag{7.2b}$$

In the problem (7.2a)-(7.2b), by choosing representatives of the coefficients in the objective function and constraints we may form an ordinary linear programming as follows:

$$F(x) = \sum_{i=1}^{n} c_i x_i = \langle c, x \rangle \to \min \tag{7.3a}$$

subject to the constraints

$$D_R = \sum_{i=1}^{n} a_{ij} x_j \geq b_i,\ i \in \{1, \dots, m\}, \tag{7.3b}$$

where $a_{ij}, b_i, c_j \in \mathbb{R}$ are representatives of $\beta_{ij}, \gamma_i, \alpha_j$, respectively.

The problem (7.3a)-(7.3b) is said to be an *associated linear programming problem* of the nearly linear programming problem (7.2a)-(7.2b).

It is worth to note that, in some cases, finding optimal solutions of an associated linear programming problem of the nearly linear programming problem (7.2a)-(7.2b) does not give full information on optimal solutions of the nearly linear programming. As shown in the next example, there are optimal solutions of a linear programming problem of the form (7.3a)-(7.3b) which are not optimal solutions of the nearly linear programming (7.3a)-(7.3b).

Example 7.1.3. Let $\epsilon > 0$ be infinitesimal. Consider the nearly linear programming problem

$$F(x) = (\epsilon + \oslash)x \to \min \tag{7.4a}$$

subject to the constraint

$$0 \leq x \leq 1. \tag{7.4b}$$

Clearly $x_0 = 1$ is an optimal solution of the problem and $F(1) = \oslash$. Also $x = 0$ is not an optimal solution since $F(0) = 0$. We now consider the associated linear programming problem

$$f(x) = \epsilon x \to \min \tag{7.5a}$$

subject to

$$0 \leq x \leq 1. \tag{7.5b}$$

Obviously $x_0 = 0$ is a minimizer of the problem and $f(0) = 0$. Also $x = 1$ is not a minimizer of the problem

(7.5a)-(7.5b).

7.2 Nearly linear programming with a precise domain

In this section we will study a special form of the nearly linear programming problem (7.2a)-(7.2b) in which the domain is a subset of $\mathbb{R}^n$. This means that the variables are real numbers. To be more precise, we will investigate the nearly linear programming of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i = \langle \alpha, x \rangle \to \min \tag{7.6a}$$

subject to constraints

$$D = \sum_{j=1}^{n} a_{ij} x_j \geq b_i \text{ for all } i \in \{1, \ldots, m\}, \tag{7.6b}$$

where $\alpha_i = c_i + B_i \in \mathbb{E}$ and $a_{ij}, b_j \in \mathbb{R}$ for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. We write

$$c = (c_1, c_2, ..., c_n),\ B = (B_1, ..., B_n). \tag{7.7}$$

Recall that in linear programming, if an objective function is bounded from below on the domain D and D contains no line, the linear programming problem has an optimal solution at some vertices of D. We state below a similar result.

Theorem 7.2.1. *Consider the nearly linear programming problem* (7.6a)-(7.6b)*. Assume that F is bounded from below on D and the domain D contains no line. Then the nearly linear programming problem* (7.6a)-(7.6b) *has optimal solutions. Also, there exists an optimal solution which is an extreme point of D.*

To prove this theorem we recall some definitions and results in the linear programming theory. For more details, we refer to [29, 2] .

Definition 7.2.2. A *face* of a convex set C is a convex subset C' of C such that every line segment in C with a relative interior point in C' has both endpoints in C'. That is, for all $x, y \in C$, if there exists $z = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$ and $z \in C'$ then $x, y \in C'$. The zero-dimensional faces of C are called the *extreme points* or the *vertices* of C. So a point x is an extreme point of C if from expression $x = (1-\lambda)y + \lambda z$, where $y, z \in C$ and $0 < \lambda < 1$ we obtain that $x = y = z$.

A *direction* of $\mathbb{R}^n$ is an equivalence class of the set of all closed half-lines of $\mathbb{R}^n$ under the equivalence relation of being a translation. The direction of the half-line $\{x + \lambda y | \lambda \geq 0\}$, where $y \neq 0$, is then by definition the set of all translates of the half-line, and this set does not depend on x. We will also call this the *direction* of y. Two vectors in $\mathbb{R}^n$ have the same direction if and only if they are positive scalar multiples of each other. The zero vector has no direction.

Definition 7.2.3. Let C' be a half-line face of a convex set C. We call a direction of C' an *extreme direction* of C.

The following result is an adapted version of Theorem 1.1 of [2].

Theorem 7.2.4. *Let C be a closed convex set, containing no line. We denote by $V(C)$ the set of all extreme points of C and $U(C)$ the set of all extreme directions of C. Then*

$$C = \operatorname{co} V(C) + \operatorname{cone} U(C),$$

where $\operatorname{co} V(C)$ *is the convex hull of* $V(C)$ *and* $\operatorname{cone} U(C)$ *is the convex cone hull of* $U(C)$.

Theorem 7.2.5 (Krein-Milman)**.** *Every non-empty line-free closed convex set has at least an extreme point.*

We next recall some properties of a polyhedral convex set. A polyhedral convex set in $\mathbb{R}^n$ is a set which can be expressed as the intersection of a finite collection of closed half-spaces, i.e. as the set of solutions to a finite system of inequalities of the form

$$D = \left\{x \in \mathbb{R}^n \middle| \langle a_i, x\rangle \geq b_i,\ i = 1, \ldots, m\right\}. \tag{7.8}$$

For polyhedral convex sets of the form (7.8) we can characterize their faces as follows. Denote

$$I_m := \{i | 1 \leq i \leq m\} \tag{7.9}$$

and

$$I_0 := \{i \in I_m |\ \langle a_i, x\rangle = b_i, \forall x \in D\}. \tag{7.10}$$

It is clear that I_0 may be an empty set. For each set of indices I such that $I_0 \subseteq I \subseteq I_m$, we write

$$F_I := \{x |\ \langle a_i, x\rangle = b_i, i \in I, \langle a_i, x\rangle \geq b_i, i \in I_m \setminus I\}$$

and

$$M_i := \{x |\ \langle a_i, x\rangle = b_i, i \in I\}.$$

It is easy to see that $F_I \subseteq M_I$ and $D = F_{I_0}$.

Proposition 7.2.6. *Assume that D is a polyhedral convex set defined by* (7.8)*. Then a set F is a face of D if and only if it is of the form*

$$F := \{x |\ \langle a_i x\rangle = b_i, i \in I, \langle a_i, x\rangle \geq b_i, i \notin I\},$$

where $I_0 \subseteq I \subseteq I_m$.

Proposition 7.2.7. *Let $D \subseteq \mathbb{R}^n$ is a polyhedral defined by* (7.8) *and I is a set of indices satisfying $I_0 \subseteq I \subseteq I_m$, where I_0, I_m is denoted as* (7.9) *and* (7.10)*. Let*

$$F_I := \{x |\ \langle a_i, x\rangle = b_i, i \in I, \langle a_i, x\rangle \geq b_i, i \in I_m \setminus I\}$$

be a face of D. Then

$$\dim F_I = n - \operatorname{rank}\{a_i |\ i \in I\}.$$

Definition 7.2.8. Let $D \subset \mathbb{R}^n$ be a polyhedral defined by (7.8). A point x_0 is said to satisfy *the i-th constraint strictly* if $\langle x_0, a_i \rangle = b_i$.

Corollary 7.2.9. *Assume that $D \subseteq \mathbb{R}^n$ is a polyhedral convex set defined by* (7.8). *Then x_0 is an extreme point of D if and only if it satisfies strictly at least n independent constraints.*

Remark 7.2.10. If the number of constraints m is standard and D is a non-empty line-free, polyhedral convex set, the number q of extreme points of D is also standard.

Proof of Theorem 7.2.1. Because D is a closed, convex set containing no line, by Theorem 7.2.4, for each $x \in D$ there exist $\lambda_i \geq 0, \mu_j \geq 0$ with $\sum\limits_{i=1}^{p} \lambda_i = 1$ such that $x = \sum\limits_{i=1}^{p} \lambda_i x^{(i)} + \sum\limits_{j=1}^{k} \mu_j u^{(j)}$, where $x^{(i)}$ are extreme points and $u^{(j)}$ are extreme directions. We have

$$\begin{aligned} F(x) &= \left\langle \alpha, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle \\ &= \left\langle c, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle + \left\langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle, \end{aligned}$$

where $x^{(i)} = (x_{i1}, \ldots, x_{in})$ for all $i \in \{1, \ldots, p\}$ and $u^{(j)} = (u_{j1}, \ldots, u_{jn})$ for all $j \in \{1, \ldots, k\}$. Then

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i x^{(i)} =& \lambda_1 (x_{11}, \ldots, x_{1n}) + \cdots + \lambda_p (x_{p1}, \ldots, x_{pn}) \\ =& \Big((\lambda_1 x_{11} + \cdots + \lambda_p x_{p1}), \ldots, (\lambda_1 x_{1n} + \cdots + \lambda_p x_{pn}) \Big). \end{aligned}$$

Similarly

$$\sum_{j=1}^{k} \mu_j u^{(j)} = \Big((\mu_1 u_{11} + \cdots + \mu_k u_{k1}), \ldots, (\mu_1 u_{1n} + \cdots + \mu_k u_{kn}) \Big).$$

It follows that

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} = \\ &\Big((\lambda_1 x_{11} + \cdots + \lambda_p x_{p1}) + (\mu_1 u_{11} + \cdots + \mu_k u_{k1}), \ldots, (\lambda_1 x_{1n} + \cdots + \lambda_p x_{pn}) + (\mu_1 u_{1n} + \cdots + \mu_k u_{kn}) \Big). \end{aligned}$$

So

$$\begin{aligned} \left\langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle =& B_1 \Big((\lambda_1 x_{11} + \cdots + \lambda_p x_{p1}) + (\mu_1 u_{11} + \cdots + \mu_k u_{k1}) \Big) + \cdots + \\ & B_n \Big((\lambda_1 x_{1n} + \cdots + \lambda_p x_{pn}) + \cdots + (\mu_1 u_{1n} + \cdots + \mu_k u_{kn}) \Big). \end{aligned}$$

Observe that for each $B_r \neq 0$, one has $u_{1r} = \cdots = u_{kr} = 0$. Indeed, suppose on contrary that there exists an index s such that $\mu_s u_{sr} \neq 0$, by the definition of extreme direction it holds that

$$(\lambda_1 x_{1r} + \cdots + \lambda_p x_{pr}) + (\mu_1 u_{1r} + \cdots + \mu_s u_{sr} + \cdots + \mu_p u_{pr}) \in D$$

for all $\mu_s \geq 0$. Taking $\mu_s \to \infty$, one has

$$B_r\Big((\lambda_1 u_{1r} + \cdots + \lambda_p u_{pr}) + (\lambda_1 x_{1r} + \cdots + \lambda_s u_{sr} + \cdots + \lambda_p u_{pr})\Big) \to \mathbb{R}.$$

It follows that $\left\langle B, \sum\limits_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle \longrightarrow \mathbb{R}$. That is, $F(x) \longrightarrow \mathbb{R}$, which is a contradiction to the lower boundedness of F. This implies that $\left\langle B, \sum\limits_{j=1}^{k} \mu_j u^{(j)} \right\rangle = 0$. As a consequence,

$$\left\langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle = \left\langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle.$$

Hence

$$F(x) = \left\langle c, \sum_{i=1}^{p} \lambda_i x^{(i)} + \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle + \left\langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle. \tag{7.11}$$

We will show that

$$\left\langle c, \sum_{j=1}^{k} \mu_j u^{(j)} \right\rangle \geq 0. \tag{7.12}$$

One has $\left\langle c, \sum\limits_{j=1}^{k} \mu_j u^{(j)} \right\rangle = \sum\limits_{j=1}^{k} \left\langle c, \mu_j u^{(j)} \right\rangle$. Suppose that $\left\langle c, \sum\limits_{j=1}^{k} \mu_j u^{(j)} \right\rangle < 0$. Then there is an index j_0 such that

$$\left\langle c, \mu_{j_0} u^{(j_0)} \right\rangle < 0.$$

Because $u^{(j_0)}$ is an extreme direction of D, we can take $\mu_{j_0} \to \infty$ and fix other factors. By formula (7.11) one derives that $f(x) \to -\infty$, which is a contradiction to the assumption that f is bounded from below on D. By formulas (7.11) and (7.12), one has

$$f(x) \geq \left\langle c, \sum_{i=1}^{p} \lambda_i x^{(i)} \rangle + \langle B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle = \left\langle c + B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle.$$

Moreover, by subdistributivity,

$$\left\langle c + B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle \subseteq \sum_{i=1}^{p} \lambda_i \left\langle c + B, x^{(i)} \right\rangle = \sum_{i=1}^{p} \lambda_i \Big((c_1 + B_1) x_{i1} + \cdots (c_n + B_n) x_{in}\Big).$$

Hence

$$\left\langle c + B, \sum_{i=1}^{p} \lambda_i x^{(i)} \right\rangle \geq \sum_{i=1}^{p} \lambda_i \Big((c_1 + B_1) x_{i1} + \cdots (c_n + B_n) x_{in}\Big).$$

Let q be the number of extreme points of D. Then q is a standard number by Remark 7.2.10. We choose an extreme point x^0 of D such that

$$\langle \alpha, x^0 \rangle = \min_{i \in \{1,\dots,q\}} \Big((c_1 + B_i)x_{i1} + \cdots (c_n + B_n)x_{in} \Big).$$

That is,

$$\Big((c_1 + B_i)x_{i1} + \cdots + (c_n + B_n)x_{in} \Big) \geq \ \langle \alpha, x^0 \rangle$$

for all $i = 1, \dots, q$.

One concludes that x^0 is an optimal solution of the nearly linear programming problem (7.6a)-(7.6b). $\square$

Remark 7.2.11. In case all $B_i \neq 0, i = 1, \dots, n$, the nearly linear programming has a solution (such that the optimal value differs from $\mathbb{R}$) if and only if the domain D is compact. Indeed, in this case, it is clear from the proof that $u^{(j)} = 0$ for all j. Hence D is bounded. This implies that D is compact because D is a polyhedral.

7.3 Nearly linear programming with flexible objective and constraints

In this section we study a nearly linear programming in which coefficients in both the objective function and the constraints are external numbers. To be more detailed we investigate a problem of the form

$$f(x) = \sum_{j=1}^{n} \lambda_j x_j \to \min \tag{7.13a}$$

subject to the constraints

$$D = \sum_{j=1}^{n} \alpha_{ij} x_j \geq \beta_i = b_i + B, i \in \{1, \dots, m\}, \tag{7.13b}$$

where $\lambda_j \in \mathbb{E}$ and $\alpha_{ij}, \beta_i \in \mathbb{E}$.

Consider a nearly linear programming of the form (7.13a)- (7.13b). Taking $a_{ij} \in \alpha_{ij}$, $b_i \in \beta_i$ and $c_j \in \lambda_j$ for all $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ we form the classical linear programming problem

$$f(x) = \sum_{j=1}^{n} c_j x_j \to \min \tag{7.14a}$$

$$\text{subject to the constraints } D_R = \sum_{j=1}^{n} a_{ij} x_j \geq b_i, i \in \{1, \dots, m\}. \tag{7.14b}$$

Next, we investigate the relationship between the sets of optimal solutions of these two problems. The next two examples show that this relationship is not always obvious. The first example shows that the two sets of optimal solutions are different. The second example shows that a solution of the problem (7.14a)-(7.14b) does not need to satisfy (7.13b).

Example 7.3.1. Let $\epsilon > 0$ be infinitesimal. Consider the nearly linear programming problem

$$F(x) = (\epsilon + \oslash)x \to \min \tag{7.15a}$$

subject to the constraint

$$0 \leq x \leq 1. \tag{7.15b}$$

Clearly $x_0 = 1$ is an optimal solution of the problem and $F(1) = \oslash$. Also $x = 0$ is not an optimal solution since $F(0) = 0$. We now consider the associated linear programming problem

$$f(x) = \epsilon x \to \min \tag{7.16a}$$

subject to

$$0 \leq x \leq 1. \tag{7.16b}$$

Obviously $x_0 = 0$ is a minimizer of the problem and $f(0) = 0$. Also $x = 1$ is not a minimizer of the problem (7.16a)-(7.16b).

Example 7.3.2. Consider the nearly linear programming problem

$$f(x, y) = x - y \to \min \tag{7.17a}$$

subject to the constraints

$$D = \begin{cases} (1 + \epsilon\oslash)x + (\epsilon + \epsilon\oslash)y \leq 1 + \epsilon\pounds \\ x, y \geq \epsilon\pounds, \end{cases} \tag{7.17b}$$

and the associated linear programming problem

$$f(x, y) = x - y \to \min \tag{7.18a}$$

subject to the constraints

$$D_R = \begin{cases} x + \epsilon y \leq 1 \\ x, y \geq 0. \end{cases} \tag{7.18b}$$

Geometrically, it is easy to see that the point $A\left(0, \dfrac{1}{\epsilon}\right)$ is the unique optimal solution of the problem (7.18a)-(7.18b). However, the point A does not belong to D.

Below we modify some notions in the theory of linear programming in away that we can apply them to nearly linear programming problems.

Definition 7.3.3. Consider the nearly linear programming problem (7.13a)-(7.13b). Let $I_m = \{1, \ldots, m\}$ and $I_0 = \{i \in I : \langle \alpha_i, x \rangle \subseteq \beta_i, \forall x \in D\}$.

For each set $I, I_0 \subseteq I \subseteq I_m$ the set

$$F_I = \left\{ x \in D : \sum_{j=1}^{n} \alpha_{ij} x_j \subseteq \beta_i, i \in I \right\}$$

is called a *pseudo-face* of D.

We say that the Pseudo-face F has *dimension* $n-k$, denoted by dim $F = n-k$, if the flexible system $\sum\limits_{j=1}^{n} \alpha_{ij} x_j \subseteq \beta_i, i \in I$ determining F has the rank k.

We next present conditions such that an optimal solution of a nearly linear programming problem may be determined through a solution of an associated linear programming problem. One of the condition is that the neutrix parts of constant terms of constraints are identical. This is consistent with our conditions to solve a flexible system of linear equations by the Gauss-Jordan method.

Theorem 7.3.4. *Assume that the problem*

$$f(x) = \sum_{j=1}^{n} x_i \lambda_i \to \min \tag{7.19a}$$

subject to the constraints

$$D_R = \sum_{j=1}^{n} a_{ij} x_j \geq b_i, i \in I_m = \{i, \ldots, m\} \tag{7.19b}$$

has an optimal solution x_{opt}, where x_{opt} is a vertex of D_R, i.e. x_{opt} is a solution of the system

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i \in \{i_1, \ldots, i_n\}, \tag{7.20}$$

and that

(i) The flexible system corresponding to the system (7.20)*, of the form*

$$\sum \alpha_{ij} x_j \subseteq \beta_i, i \in \{i_1, \ldots, i_n\} \equiv P_0 \tag{7.21}$$

satisfies the Cramer conditions.

(ii) $\overline{x}_{opt} \cdot \overline{A} \subseteq B$ and $\overline{\alpha} \cdot B \subseteq B$.

(iii) Every singular flexible system defining the face $F_I, I_0 \subseteq I \subseteq I_m$ is solvable in the sense of Definition 4.5.10.

Then $\alpha_{opt} = x_{opt} + B$ is an optimal solution of problem (7.13a) (7.13b).

Proof. We observe that the system (7.20) satisfies the Cramer conditions, so by Remark 4.3.16, $\alpha_{opt} = x_{opt} + B$ is a solution of the system (7.21). Also, condition (ii) yields $\alpha_{opt} = x_{opt} + B \in D$. We first assume that

$\gamma = (d_1 + D_1, \ldots, d_n + D_n)$ is an interior point of D, that is for all $i \in \{1, \ldots, m\}$ we have

$$\sum_{j=1}^{n} \alpha_{ij}\gamma_i > b_i + B.$$

We shall prove that $f(\gamma) \geq f(\alpha_{opt})$. Let $z = (z_1, \ldots, z_n) \in \gamma$. Then

$$\sum_{j=1}^{n} \alpha_{ij} z_i > b_i + B \text{ for all } i \in \{1, \ldots, m\}.$$

It follows that $\sum\limits_{j=1}^{n} a_{ij} z_i > b_i$ for all $i \in \{1, \ldots, m\}$. So z is an interior point of D_R. Hence, $f(z) \geq f(x_{opt}) \geq f(\alpha_{opt})$.

Next we we assume that $\gamma \in F_I$ which is defined by the system

$$\sum_{j=1}^{n} \alpha_{ij} x_j \subseteq \beta_i, i \in I. \tag{7.22}$$

We will show that $f(\gamma) \geq f(\alpha_{opt})$. Assume that $\dim F_I = n - k$. By Theorem 4.5.22 there exists a set of indices $K = \{i_1, \ldots, i_k\} \subseteq I \cap P_0$ such that the system (7.22) is equivalent to

$$\alpha_{i1}x_1 + \cdots + \alpha_{ik}x_k \subseteq b_i - a_{ik+1}x_{k+1} - \cdots - a_{in}x_n + B,\ i \in K. \tag{7.23}$$

Also, by Theorem 4.5.27 the set of solutions of the system (7.23) is given by

$$S = \left\{ (x_1 + B, \ldots, x_k + B, x_{k+1}, \ldots, x_n) \middle| x_i \in N_i \equiv B : A_i, i \in \{1 \ldots, n\} \right\},$$

where $(x_1, \ldots, x_n)$ is a solution of the linear system

$$a_{i1}x_1 + \cdots + a_{ik}x_k + a_{ik+1}x_{k+1} + \cdots + a_{in}x_n = b_i, i \in K. \tag{7.24}$$

In addition, each point $y = (y_1, \ldots, y_n) \in \gamma$ is a solution of (7.22) since $\gamma \in F_I$. Hence $y_i \in x_i + B, i = 1, \ldots, k$ and $y_i = x_i, i = k+1, \ldots, n$ with $x = (x_1, \ldots, x_n) \in D_R$. It follows that $f(y) \subseteq f(x_1 + B, \ldots, x_k + B, x_{k+1}, \ldots, x_n)$. Hence $f(y) \geq f(x_1 + B, \ldots, x_k + B, x_{k+1}, \ldots, x_n)$. Also, $x \in D_R$ implies $f(x) \geq f(x_{opt})$. Because $\sum\limits_{i=1}^{k} \lambda_i B \subseteq \sum\limits_{i=1}^{n} \lambda_i B$, we have

$$f(y) \geq f(x_1 + B, \ldots, x_k + B, x_{k+1}, \ldots, x_n) = f(x) + \sum_{i=1}^{k} \lambda_i B \geq f(x_{opt}) + \sum_{i=1}^{n} \lambda_i B = f(x_{opt} + B).$$

This equality is true for all $y \in \gamma$, we conclude that $f(\gamma) \geq f(\alpha_{opt})$. □

Remark 7.3.5. In particular, if the coefficients of the objective function in the nearly linear programming problem (7.19a)-(7.19b) are real numbers, the problem reduces to the ordinary linear programming problem (7.14a)-(7.14b). In order to find an optimal solution of this problem we can use classical methods, for instance the complex method to find an optimal solution x_{opt} of the problem (7.14a)-(7.14b). Also, if all conditions (i)-(iii) are satisfied, we conclude that $x_{opt} + B$ is an optimal solution of the original problem.

Corollary 7.3.6. *Assume that the flexible function* $f(x) = \sum_{j=1}^{n} \lambda_j x_j$ *is bounded from below on the domain* D_R *given by* (7.14b) *and* D_R *does not contain a line. Let* x_{opt} *be an optimal solution of the problem* (7.19a)-(7.19b) *which is a vertex of* D_R*. Then* $\alpha_{opt} = x_{opt} + B$ *is an optimal solution of of the problem* (7.13a)-(7.13b).

Proof. By Theorem 7.2.1, the problem (7.19a)-(7.19b) has an optimal solution x_{opt} at vertex of D_R. Using Theorem 7.3.4 we conclude that $\alpha_{opt} = x_{opt} + B$ is an optimal solution of of the problem (7.13a)-(7.13b). □

Example 7.3.7. Let $\epsilon > 0$ be infinitesimal. Consider the nearly linear programming problem

$$f(x, y) = -x + y \to \min \tag{7.25a}$$

subject to the constraints

$$D = \begin{cases} (1 + \epsilon\oslash)x + (1 + \epsilon£)y \leq 1 + \epsilon£ \\ x, y \geq \epsilon£ \end{cases} \tag{7.25b}$$

and the associated linear programming problem

$$f(x, y) = -x + y \to \min \tag{7.26a}$$

subject to the constraints

$$D = \begin{cases} x + y \leq 1 \\ x, y \geq 0. \end{cases} \tag{7.26b}$$

By a geometrical method, we find that the point $A(1, 0)$ is an optimal solution of the problem (7.26). Also A is a vertex of D_R defined by the system

$$\begin{cases} x \;+\; y \;=\; 1 \\ \qquad\;\; y \;=\; 0. \end{cases}$$

We verify that the problem satisfies all conditions in Theorem 7.3.4.

The flexible system corresponding to this system

$$\begin{cases} (1+\epsilon\oslash)x \;+\; (1+\epsilon£)y \;\subseteq 1 \;+\; \epsilon£ \\ \qquad\qquad\qquad\qquad\qquad\;\; y \;\subseteq\; \epsilon£ \end{cases}$$

satisfies the Cramer conditions

(i) $R(\mathcal{A}) \subseteq P(\mathcal{B})$

(ii) $\Delta = 1 + \epsilon\oslash$ is not an absorber of $B = \epsilon£$ and $\underline{B} = \overline{B}$.

This flexible system define an point $A' = (1 + \epsilon£, \epsilon£)$.

Also $\overline{x}_{opt} = 1$, $\mathcal{A} = \epsilon£$, $B = \epsilon£$. So the condition $\overline{x}_{opt} \cdot \overline{A} \subseteq B$ is satisfied.

Moreover, subsystems defining pseudo-faces of D are solvable.

Using Theorem 7.3.4 we conclude that the point $A' = (1 + \epsilon£, \epsilon£)$ is an optimal solution of the near linear programming (7.25) and the minimal value is

$$f(A') = 1 + \epsilon£.$$

Remark 7.3.8. It is natural to study nearly linear programming problems with different imprecisions of constants terms of constraints. Indeed, on some quantities we may have very precise information while other quantities may only be roughly known. To do this it seems to be necessary to develop more the theory of exact solutions of flexible systems of linear equations of Section 4.6.

Below we illustrate this by an example with different imprecisions of the constant terms to see how it works.

Example 7.3.9. Let $\epsilon > 0$ be infinitesimal. Consider the nearly linear programming problem

$$f(x, y) = -x + y \to \min \tag{7.27a}$$

subject to the constraints

$$D = \begin{cases} (1 + \epsilon\oslash)x + (1 + \epsilon£)y \leq 1 + \epsilon£ \\ x \geq \epsilon£ \\ y \geq \oslash \end{cases} \tag{7.27b}$$

and the associated linear programming problem

$$f(x, y) = -x + y \to \min \tag{7.28a}$$

subject to the constraints

$$D = \begin{cases} x + y \leq 1 \\ x, y \geq 0. \end{cases} \tag{7.28b}$$

By a geometrical method, we find that the point $A(1, 0)$ is an optimal solution of the problem (7.28). Also A is a vertex of D_R defined by the system

$$\begin{cases} x & + & y & = & 1 \\ & & y & = & 0. \end{cases}$$

We verify that the problem satisfies all conditions in Theorem 7.3.4.

The flexible system corresponding to this system

$$\begin{cases} (1 + \epsilon\oslash)x & + & (1 + \epsilon£)y & \subseteq 1 & + & \epsilon£ \\ & & y & \subseteq & & \oslash \end{cases}$$

has a solution

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon £ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \oslash \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{7.29}$$

By a geometrical method we see that the vector (x_0, y_0) given by (7.29) is a minimizer of the problem (7.27). Substituting this vector in the objective function we find the optimal value is given by

$$f_{opt} = -1 + \oslash.$$

8

Non-linear optimizations with flexible objectives

In this chapter we investigate non-linear optimization with flexible objective functions. We only consider the case in which variables are precise, this means values of variables are real numbers. We will introduce not only the notion of optimal solution, but also of N-optimal solution, where N is a neutrix. The latter is a kind of “flexible” optimal solution, in the sense that it is approximate optimal. In some cases we can not find exact optimal solutions, yet we can find such approximate solutions. We will consider both global and local optimality. Necessary and sufficient conditions for the existence of both optimal and nearly optimal solutions are presented.

Firstly, we will extend the well-known result which says that the derivative of a differentiable function at an extreme point vanishes. To this end we introduce the notion of N-derivative. We also state a similar result for an extreme point of a function of several variables by using N-partial derivatives.

Secondly, optimality conditions will be expressed through representatives of an objective function. After studying the general case, we will continue with a special form of objective functions. In fact, we consider the optimization problems in which the objective functions have the expansion $F(x) = f(x) + g_1(x)N_1 + \cdots + g_n(x)N_n$, where $f, g_1, \ldots, g_n$ are real functions and $N_1, \ldots, N_n$ are given neutrices. An optimal solution of an optimization problem with objective function F is characterized via an optimal solution for f, where the neutrix part is determined by the maximal values of $|g_i|$ and N_i with $1 \leq i \leq n$.

Thirdly, we will apply a parameter method to study this kind of optimization problem. In fact, we will treat external numbers as a collection of parameters. For each value of parameter we obtain a conventional optimization problem. Conditions which enable us to find optimal or approximate optimal solutions of the original problem through the sets of optimal solutions of problems corresponding to values of parameters are given.

Fourthly, we will use techniques of the theory of set-valued mapping to investigate these problems. Because the values of an objective function are external sets, some results in set-valued mapping theory do not fit completely for these functions. So we will modify notions in the theory of set-valued mapping, for instance, the notion of derivative, so that we can apply them to our problems.

Finally we present a necessary condition for optimality which is similar to the Lagrange multiplier. In fact we will show that there exist multipliers such that N-partial derivatives of the Lagrange function of an optimization problem are not zero but included in a suitable neutrix.

Convention 8.0.1. Through the whole chapter, unless otherwise stated, we always assume that $n \in \mathbb{N}$ be standard and N is a neutrix.

8.1 Some notions and elementary properties

We study optimization problems with flexible objective functions which have the form

$$\min_{x \in X} F(x) \text{ or } \max_{x \in X} F(x) \tag{8.1}$$

where $F(x)$ is a given flexible function defined on X with $X \subseteq \mathbb{R}^n, X \neq \emptyset$.

An optimal solution of an optimization problem with a flexible objective function is defined as follows.

Definition 8.1.1. Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$, $x_0 \in X$ and F be a flexible function defined on X. The point x_0 is called

(i) a *minimal solution* of the minimization problem $\min\limits_{x \in X} F(x)$ if $F(x) \geq F(x_0)$ for all $x \in X$. Then we also call x_0 a *minimizer* and $F(x_0)$ the *minimal value* or the *minimum*.

(ii) a *maximal solution* of the maximization problem $\max_{x \in X} F(x)$ if $F(x) \leq F(x_0)$ for all $x \in X$. Then we also call x_0 a *maximizer* of the problem and $F(x_0)$ the *maximal value* or the *maximum*.

A minimal or maximal solution is called an *optimal solution*.

Remark 8.1.2. By the definition of order relationship on the set of external numbers, a point $x_0 \in X$ is:

(i) a minimizer of F on X if and only if for each $x \in X$ and for all $y \in F(x)$ there exists $z \in F(x_0)$ such that $y \geq z$. Also for all $x \in X$ it holds that $F(x) > F(x_0)$ or $F(x) \subseteq F(x_0)$.

(ii) a maximizer of F on X if and only if for each $x \in X$ and for all $y \in F(x)$ there exists $z \in F(x_0)$ such that $y \leq z$. Also for all $x \in X$ it holds that $F(x) < F(x_0)$ or $F(x) \subseteq F(x_0)$.

Let $F\colon X \to \mathcal{P}(\mathbb{R})$ be an internal set-valued mapping. Recall that a point $x_0 \in X$ is said to be a minimizer of F on X if there exists $y_0 \in F(x_0)$ such that y_0 is a minimizer of $F(X)$, that is for all $x \in X$, $y \in F(x)$ we have $y \geq y_0$ (see [7]). In our approach, values of F are external sets, in general, there does not exist an element y_0 satisfying this definition. However, the definition of minimizer in our model is similar to the classical one in some sense, as shown in Remark 8.1.2(i).

Example 8.1.3. Let $F\colon [0, +\infty) \equiv X \longrightarrow \mathbb{E}$ be a flexible function defined by $F(x) = x^3 + (x^2+1)\oslash + \dfrac{\epsilon £}{e^x}$, $x \in [0, +\infty)$. Consider the optimization problem $\min\limits_{x \in X} F(x)$.

For every $x \in [0, +\infty)$ one has $F(x) = x^3 + (x^2+1)\oslash + \dfrac{\epsilon £}{e^x} \geq \oslash = F(0)$. Then $x_0 = 0$ is a minimizer of F on $[0, +\infty)$. Also every point $x_0 \in \oslash, x_0 > 0$ is a minimizer of F on $[0, +\infty)$.

However we can not find a real number $y_0 \in \mathrm{Im}F$ satisfying the condition $y \geq y_0$ for all $y \in \mathrm{Im}F$.

Assume that x_0 is a minimizer of F on X and $F(x_0)$ is the minimal value. Two cases may occur: $F(x_0) < F(x)$ or $F(x) \subseteq F(x_0)$. In the second case we have $N_F(x) \subseteq N_F(x_0)$. So, in our context, if we have the same value with different uncertainties, we will choose a value with a larger uncertainty. A similar argument holds for maximal solutions. We also note that if the inequality $F(x_0) \leq F(x)$ holds for all $x \in X$, we can not conclude that x_0 is a minimizer. For example, let $F\colon [0,1] \longrightarrow \mathbb{E}$ be given by

$$F(x) = \begin{cases} x + \oslash & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $F(0) = 0 \leq F(x)$ for all $x \in [0,1]$. However, $x = 0$ is not a minimizer of F on $[0,1]$ since it does not satisfy $F(x) \geq F(0)$ for all $x \in [0,1]$. Indeed, one has $F(\epsilon) = \oslash \ngeq 0 = F(0)$, where $\epsilon > 0$ is an infinitesimal. Similarly, if $F(x_0) \geq F(x)$ for all $x \in X$, it does not mean that x_0 is a maximizer of F on X.

Often an optimization problem with a flexible objective function does not have an optimal solution. We may, however, find "nearly" optimal solutions. Recall that a point x_0 is a minimizer of an internal function f on X with $X \subseteq \mathbb{R}^n$ if $f(x) - f(x_0) \geq 0, \forall x \in X$, and an ϵ-minimizer of f on X if for all $x \in X$ one has $f(x_0) \leq f(x) + \epsilon$ or $f(x_0) - f(x) \leq \epsilon$. Substituting 0 or ϵ above by a neutrix N, we can generalize approximate optimal solutions for optimization with flexible objectives. We call it a *N-optimal solution*.

Definition 8.1.4. Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function. Let N be a neutrix and $x_0 \in X$. The point x_0 is called

(i) an *N-minimizer* or an *N-minimal solution* of the minimization problem $\min\limits_{x \in X} F(x)$ if for all $x \in X$ one has $F(x) - F(x_0) \geq N$. Then $F(x_0)$ is called the *N-minimal value* or the *N-minimum*.

(ii) The point x_0 is called an *N-maximizer* or an *N-maximal solution* of the maximization problem $\max\limits_{x \in X} F(x)$ if $F(x) - F(x_0) \leq N$ for all $x \in X$. Then $F(x_0)$ is called the *N-maximal value* or the *N-maximum*.

The N-minimal or N-maximal value is called the *N-optimal value* and an N-minimal or N-maximal solution is called a *N-optimal solution* or a *N-optimal point*. It is also called a *N-extreme point*.

Example 8.1.5. Let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by $F(x) = x^2 + \oslash \cdot x,\ x \in \mathbb{R}$. Consider the optimization problem $\min\limits_{x \in \mathbb{R}} F(x)$.

This problem has no solution. Indeed, suppose on contrary that it has an optimal solution $x_0 \in \mathbb{R}$. Clearly $x = 0$ is not a minimizer because of Remark 8.1.2 and the fact that for $\epsilon \in \oslash, \epsilon > 0$ one has $F(0) \subset F(\epsilon)$. So $x_0 \neq 0$. If $x_0 \notin \oslash$ it holds that $x_0^2 + \oslash \cdot x_0 > F(0)$, which is a contradiction. Consequently, $x_0 \in \oslash, x_0 \neq 0$ and hence $F(x_0) = x_0 \cdot \oslash$. Let $\epsilon = \sqrt{|x_0|} \in \oslash$. Then $x_0 \oslash \subset \epsilon \oslash$. It follows that $F(x_0) = x_0^2 + \oslash \cdot x_0 = x_0 \oslash \subset \epsilon \oslash = \epsilon^2 + \epsilon \cdot \oslash = F(\epsilon)$, which is a contradiction since x_0 is a minimal solution. So the problem has no solution.

However, this problem has $\oslash$-optimal solutions. We will show that every $x_0 \in \oslash$ is an $\oslash$-minimizer of F on $\mathbb{R}$. For each $x \in \oslash, F(x) = x^2 + \oslash \cdot x \subseteq \oslash$ and for $x \notin \oslash, F(x) = x^2 + x \cdot \oslash > \oslash$. Since $x_0 \in \oslash$, it follows that $F(x) - F(x_0) > \oslash$ for all $x \notin \oslash$ and $F(x) - F(x_0) \subseteq \oslash$ for all $x \in \oslash$. Hence $F(x) - F(x_0) \geq \oslash$ for all $x \in \mathbb{R}$. This means that $x_0 \in \oslash$ is an $\oslash$-minimizer of the problem.

The following result shows that an $N_F(x_0)$-optimal solution is an optimal solution and vice versa.

Proposition 8.1.6. *Consider the optimization with flexible objective of the form* (8.1)*. A point $x_0 \in X$ is an $N_F(x_0)$-minimizer (maximizer) of F on X if and only if x_0 is a minimizer (maximizer) of F on X.*

Proof. We write $F(x) = f(x) + N_F(x)$ for all $x \in X$. We prove the case of minimization problem, the another case is done similarly. Since $x_0 \in X$ is an $N_F(x_0)$-minimal solution, one has $F(x) - F(x_0) \geq N_F(x_0)$ for all $x \in X$. It follows that $F(x) + N_F(x_0) \geq f(x_0) + N_F(x_0) = F(x_0)$ for all $x \in X$. On the other hand, $F(x) \subseteq F(x) + N_F(x_0)$ for all $x \in X$. So $F(x) \geq F(x) + N_F(x_0) \geq F(x_0)$ for all $x \in X$. Hence x_0 is a minimizer of F on X.

Conversely, assume that $x_0 \in X$ is a minimizer of F on X. Then $F(x) \geq F(x_0)$ for all $x \in X$. This implies that $F(x) - f(x_0) \geq N_F(x_0)$ for all $x \in X$. We consider two cases. For the case $F(x) - f(x_0) \subseteq N_F(x_0)$ we have $F(x) - f(x_0) + N_F(x_0) = N_F(x_0)$, so $F(x) - F(x_0) = N_F(x_0)$. For the case $F(x) - f(x_0) > N_F(x_0)$ we have $F(x) - f(x_0) + N(x_0) = F(x) - F(x_0) > N_F(x_0)$. So $F(x) - F(x_0) \geq N_F(x_0)$ for all $x \in X$. We conclude that x_0 is an $N_F(x_0)$-minimizer of F on X. □

For local optimal solutions, in some cases, we would like to have information on the size of the neighbourhood of a local solution in which an objective function reaches the minimum. For this approach we can classify different orders of magnitudes of the size of the neighbourhood by using neutrices. We call it an *M-local N-optimal solution*, where M, N are two neutrices.

Definition 8.1.7. Let $F\colon X \to \mathbb{E}$ be a flexible function and x_0 be a point in X. The point $x_0 \in X$ is called

(i) an *M-local N-minimizer* or *M-local N-minimal solution* of the problem $\min\limits_{x\in X} F(x)$ if there exists $\delta > M$ such that $(x_0 - \delta, x_0 + \delta) \subseteq X$ and $F(x) - F(x_0) \geq N$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

(ii) an *M-local N-maximizer* or a *M-local N-maximal solution* of the problem $\max\limits_{x\in X} F(x)$ if there exists $\delta > M$ such that $(x_0 - \delta, x_0 + \delta) \subseteq X$ and $F(x) - F(x_0) \leq N$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

In particular, for minimization problems, if $N = N_F(x_0)$ we call x_0 a *M-local minimizer*. If $M = 0$, we call x_0 a *local N-minimizer*. In addition, if both N and M are zeros, we call x_0 a *local minimizer*.

We have similar definitions for maximization problems.

Example 8.1.8. Let F: $\mathbb{R} \to \mathbb{E}$ be a flexible function defined by $F(x) = x^3 - 3x + 1 + \oslash x$. We will show that every $x \simeq -1$ is an $\oslash$-local maximizer and every $x \simeq 1$ is an $\oslash$-local minimizer of F. We first prove that $x = -1$ is an $\oslash$-local maximizer of F and then we do it for $x \simeq -1, x \neq 1$. The case $x \simeq 1$ is done similarly.

Put $f(x) = x^3 - 3x + 1$ for all $x \in \mathbb{R}$. A short calculation shows that $x_1 = -1$ is a local maximizer of f and $x_2 = 1$ is a local minimizer of f. In fact, for $x \in (-\infty, 1 + \oslash)$ we have $f(x) < f(-1)$ and $x \in -1 + \oslash$ we have $F(x) - F(-1) \subseteq \oslash$. This means that

$$F(x) \leq F(-1),\ \forall x \simeq -1. \tag{8.2}$$

On the other hand, for each $x \in (-2, 1 + \oslash) \setminus \{-1 + \oslash\}$, it holds that

$$F(x) \leq F(-1). \tag{8.3}$$

Indeed, we consider two cases. For the case in which x is standard, because $f(x)$ is standard and -1 is a local maximal point of f, we have $f(x) - f(-1) = z < \oslash$, where z is standard. It follows that $F(x) - F(-1) = f(x) - f(-1) + \oslash = z + \oslash < \oslash$. For the case in which x is not standard, let y be the standard part of x. Then for all $x \in (2, 1 + \oslash)$ we have $F(x) \subseteq F(y) + \oslash$. The first case shows that $F(y) - F(-1) < \oslash$ and hence $F(x) - F(-1) < \oslash$. Thus $F(x) - F(-1) \leq \oslash$ for all $x \in (-2, 1 + \oslash) \setminus \{-1 + \oslash\}$. This implies that

$$F(x) \leq F(-1) \quad \text{for all} x \in (-2, 1 + \oslash) \setminus \{-1 + \oslash\}. \tag{8.4}$$

From (8.2) and (8.4) we conclude that $x_0 = -1$ is an $\oslash$-local maximizer of F.

Secondly, we will show that points $x_0 \in -1 + \oslash, x_0 \neq -1$ are also $\oslash$-local maximizers of F on $\mathbb{R}$. We note that $F(x_0) - F(-1) = \oslash$ or $F(x_0) = F(-1) + \oslash$, so $F(x) - F(x_0) = F(x) - F(-1) + \oslash \leq \oslash$ for all $x \in (-2, 1 + \oslash)$. Hence $F(x) \leq F(-1)$ for all $x \in (-2, 1 + \oslash)$. So x_0 is an $\oslash$-local maximizer of F.

Similarly, we conclude that $x_0 \in 1 + \oslash$ is an $\oslash$-local minimizer of F.

It is easy to see that $x_1 = -1$ is not an £-local maximizer of F and $x_2 = 1$ is not an £-local minimizer of F.

An M-local optimal solution, as shown by the next proposition, is also an M'-local optimal solution with $M' < M$. As a consequence, in practice we tend to determine M-local optimal solutions with the largest possible M.

Proposition 8.1.9. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and N, M, M' be neutrices such that $M' < M$. Consider the optimization problem* (8.1). *Assume that $x_0 \in X$ is an M-local N-optimal solution of* (8.1). *Then x_0 is also an M'-local N-optimal solution of this problem.*

Proof. Without loss of generality we assume that x_0 is an M-local N-minimal solution of problem (8.1). Then there exists $\delta > M$ such that $(x_0 - \delta, x_0 + \delta) \subseteq X$ and $F(x) - F(x_0) \geq N$ for all $x \in (x_0 - \delta, x_0 + \delta)$. On the other hand, $M' \subseteq M$ and $\delta > M$ yield $\delta > M'$. By Definition 8.1.7 one concludes that x_0 is an M'-local N-minimal solution of the problem (8.1). □

Proposition 8.1.10. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and N, N' be neutrices such that $N \leq N'$. Consider the optimization problem* (8.1). *Assume that $x_0 \in X$ is an N-optimal solution of* (8.1). *Then x_0 is an N'-optimal solution of* (8.1).

Proof. We will prove the proposition for N-minimizers, the case of N-maximizers is done similarly. Because x_0 is an N-minimizer of F on X, we have $F(x) - F(x_0) \geq N$ for all $x \in X$. This implies that $F(x) - F(x_0) \subseteq N$ or $F(x) - F(x_0) > N$. We will show that $F(x) - F(x_0) \subseteq N'$ or $F(x) - F(x_0) > N'$ for all $x \in X$.

If $F(x) - F(x_0) \subseteq N$ then $F(x) - F(x_0) \subseteq N'$, since $N \subseteq N'$. If $F(x) - F(x_0) > N$ then $F(x) - F(x_0) > N'$ or $F(x) - F(x_0) \subseteq N'$. Indeed, otherwise, we have two cases: (i) $N' \subset F(x) - F(x_0)$ or (ii) $F(x) - F(x_0) < N'$. If (i) happens then $N \subseteq N' \subset F(x) - F(x_0)$, which is a contradiction. If (ii) happens, we have $F(x) - F(x_0) < N$, which is a contradiction. □

Remark 8.1.11. The conclusion is also true for M-local N-optimal solutions.

Combining these two results we obtain the following.

Proposition 8.1.12. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and N, N', M, M' be neutrices such that $M' \leq M$ and $N \leq N'$. Consider the optimization problem* (8.1). *Assume that a point x_0 is an M-local N-optimal solution of* (8.1). *Then x_0 is an M'-local N'-optimal solution of* (8.1).

Proof. Because of Proposition 8.1.9 the point x_0 is an M'-local N-optimal solution of the optimization problem (8.1). By Proposition 8.1.10 we conclude that x_0 is an M'-local N'-optimal solution of the optimization problem (8.1). □

We end by proving that a maximum problem can be transformed into a minimum problem and vice versa.

Proposition 8.1.13. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$. A point $x_0 \in X$ is a minimizer of F on X if and only if x_0 is a maximizer of $-F$ on X.*

Proof. By Lemma 2.2.35, it holds that $F(x) \geq F(x_0)$ if and only if $-F(x) \leq -F(x_0)$ for all $x \in X$. Then x_0 is a maximizer of $-F$ on X if and only if x_0 is a minimizer of F on X. □

Note that the conclusion above also holds for M-local N-optimal solutions.

8.2 Nearly optimal points and N-derivatives

The necessary condition of an N-optimal point says that the neutrix part of the N-optimal value must be included in N.

Proposition 8.2.1. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and N be a neutrix. Consider the optimization problem* (8.1). *Assume that $x_0 \in X$ is an N-optimal solution of the optimization problem* (8.1). *Then $N_F(x_0) \subseteq N$.*

Proof. Suppose on contrary that $N \subset N_F(x_0)$. Because x_0 is an N-optimal point, without loss of generality, we assume that x_0 is an N-minimizer of F on X. By the definition, it holds that $F(x) - F(x_0) \geq N$ for all $x \in X$ and $x_0 \in X$. For $x = x_0 \in X$ we obtain that $F(x) - F(x_0) = N_F(x_0) \supset N$. This implies that $F(x) - F(x_0) \not\geq N$, which is a contradiction. □

Note that the equality $F(x) - F(x_0) \geq N$ for all $x \in X$ is not equivalent to $F(x) \geq F(x_0) + N$ for all $x \in X$. For example, let $N = 0$ and $F(x) = x^2 + \oslash$. Then $F(x) \geq F(0) + N = F(0) + 0$ for all $x \in \mathbb{R}$. However, $F(\epsilon) - F(0) = \oslash \not\geq 0 = N$ where $\epsilon 0$ is infinitesimal. So, from the expression $F(x) \geq F(x_0) + N$ for all $x \in X$ we can not conclude that x_0 is an N-optimal solution of F on X. Yet, as a consequence of Proposition 8.2.1, a point $x_0 \in X$ is

(i) an N-minimizer of F on X if and only if

$$\begin{cases} F(x) \geq F(x_0) + N, \ \forall x \in X, \\ N_F(x_0) \subseteq N. \end{cases}$$

(ii) an N-maximizer of F on X if and only if

$$\begin{cases} F(x) \leq F(x_0) + N, \ \forall x \in X, \\ N_F(x_0) \subseteq N. \end{cases}$$

In contrast to classical continuity of an internal function, even we have an $M \times N$-inner continuous function it does not guarantee that this function obtains the maximum and minimum on a closed interval.

For example, consider a function $F\colon [-1, 1] \longrightarrow \mathbb{E}$ given by $F(x) = \begin{cases} x & \text{if } -1 \leq x \lnapprox 0, \\ -1 - x & \text{if } 0 \lnapprox x \leq 1 \end{cases}$. This function is $\oslash \times \oslash$-inner continuous, but it does not have the maximal value, even the $\oslash$-maximal value. Also observe that F is not $\oslash \times \oslash$-continuous at $x = 0$.

In classical mathematics it is well-known that the derivative of a differentiable function vanishes at an extreme point. For a function of several variables, the partial derivatives also vanish at an extreme point. Here, by using the notion of $M \times N$-derivative for a flexible function of one variable and $M \times N$-partial derivatives for a flexible function of several variables we obtain similar results.

Theorem 8.2.2. *Let N, M be neutrices such that $\oslash_N \subseteq M$ and $X \subseteq \mathbb{R}, X \neq \emptyset$. Let F be a flexible defined on X. Assume that x_0 is an M-local N-minimizer of F on X and x_0 is an M-interior point of X. Assume also that F is $M \times N$-differentiable at x_0. Then*

$$\frac{d_N F}{d_M x}(x_0) = N.$$

Proof. Assume that $\frac{d_N F}{d_M x}(x_0) = N\text{-}\lim_{x \to x_0 + M} \frac{F(x) - F(x_0)}{x - x_0} = a + N = \alpha$. So, for all $\epsilon > N$ there exists $\delta_0 > M$ such that for all $x \in X, M < |x - x_0| < \delta_0$ one has

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - a + N\right| < \epsilon. \tag{8.5}$$

This implies that

$$a + N - \epsilon < \frac{F(x) - F(x_0)}{x - x_0} + N \tag{8.6}$$

for all $M < x_0 - x < \delta_0$ and

$$\frac{F(x) - F(x_0)}{x - x_0} + N < \epsilon + a + N \tag{8.7}$$

for all $M < x - x_0 < \delta_0$.

On the other hand, the point x_0 is an M-local N-minimizer of F on X, so

$$F(x) - F(x_0) \geq N \tag{8.8}$$

for all $x \in X, |x - x_0| \leq \delta_1$. Put $\delta = \min\{\delta_0, \delta_1\}$. Note that $\oslash_N \subseteq M$, so for $|x| > M$ it holds that x is not an absorber of N. As a result, for all $x \in X, M < x_0 - x < \delta$, we have

$$\frac{F(x) - F(x_0)}{x - x_0} \leq \frac{N}{x - x_0} \leq N.$$

This implies

$$\frac{F(x) - F(x_0)}{x - x_0} + N \leq N + N = N \tag{8.9}$$

for all $x \in X, M < x_0 - x < \delta$. From (8.6) and (8.9) imply

$$a + N - \epsilon < N \tag{8.10}$$

Similarly, from (8.8) one has $\frac{F(x) - F(x_0)}{x - x_0} \geq \frac{N}{x - x_0} \geq N$ for all $x \in X, M < x - x_0 < \delta$. Using (8.7) and analogous arguments we obtain

$$a + N + \epsilon > N \tag{8.11}$$

for all $\epsilon > N$ and $x \in X, M < x_0 - x < \delta$.

Formulas (8.10) and (8.11) imply $a \in N$. Indeed, if $a < N$, we choose $\epsilon = -a/2 > N$ then $a + \epsilon + A = a/2 + A < N$, which is a contradiction to (8.11). If $a > N$, taking $\epsilon = a/2 > N$ then $a + A - \epsilon = a/2 + A > N$, which is contradictory to (8.10). Thus $\frac{d_N F}{d_M x}(x_0) = N$. □

By this result, if $M \times N$-derivative of F at a point differs from N, we conclude that this point is not an M-local N-extreme point of F.

Example 8.2.3. Let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible function given by $F(x) = x^2 + \oslash$ for all $x \in \mathbb{R}$. We have $\frac{d_\oslash F}{d_\oslash x}(x) = 2x + \oslash$. So, for all $x \notin \oslash$, $\frac{d_\oslash F}{d_\oslash x}(x) = 2x + \oslash \nsubseteq \oslash$ and hence, they are not $\oslash$-local $\oslash$-minimizers of F.

Next we present a necessary condition for the existence of an N-extreme point of a flexible function of several variables.

Theorem 8.2.4. *Let M, N be neutrices such that $\oslash_N \subseteq M$, $F\colon X \subseteq \mathbb{R}^n \to \mathbb{E}$ be a flexible function and $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \in X$ is an M-interior point of X. Assume that $x^{(0)}$ is an M-local N-extreme point of F and that F is $M \times N$-total differential at $x^{(0)}$. Then $\frac{\partial_N F}{\partial_M x_i}(x^{(0)}) = N$.*

Proof. For each $i \in \{1, \ldots, n\}$, put $G(x_i) = F(x_1^{(0)}, \ldots, x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, \ldots, x_n^{(0)})$. Since $x^{(0)}$ is an M-local N-extreme point of F, it holds that $x_i^{(0)}$ is an N-extreme point of G. By Theorem 8.2.2, ${}_M^N DG(x_i^{(0)}) = \frac{\partial_N F}{\partial_M x_i}(x^{(0)}) = N$. □

8.3 Conditions for optimality via representatives of objective functions

In this section we will use representatives of the objective function to construct optimality conditions. An optimal/ approximate optimal solution of an optimization problem with flexible objective function will be charactered through optimal solutions or the optimal value of a conventional optimization problem in which the objective function is a representative of the original objective function. The latter problem is called an *associated optimization problem* of (8.1). This is an ordinary optimization problem, we may use advantage results in the classical theory of optimization to deal with it. We will start with a general form and then with a special form of objective function. We will present two kinds of conditions. We first use the relationship between the external infimum (supremum) of the image of an objective function and the minimal (maximal) value of a representative to construct conditions. With this approach we will overcome some drawbacks of using the $M \times N$-derivative, for instance, the optimality condition based on the notion of $M \times N$-derivative does not work when $M = 0$, because the ratio $\frac{N}{x - x_0}$ tends to $\mathbb{R}$ when $x - x_0$ approaches to 0.

8.3.1 General forms

Definition 8.3.1. Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function. Consider the optimization problem

$$\min_{x \in X} F(x). \tag{8.12}$$

Let f be an internal representative of F. The minimization problem

$$\min_{x \in X} f(x) \tag{8.13}$$

is called a *representative problem* of (8.12).

The results below give conditions so we can find an optimal solution of an optimization problem with flexible objective function by solving a representative problem.

Theorem 8.3.2. *Let N be a neutrix. Consider the optimization problem* (8.12) *and the representative problem* (8.13). *Assume that x_0 is an N-optimal solution of* (8.13) *and that $N_F(x) \subseteq N$ for all $x \in X$. Then x_0 is an N-optimal solution of the optimization problem* (8.12).

Proof. We write $F(x) = f(x) + N_F(x)$. By the assumptions, we have $F(x) = f(x) + N_F(x) - f(x_0) + N_F(x_0) = f(x) - f(x_0) + N_F(x) + N_F(x_0) \geq N + N_F(x) + N_F(x_0) = N$ for all $x \in X$. We conclude that x_0 is an optimal solution of the optimization problem (8.12). □

In particular, we obtain a sufficient condition for the existence of optimal solution as follows.

Theorem 8.3.3. *Consider the optimization problem* (8.12) *and the representative problem* (8.13). *Assume that x_0 is a optimal solution of* (8.13) *and that $N_F(x) \subseteq N_F(x_0)$ for all $x \in X$. Then x_0 is an optimal solution of the optimization problem* (8.12).

Proof. By assumption we have $F(x) = f(x) + N_F(x) \geq f(x_0) + N_F(x) \geq f(x_0) + N_F(x_0)$ for all $x \in X$. We conclude that x_0 is an optimal solution of (8.12). □

Similarly, we have a sufficient condition for the existence of local optimal solutions.

Theorem 8.3.4. *Let $X \subseteq \mathbb{R}^n$, $F\colon X \longrightarrow \mathbb{E}$ be a flexible function, f be a representative of F and N_F be the neutrix part of F. Let M, N be neutrices. Assume that*

(i) $x_0 \in X$ is an M-local optimal point of f on X and that there exists an M-neighbourhood U of x_0 such that $N_F(x) \subseteq N_F(x_0)$ for all $x \in U$. Then x_0 is an M-local optimal point of F on X.

(ii) $x_0 \in X$ is an M-local N-optimal point of f on X and that there exists an M-neighbourhood U of x_0 such that $N_F(x) \subseteq N$ for all $x \in U$. Then x_0 is an M-local N-optimal point of F on X.

Theorem 8.3.5. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and f be a representative of F. Assume that $x_0 \in X$ is a minimizer of f on X. Let $X_0 = \left\{x \in X \middle| F(x) \cap F(x_0) \neq \emptyset\right\}$. If there exists $x_1 \in X_0$ such that $F(x_1) = \bigcup_{x \in X_0} F(x)$ then $F(x_1)$ is a minimizer of F on X.*

Proof. We will show that for all $x \in X$, $F(x) \geq F(x_1)$. For all $x \in X_0$ we have $F(x) \subseteq F(x_1)$, in particular, $F(x_1) = f(x_0) + N_F(x_1)$. For $x \in X \setminus X_0$ it holds that (i) $F(x) \cap F(x_1) \neq \emptyset$ or (ii) $F(x) > F(x_1)$. For the case (i), this implies $F(x) \subseteq F(x_1)$. Indeed, suppose on contrary that $F(x_1) \subset F(x)$. Then $F(x_0) \subseteq F(x_1) \subseteq F(x)$. It follows that $x \in X_0$ and hence $F(x) \subseteq F(x_1)$, which is a contradiction.

So for all $x \in X$ we have $F(x) \geq F(x_1)$. This means x_1 is a minimizer of F on X. □

With a similar arguments we obtain

Theorem 8.3.6. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$ and f be a representative of F. Assume that $x_0 \in X$ is a maximizer of f on X. Let $X_0 = \left\{x \in X \middle| F(x) \cap F(x_0) \neq \emptyset\right\}$. If there exists $x_1 \in X_0$ such that $F(x_1) = \bigcup_{x \in X_0} F(x)$ then $F(x_1)$ is a maximizer of F on X.*

Applying this result, in order to find an optimal solution of an optimization problem with a flexible objective function we will make the following steps.

We first choose an appropriate representative f of F, for instance, continuous, differentiable, convex, etc. Then we use classical methods, for instance the Lagrange multiplier method, to solve the optimization problem with objective function f. Next we calculate the set $X_0 = \{x \in X | F(x) \cap F(x_0) \neq \emptyset\}$. In the last step we verify if there exists a point $x_1 \in X_0$ such that $F(x_1) = \bigcup_{x \in X_0} F(x)$, we conclude that x_1 is an optimal solution of the given problem.

In the following result we use the notions of external infimum and external supremum to give sufficient conditions for N-optimal solutions.

Notation 8.3.7. Let $X \subseteq \mathbb{R}^n$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function defined on X. We denote $\mu = \inf\left(\mathrm{Im}F\right), \sigma = \sup\left(\mathrm{Im}F\right)$ and N_μ the neutrix part of μ, N_σ the neutrix part of σ.

Recall that $\mathrm{Im}F = \bigcup_{x \in X} F(x) \equiv F(X)$.

Theorem 8.3.8. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n, X \neq \emptyset$. Both of the following statements hold.*

(i) If $\mu \cap \overline{\mathrm{conv}}(\mathrm{Im}F) \neq \emptyset$ then F achieves N_μ-minimum on X, i.e. there is $x_0 \in X$ such that for all $x \in X$ one has $F(x) - F(x_0) \geq N_\mu$. In fact, every $x_0 \in X$ such that $F(x_0) \subseteq \mu$ is an N_μ-minimal solution.

(ii) If $\sigma \cap \underline{\mathrm{conv}}(\mathrm{Im}F) \neq \emptyset$ then F achieves N_σ-maximum on X, i.e. there exists $x_0' \in X$ such that for all $x \in X$ $F(x) - F(x_0') \leq N_\sigma$. In fact, every $x_0' \in X$ such that $F(x_0') \subseteq \sigma$ is an N_σ-maximal solution.

Proof. We will prove the first statement, the second is done similarly. We first show that there exists $\delta_0 \in \mathrm{Im}F$ such that $\delta_0 \cap \mu \neq \emptyset$. Indeed, suppose on contrary that for all $\delta \in \mathrm{Im}F, \mu < \delta$. For all $\xi \in \overline{\mathrm{conv}}(\mathrm{Im}F)$, by Definition 2.12, there exists $\delta \in \mathrm{Im}F$ such that $\xi \geq \delta > \mu$, which is a contradiction to the assumption $\mu \cap \overline{\mathrm{conv}}(\mathrm{Im}F) \neq \emptyset$.

In particular, from Proposition 2.4.12(ii) we have

$$F(x_0) = \delta_0 \subseteq \mu, \quad \text{for some} \quad x_0 \in X. \tag{8.14}$$

On the other hand, for all $x \in X$ such that $F(x) \subseteq \mu$ one has $F(x) - F(x_0) \subseteq \mu - \mu = N_\mu$, i.e. $F(x) - F(x_0) \geq N_\mu$. For $x \in X$ and $F(x) \not\subseteq \mu$, the fact $F(x) \geq \mu$ implies $F(x) > \mu$. So $F(x) - \mu > N_\mu$. Also, by formula (8.14) we have $\mu = F(x_0) + N_\mu$. It follows that $F(x) - F(x_0) \geq F(x) - F(x_0) + N_\mu = F(x) - \mu > \mathbb{N}_\mu$.

Hence for all $x \in X$ one has $F(x) - F(x_0) \geq N_\mu$. We conclude that x_0 is an N_μ-minimizer of F on X. $\square$

Example 8.3.9. Let $F\colon X \equiv [0, +\infty)$ be given by $F(x) = e^x + \epsilon £ x^2 + \oslash x$, $x \in [0, +\infty)$, here $\epsilon > 0$ is an infinitesimal. Consider the optimization problem $\min\limits_{x \in X} F(x) \equiv \min\limits_{x \in X}(e^x + \epsilon £ x^2 + \oslash x)$. A short calculation shows that $\inf\limits_{x \in \mathbb{R}} F(x) = 1 + \oslash$ and $F(0) = 1 \in 1 + \oslash$. Using Theorem 8.3.8 we conclude that $x_0 = 0$ is an $\oslash$-minimizer of F on $[0, +\infty)$.

Remark 8.3.10. In classical mathematics, the fact that $\inf(\mathrm{Im}F) \cap \mathrm{Im}F \neq \emptyset$ implies that $\inf(\mathrm{Im}F) \in \mathrm{Im}F$. It does not hold anymore in our context. Indeed, let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by $F(x) = x^2 + x\oslash$ for all $x \in \mathbb{R}$. Then we have $\inf\limits_{x \in \mathbb{R}}(F(x)) = \oslash$ and hence $\inf\big(\mathrm{Im}F\big) \cap \mathrm{Im}F = \oslash \cap \mathrm{Im}F \neq \emptyset$ because $0 \in \oslash \cap F(0)$. However, $\oslash \notin \mathrm{Im}F$.

Next we provide characterizations through representatives of F such that the conditions $\mu \cap \overline{\mathrm{conv}}\big(\mathrm{Im}F\big) \neq \emptyset$ and $\sigma \cap \underline{\mathrm{conv}}\big(\mathrm{Im}F\big) \neq \emptyset$ of Theorem 8.3.8 is satisfied.

Proposition 8.3.11. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$. If there exists a representative f of F such that f has a minimizer x_0 on X,*

$$\inf\big(\mathrm{Im}F\big) \cap \mathrm{Im}F \neq \emptyset.$$

As a consequence, $\inf(\mathrm{Im}F) \cap \overline{\mathrm{conv}}\,(\mathrm{Im}F) \neq \emptyset$.

Proof. For each $x \neq x_0$, there are three cases as follows: $F(x) > F(x_0)$, $F(x) < F(x_0)$ or $F(x) \cap F(x_0) \neq \emptyset$. Note that $F(x) \cap F(x_0) \neq \emptyset$ if and only if $F(x) \subseteq F(x_0) \vee F(x_0) \subseteq F(x)$. However, the case $F(x) < F(x_0)$ can not happen. Indeed, if $F(x) < F(x_0)$, it follows that $f(x) < f(x_0) + N_F(x_0)$ and hence $f(x) < f(x_0)$, which is a contradiction to the fact that $f(x_0)$ is the minimal value.

Denote $X_0 = \{x \in X | F(x) \cap F(x_0) \neq \emptyset\}$. Since $x_0 \in X_0$ we have $X_0 \neq \emptyset$. Also $F(x) = f(x_0) + N_F(x)$ for all $x \in X_0$. Because $F(x) > F(x_0)$ or $F(x) \cap F(x_0) \neq \emptyset$, one has

$$\inf(\mathrm{Im}F) = \inf\Big\{F(x) : x \in X_0\Big\} = \inf\left\{\bigcup_{\substack{x \in X_0 \\ F(x_0) \subseteq F(x)}} F(x)\right\} = f(x_0) + \inf_{x \in X_1} N_F(x),$$

where $X_1 = \{x \in X_0, F(x_0) \subseteq F(x)\}$. On the other hand

$$\inf_{x \in X_1} N_F(x) \equiv N_0 \quad \text{is a neutrix.} \tag{8.15}$$

Indeed, suppose on contrary that $\inf\limits_{x \in X_1} N_F(x) = \alpha = a + A$ is zeroless. Then $\alpha \cap N_F(x) = \emptyset$ for all $x \in X_1$. It follows that $2\alpha \cap N_F(x) = \emptyset$ for all $x \in X_1$ since 2α is zeroless and $N_F(x)$ is a neutrix. In particular

$\alpha < 2\alpha < N_F(x)$ for all $x \in X_1$, which is contradiction to Proposition 2.4.12(i). The fact (8.15) implies $N_F(x) \subseteq N_0$ for all $x \in X_1$. Consequently, $F(x) \subseteq f(x_0)+N_0$ for all $x \in X_1$ and hence $F(x) \subseteq f(x_0)+N_0$ for all $x \in X_0$ since $F(x) \subseteq F(x_0)$ for all $x \in X_0 \setminus X_1$. So $f(x_0) \in \inf\big(\mathrm{Im}F\big) \cap F(x_0)$. This means that $\inf(\mathrm{Im}F) \cap \mathrm{Im}F \neq \emptyset$. The conclusion $\inf(\mathrm{Im}F) \cap \overline{\mathrm{conv}}\,(\mathrm{Im}F) \neq \emptyset$ follows by $\mathrm{Im}F \subseteq \overline{\mathrm{conv}}(\mathrm{Im}F)$. □

For maximization problems, we have a similar result.

Proposition 8.3.12. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and f be a representative of F. If f achieves a maximum on X at x_0,*

$$\sup\big(\mathrm{Im}F\big) \cap \mathrm{Im}F \neq \emptyset.$$

As a consequence, $\sup(\mathrm{Im}F) \in \underline{\mathrm{conv}}(\mathrm{Im}F)$.

Proof. We use analogous arguments as in the proof of Proposition 8.3.11. □

Next theorem shows that an optimization with flexible objective function has an approximate optimal solution if a representative has a minimum or a maximum.

Theorem 8.3.13. *Let F be a flexible function defined on $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and f be a representative of F. Then following statements are true:*

(i) If f achieves a minimum on X, the function F achieves an N_μ-minimum on X.

(ii) If f achieves a maximum on X, the function F achieves an N_σ-maximum on X.

Proof. Theorem follows directly from Proposition 8.3.11, Proposition 8.3.12 and Theorem 8.3.8. □

Corollary 8.3.14. *Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ be an internal set and F be a flexible function defined on X. Assume that f is a continuous internal representative of F on X and X is a compact set. Then F achieves an N_μ-minimum and an N_σ-maximum on X.*

Proof. Because f is a continuous internal function and X is an internal compact set, it achieves its minimum and maximum on X. The conclusion follows by Theorem 8.3.13. □

We can use results of the ordinary optimal theory to verify whether a representative problem has the minimum (maximum) and find its optimal solutions. For instance, we can apply the Lagrange multiplier method to find approximate solutions. To be more specific, we consider the optimization problem with flexible objective

$$\min_{x\in\mathbb{R}^n} F(x) = \min_{x\in\mathbb{R}^n} \big(f(x) + N(x)\big)$$

subject to constraints

$$\begin{cases} g_1(x_1,\ldots,x_n) &= & 0 \\ \quad\vdots & & \vdots \\ g_m(x_1,\ldots,x_n) &= & 0, \end{cases}$$

where $f, g_i, i \in \{1, \ldots, m\}$ are internal functions whose partial derivatives are continuous.

In this case we can apply the Lagrange multiplier method to find an minimizer of

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to the constraints

$$\begin{cases} g_1(x_1, \ldots, x_n) &= 0 \\ \vdots & \vdots \\ g_m(x_1, \ldots, x_n) &= 0. \end{cases}$$

Assume that x_0 is a minimizer of f satisfying the constraints. Then, by Proposition 8.3.11, the given problem has N_μ-solution and this solution y_0 satisfies $F(y_0) \cap F(x_0) \neq \emptyset$.

The following theorem gives some characterizations of optimal solutions.

Theorem 8.3.15. *Let $F \colon X \longrightarrow \mathbb{E}$ be a flexible function defined on $X \subseteq \mathbb{R}^n$, f be an internal representative of F and N_F be the neutrix part of F. Assume that $x_0 \in X$ is a minimizer of* (8.12) *and f is continuous at x_0. The followings three statements are true:*

(i) If $N_F(x_0) \neq 0$ then there is a real positive number δ such that for all $x \in B(x_0, \delta)$ we have $F(x) - F(x_0) = N_F(x_0)$. This means that

$$\begin{cases} f(x) - f(x_0) \in N(x_0) \\ N_F(x) \subseteq N_F(x_0) \end{cases}$$

for all $x \in B(x_0, \delta)$.

(ii) For $x \in X$ such that $N_F(x_0) \subset N_F(x)$, it holds that $f(x) - f(x_0) > N_F(x)$ for all representatives $f(x)$ of F.

(iii) If $N_F(x_0) = 0$, the partial derivative of each totally differentiable representative $f(x)$ of $F(x)$ at x_0 is 0.

Proof. (i). Since x_0 is a minimizer of (8.12), we have $F(x) \geq F(x_0)$. So

$$F(x) - F(x_0) \geq N_F(x_0). \tag{8.16}$$

On the another hand $N_F(x_0) \neq 0$, so $N_F(x) + N_F(x_0) = \max\{N_F(x), N_F(x_0)\} \neq 0$. This implies that $f(x_0) + N_F(x) + N_F(x_0)$ is a neighbourhood of $f(x_0)$. By continuity of f at x_0, there is a $\delta > 0$ such that $f(x) \in f(x_0) + N_F(x_0) + N_F(x)$ for all $x \in B(x_0, \delta)$. Hence

$$F(x) - F(x_0) = N_F(x) + N_F(x_0) \text{ for all } x \in B(x_0, \delta) \tag{8.17}$$

Suppose on contrary that $N_F(x_0) \subset N_F(x) + N_F(x_0) = N_F(x)$. Then there exists $y \in N_F(x) = F(x) - F(x_0)$ such that $y < z$ for all $z \in N_F(x_0)$, which is a contradiction to (8.16). Thus $N_F(x) + N_F(x_0) \subseteq N_F(x_0)$, i.e. $N_F(x) \subseteq N_F(x_0)$. By formula (8.17) this implies $F(x) - F(x_0) \subseteq N_F(x_0)$ for all $x \in B(x_0, \delta)$.

(ii). We suppose that there exists a representative f of F such that $N_F(x_0) \subset N_F(x)$ and $f(x) - f(x_0) \in N_F(x)$. Then $F(x_0) = f(x_0) + N_F(x_0) \geq f(x_0) + N_F(x) = f(x) + N_F(x) = F(x)$, which is a contradiction.

(iii). Let f be a totally differentiable representative of F. If $\dfrac{\partial f}{\partial x_i}(x_0) \neq 0$, the point x_0 is not a minimizer of f. It follows that there exists $x_1 \in X$ such that $f(x_1) < f(x_0)$ and $N_F(x_1) \supseteq N_F(x_0)$. Consequently, $f(x_1) + N_F(x_1) \ngeq f(x_0) + N_F(x_0)$. We conclude that $\dfrac{\partial f}{\partial x_i}(x_0) = 0$. □

Remark 8.3.16. By the theorem above, if a point x_0 is a minimizer, the neutrix-function $N_F(x)$ achieves local maximum at x_0 or $N_F(x_0) = 0$.

8.3.2 Special forms

In this part we will present optimality conditions of an optimization problem with flexible objective function in which the objective function has the form $F(x) = f(x) + g_1(x)N_1 + \cdots + g_k(x)N_k$, where $N_i, 1 \leq i \leq k$ are neutrices and g_i, f are real function, for $1 \leq i \leq k$. We will show that under some appropriate conditions, an solution of the optimization problem with the objective function F can be determined via a solution of the problem with objective function f and maximizers of g_i on the domain, for $1 \leq i \leq k$. We start with the function F of the form $F(x) = f(x) + g(x)N$ and then with $F(x) = f(x) + +g_1(x)N_1 + \cdots + g_k(x)N_k$.

Theorem 8.3.17. *Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function defined by $F(x) = f(x) + g(x) \cdot N$ for all $x \in X$, where f, g are real functions defined on X and N is a scalar neutrix. Assume that $x_0 \in X$ is a minimizer of f on X and $x_1 \in X$ is a maximizer of $|g|$ on X. If $|g(x_0)|/|g(x_1)|$ is not an absorber of N then x_0 is a minimizer of F on X. In particular, if $x_0 = x_1$ then x_0 is optimal solution of* (8.12).

Proof. If $g(x) = 0$ for all $x \in X$ or $N = 0$, the conclusion is trivial. We assume that there is $x \in X$ such that $g(x) \neq 0$ and $N \neq 0$. It implies that $\max_X |g(x)| \neq 0$, i.e. $g(x_1) \neq 0$. Because x_0 is a minimizer of $f(x)$ on X, we have $f(x_0) \leq f(x)$. On the other hand, $|g(x)| \leq |g(x_1)|$ for all $x \in X$. It follows that $g(x) \cdot N \subseteq g(x_1) \cdot N$. Also note that, $0 \leq |g(x_0)|/|g(x_1)| \equiv \lambda \leq 1$ is not an absorber of N, it follows that $N/\lambda = N$ by Proposition 2.2.26. Consequently, $g(x) \cdot N \subseteq g(x_1) \cdot N = (g(x_0)/\lambda) \cdot N = g(x_0) \cdot N$. Hence, $f(x) + g(x) \cdot N \geq f(x_0) + g(x_0) \cdot N$ for all $x \in X$. In particular, when $x_0 = x_1$ we have $|g(x_0)|/g(x_1)| = 1$ which is not an absorber of N. This implies the last conclusion. □

Remark 8.3.18. In this theorem we need the condition that $|g(x_0)|/|g(x_1)|$ is not an absorber of N. In some special cases this condition is easy to verify. We list here some these cases.

(i) If $g(x_0), g(x_1) \in \pm@$ then $g(x_0)|/|g(x_1)| \in @$ are not an absorber of N.

(ii) If g is a standard function on X and x_0, x_1 are standard points, then the condition is true if $g(x_0) \neq 0$.

(iii) Assume that g is standard and continuous on X, $g(x) \neq 0$ for all $x \in X$, and $|g(x_1)| \in @$. If $x_0 \in X \cap £$ then $g(x_0)/g(x_1)$ is not an absorber of N. Indeed, since $g(x)$ is standard, for all $st(x) \in X$, $g(x)$ is standard. Also $g(x) \neq 0$, it follows that $g(x) \in @$ for $st(x) \in X$. On the other hand, for $x_0 \in X \cap £$, there exists st(z) such that $x_0 \simeq z$. Since $g(x)$ is continuous so that $g(x_0) \simeq g(z) \in @$. It follows by remark 1 above that $g(x_0)/g(x_1)$ is not an absorber of N.

(iv) Assume that the function g is continuous, standard and satisfies the conditions $g(x) \in \oslash \Rightarrow x \in \oslash$ for all $x \in X$, $|g(x_1)| \in @$. If $x_0 \in \pm @ \cap X$ then $g(x_0)/g(x_1)$ is not an absorber of N. In particular, $x_0 \neq 0$ is standard, then the condition is true. Indeed, because $g(x)$ is standard, it follows that $g(x)$ is standard for all $st(x)$. In particular, $|g(x)| \in @$. Because x_1 is standard, $|g(x_1)| \in @$. Also $x_0 \in \pm @ \cap X$ so that there exists $st(z)$ such that $x_0 \simeq z$. On the other hand, $g(x)$ is continuous, it implies that $g(x_0) \simeq g(z) \in @$. By Remark 8.3.18(i), $g(x_0)/g(x_1)$ is not an absorber of N.

To generalize Theorem 8.3.17, where $F(x) = f(x) + g_1(x)N_1 + \cdots + g_k(x)N_k$, we need first the following lemma:

Lemma 8.3.19. *Put $G(x) = g_1(x)N_1 + \cdots + g_k(x)N_k = \max\{g_1(x)N_1, \ldots, g_k(x)N_k\}$ where $g_i(x), 1 \leq i \leq k$ are real functions on $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $N_1, \ldots, N_k$ are scalar neutrices. Assume that there exists x_i which is a maximizer of $|g_i(x)|$ on X, for $1 \leq i \leq k$. Then there exists $x_m \in \{x_1, \ldots, x_k\}$ such that $G(x_m) = \max\{G(x_1), \ldots, G(x_k)\} = g_m(x_m) \cdot N_m$.*

Proof. Assume that $G(x_m) = \max\{G(x_1), \ldots, G(x_k)\} = g_r(x_m)N_r$. Since x_r is a maximizer of g_r on X, we have $g_r(x_m)N_r \subseteq g_r(x_r)N_r$. So

$$G(x_m) = g_r(x_m)N_r \subseteq g_r(x_r)N_r \subseteq g_1(x_r)N_1 + \cdots + g_k(x_r)N_k = G(x_r). \tag{8.18}$$

On the other hand, it holds that $G(x_r) \subseteq G(x_m)$. It follows by (8.18) that $G(x_m) = G(x_r) = g_r(x_r) \cdot N_r$. □

Applying the Theorem 8.3.17 for $G(x)$ we obtain one more general result as follows:

Theorem 8.3.20. *Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $F\colon X \longrightarrow \mathbb{E}$ be a flexible function given by $F(x) = f(x) + g_1(x)N_1 + \cdots + g_k(x)N_k$, where $f(x), g_i(x), 1 \leq i \leq k$ are internal functions and $N_1, \ldots, N_k$ are scalar neutrices. Assume that $x_0 \in X$ is a minimizer of $f(x)$ on X and $x_i, 1 \leq i \leq k$ are maximizers of $|g_i(x)|$ on X, respectively. Let $x_m \in \{x_1, \ldots, x_k\}$ be a point such that $G(x_m) = \max\{G(x_1), \ldots, G(x_k))\} = g_m(x_m)N_m$. If $|g_m(x_0)|/|g_m(x_m)|$ is not an absorber of N_m, the point x_0 is an optimal solution of optimization problem* (8.12). *In particular, if $x_0 = x_m$ then x_0 is an optimal solution of* (8.12).

Proof. The theorem follows from Lemma 8.3.19 and Theorem 8.3.17. □

Example 8.3.21. Consider the optimization problem

$$\min_{x \in \mathbb{R}} F(x) = \min_{x \in \mathbb{R}} \left(x^2 + \frac{\oslash}{1 + x^2} \right)$$

The function $f(x) = x^2$ obtains the minimum at $x_0 = 0$ and $g(x) = \frac{1}{1 + x^2} > 0$ obtains the maximum at x_0. So $N_F(x) = \dfrac{\oslash}{1 + x^2} \subseteq N_F(0) = \oslash$ for all $x \in X$. Hence $x = 0$ is an optimal solution.

Example 8.3.22. Let $F\colon X \equiv [0, 1] \longrightarrow \mathbb{E}$ be given by $F(x) = e^x + \cos x \oslash + \ln(1 + x)\epsilon£$ with $\epsilon > 0$ an infinitesimal. Consider the problem $\min_{x \in [0,1]} F(x)$.

Let $f(x) = e^x, g_1(x) = \cos x, g_2(x) = \ln(1+x)$ and $G(x) = \cos x \oslash + \ln(1+x)\epsilon£$ for all $x \in [0,1]$. Now $x = 0$ is a minimizer of f on $[0,1]$ and $x = 0$ is also a maximizer of g_1 on $[0,1]$. In addition, $x = 1$ is a maximizer of g_2 on $[0,1]$. As a consequence, $\max\{G(0), G(1)\} = G_1(0) = \oslash$. By Theorem 8.3.20 we conclude that $x = 0$ is a minimizer of F on $[0,1]$.

8.4 Optimality conditions based on parameter methods

In this section we use a parameter method to study an optimization problem with flexible objective function. In fact, we will consider the optimization problem of the form

$$\min_{x\in X} F(x;\alpha_1,\dots,\alpha_n), \tag{8.19}$$

where $X \subseteq \mathbb{R}^n, X \neq \emptyset$, $\alpha_i, 1 \le i \le n$ are external numbers and

$$F(x;\alpha_1,\dots,\alpha_n) = \left\{ F(x;a_1,\dots,a_n) \middle| a_i \in \alpha_i, (1 \le i \le n) \right\}.$$

In this case we will treat external numbers as a collection of parameters. The optimization problem

$$\min_{x\in X} F(x;a_1,\dots,a_n) \tag{8.20}$$

is called an *associated optimization problem with precise objective* of the problem $\min\limits_{x\in X} F(x;\alpha_1,\dots,\alpha_n)$, where $(a_1,\dots,a_n) \in (\alpha_1,\dots,\alpha_n)$.

Next we provide conditions such that an optimal solution or an approximate optimal solution of the problem (8.19) can be determined through the sets of optimal solutions of problems of the form (8.20).

Theorem 8.4.1. *Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and $F(.,\alpha_1,\dots,\alpha_n)$ be a flexible function defined on X. Consider the optimization problem*

$$\min_{x\in X} F(x;\alpha_1,\dots,\alpha_n). \tag{8.21}$$

Let $S(a_1,\dots,\dots,a_n)$ be the set of optimal solutions of the problem

$$\min_{x\in X} F(x;a_1,\dots,a_n) \tag{8.22}$$

for each $(a_1,\dots,a_n) \in (\alpha_1,\dots,\alpha_n)$. If $\bigcap\limits_{(\alpha_1,\dots,\alpha_n)} S(a_1,\dots,a_n)$ is not empty, every point

$$x_0 \in \bigcap_{(\alpha_1,\dots,\alpha_n)} S(a_1,\dots,a_n)$$

is an optimal solution of (8.21).

Proof. We will show that $F(x;\alpha_1,\dots,\alpha_n) \ge F(x_0;\alpha_1,\dots,\alpha_n)$. Indeed, for each $x \in X$, taking $y \in$

$F(x;\alpha_1,\dots,\alpha_n)$. Then there exists $a_i \in \alpha_i, 1 \le i \le n$ such that $y = F(x;a_1,\dots,a_n)$. Since $x_0 \in \bigcap_{a_i\in\alpha_i} S(a_1,\dots,a_n)$, one has $x_0 \in S(a_1,\dots,a_n)$. It follows that $y = F(x;a_1,\dots,a_n) \ge F(x_0;a_1,\dots,a_n)$ which belongs to $F(x_0;\alpha_1,\dots,\alpha_n)$. Hence $F(x;\alpha_1,\dots,\alpha_n) \ge F(x_0;\alpha_1,\dots,\alpha_n)$. □

For each $a_i \in \alpha_i, 1 \le i \le n$ we have a conventional optimization problem $\min_{x\in X} F(x,a_1,\dots,a_n)$. So we can apply results in the classical optimal theory to solve this problem. Then we verify the condition

$$\bigcap_{(\alpha_1,\dots,\alpha_n)} S(a_1,\dots,a_n) \neq \emptyset.$$

If this set is not empty then we can find optimal solutions. As a consequence we apply this result for convex functions of one variable.

Theorem 8.4.2. *Suppose that for each $(a_1,\dots,a_n) \in (\alpha_1,\dots,\alpha_n)$, the function $F(x;a_1,\dots,a_n)$ is convex and differentiable on X. Let $S(a_1,\dots,a_n)$ be the set of solutions of the equation*

$$F'(x;a_1,\dots,a_n) = 0. \tag{8.23}$$

Assume also that $\bigcap_{(\alpha_1,\dots,\alpha_n)} S(a_1,\dots,a_n)$ is not empty. Then every point $x_0 \in \bigcap_{(\alpha_1,\dots,\alpha_n)} S(a_1,\dots,a_n)$ is an optimal solution of the problem (8.21)*.*

Proof. Since the function $F(x;a_1,\dots,a_n)$ is convex on X, $S(a_1,\dots,a_n)$ is the set of optimal solutions of $\min_{x\in X} F(x;a_1,\dots,a_n)$. This implies that x_0 is an optimal solution of the problem (8.19) by Theorem (8.4.1). □

Example 8.4.3. Consider the following problem

$$\min_{x\in\mathbb{R}} F(x) = (1+\oslash)x^2.$$

For each $\epsilon \in \oslash$, one has

$$F'(x,\epsilon) = (1+\epsilon)2x \text{ and } F''(x,\epsilon) = 2(1+\epsilon) > 0.$$

Consequently, the function $F(x,\epsilon)$ is convex and the set of optimal solution of its is $S_\epsilon = 0$. Hence the optimal solution of the given problem is $x = 0$.

Theorem 8.4.4. *Let $F\colon X \longrightarrow \mathbb{E}$ be a flexible function of the form 8.19 and N be a neutrix. Let $S(a_1,\dots,a_n)$ be the set of solutions of problem 8.20 for $(a_1,\dots,a_n) \in (\alpha_1,\dots,\alpha_n)$ and $S = \bigcup_{\substack{a_i\in\alpha_i\\1\le i\le n}} S(a_1,\dots,a_n)$. Assume that $F(u) - F(v) \subseteq N$ for all $u, v \in S$ and $S(a_1,\dots,a_n) \neq \emptyset$ for all $a = (a_1,\dots,a_n) \in \alpha = (\alpha_1,\dots,\alpha_n)$. Then every point in S is an N-optimal solution of $\min_{x\in X} F(x)$.*

Proof. Let $v \in S$. We will show that $F(x) - F(v) \ge N$ for all $x \in X$. By assumption we have $F(x) - F(v) \ge N$ for all $x \in S$. Let $x \in X\backslash S$. Pick $y \in F(x)$ and $z \in F(v)$. Then there exists $a_i, a'_i \in \alpha_i, 1 \le i \le n$ such that $y = F(x;a_1,\dots,a_n)$ and $z = F(v,a'_1,\dots,a'_n)$. Let $u_0 \in S(a_1,\dots,a_n)$. We have $y \ge F(u_0;a_1,\dots,a_n)$. So $y - z = y - F(u_0;a_1,\dots,a_n) + F(u_0;a_1,\dots,a_n) - z \ge N$. Hence $F(x) - F(v) \ge N$. □

Example 8.4.5. Let F: $(0,\infty) \longrightarrow \mathbb{E}$ be given by $F(x) = -\ln(x)(1+\epsilon £) + x$ for all $x > 0$. Consider the problem $\min\limits_{x>0} F(x)$. We will show that every point in $1+\epsilon £$ is $\oslash$-minimizer of F on $(0,\infty)$.

Let $f(x,\mu) = -\ln(x)(1+\mu)+x$ for $x > 0, \mu \in \epsilon £$. A short calculation shows that $x_\mu = 1+\mu$ is a minimizer of $f(.,\mu)$ on $(0,\infty)$. Let $S = 1+\epsilon £$. Then for $u, v \in S$ we have $F(u) - F(v) \subseteq \oslash$. By Theorem 8.4.4 we conclude that every point in S is an $\oslash$-minimizer of F on $(0,\infty)$.

Theorem 8.4.6. *Let $X \subseteq \mathbb{R}^n, X \neq \emptyset$ and*

$$\begin{aligned} F\colon X &\longrightarrow \mathbb{E} \\ x &\longmapsto F(x;\alpha_1,\ldots,\alpha_n) = \left\{ F(x;a_1,\ldots,a_n) \middle| a_i \in \alpha_i, (1 \le i \le n) \right\}. \end{aligned}$$

be a flexible function. Let M, N be neutrices. Let $S(a_1,\ldots,a_n)$ be the set of optimal solutions of

$$\min_{x\in X} F(x, a_1, \ldots, a_n) \tag{8.24}$$

for each $a = (a_1,\ldots,a_n) \in \alpha = (\alpha_1,\ldots,\alpha_n)$ and $S = \bigcap\limits_{(\alpha_1,\ldots,\alpha_n)} S(a_1,\ldots,a_n)$. Assume that F is $M \times N$-strongly continuous on X and $S \neq \emptyset$. Then every point $x_0 \in X$ being M-close to S is a minimizer or N-minimizer of F on X.

Proof. Let $x_0 \in X$ be M-close to S. Then there exists $x_0' \in S$ such that $x_0 \in x_0' + M$. Because F is $M \times N$-strongly continuous, it holds that

$$F(x_0) \subseteq F(x_0') + N. \tag{8.25}$$

By Theorem 8.4.2, the point x_0' is an optimal solution of the problem (8.19). If $N_F(x_0) \subseteq N$, from formula (8.25) we conclude that x_0 is an N-optimal solution of the problem (8.21). Otherwise we conclude that x_0 is an optimal solution of F on X. $\square$

Example 8.4.7. Consider the problem

$$\min_{x\in\mathbb{R}} F(x) = |x| + \oslash.$$

One has, for each $\epsilon \in \oslash$ the problem $\min\limits_{x\in\mathbb{R}} F(x,\epsilon)$ takes optimal solution $S_\epsilon = \{0\}$ if $\epsilon \ge 0$ and $S_\epsilon = \{\pm\epsilon\}$ if $\epsilon < 0$. Therefore, the set of optimal solutions of the problem is $S = \oslash$. We give a sufficient condition for approximate solutions.

8.5 Optimality conditions based on set-valued mapping

Each external number is an external set of real numbers, so each flexible function can be seen as a set-valued mapping. However, the values of a flexible function are usually external sets, hence some results of the theory of set-valued mapping may not apply. Below we use the notion of radical cone to construct a type of derivative of flexible functions. Then we apply this notion to formulate necessary and sufficient conditions for the existence of optimal solutions of an optimization problem with flexible objective functions.

Definition 8.5.1. Let $F\colon X \longrightarrow \mathbb{E}$ be a flexible function defined on $X \subseteq \mathbb{R}^n, X \neq \emptyset$.

(i) The *graph* of F, denoted by Gr(F), is the set of points

$$\mathrm{Gr}(F) = \left\{(x,y)\middle| x \in X, y \in F(x)\right\}.$$

(ii) The *epigraph* of F, denoted by epi(F),

$$\mathrm{epi}(F) = \left\{(x,y)\middle| \exists z \in F(x), z \leq y\right\}.$$

Definition 8.5.2. Let $X \subseteq \mathbb{R}^n$. A flexible function $F\colon X \longrightarrow \mathbb{E}$ is said to be *convex* if epi(F) is convex.

For example, the flexible function $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be given by $F(x) = e^x + \oslash$ for $x \in \mathbb{R}$. Then F is convex on $\mathbb{R}$. Indeed, we have epi$(F) = \left\{(x,y) \in \mathbb{R}^2 \middle| y \geq e^x + \oslash\right\}$. Let (x_1, y_1), $(x_2, y_2) \in$ epi(F) and $\lambda \in [0,1]$. We need to show that $(x,y) = \lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) \in$ epi(F). One has

$$\lambda y_1 + (1-\lambda y_2) \geq \lambda(e^{x_1} + \oslash) + (1-\lambda)(e^{x_2} + \oslash) = \lambda e^{x_1}(1-\lambda)e^{x_2} + \lambda \oslash + (1-\lambda) \oslash.$$

Because e^x is convex on $\mathbb{R}$, it follows that

$$\lambda e^{x_1}(1-\lambda)e^{x_2} \geq e^{\lambda x_1 + (1-\lambda)x_2}. \tag{8.26}$$

Also, we show that

$$\lambda \oslash + (1-\lambda)\oslash = \oslash. \tag{8.27}$$

For $\lambda \in \oslash, \lambda \geq 0$, it holds that $1 - \epsilon \in @$. So formula (8.27) holds. For $\lambda \in [0,1], \lambda \notin \oslash$, formula (8.27) also holds.

Formulas (8.26) and (8.27) imply that

$$\lambda y_1 + (1-\lambda y_2) \geq e^{\lambda x_1 + (1-\lambda)x_2} + \oslash.$$

Hence $\lambda(x_1,y_1) + (1-\lambda)(x_2,y_2) \in$ epi(F). We recall some notions in the theory of set-valued mapping. For more details we refer to the article [7] and books [5, 1]. The definitions below are as in [5, Def. 2.54, p. 44].

Let $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

(i) The *radical cone* is defined by

$$\mathcal{R}_S(x) = \{h \in X : \exists t^* > 0, \forall t \in [0,t^*], x + th \in S\}. \tag{8.28}$$

(ii) The *contingent (Bouligand) cone* is defined by

$$T_S(x) = \{h \in \mathbb{R}^n : \exists t_n \downarrow 0,\ d(x + t_n h, S) = o(t_n)\}. \tag{8.29}$$

(iii) The *inner tangent cone* is defined by

$$T_S^i(x) = \{h \in \mathbb{R}^n : d(x+th, S) = o(t), t \geq 0\}. \tag{8.30}$$

(iv) The *Clark tangent cone* is defined by

$$T_S^c(x) := \{h \in \mathbb{R}^n : \forall t_n \downarrow 0, \forall x_n \longrightarrow_S x, \exists h_n \longrightarrow h \quad \text{such that} \quad \forall n, x_n + t_n h_n \in S\}. \tag{8.31}$$

In the article [7] the contingent derivative is defined and written as

$$CF(x_0, y_0) = T_{\text{Gr}(F)}(x_0, y_0)$$

and derivative of F at (x_0, y_0) is defined as written as

$$DF(x_0, y_0) = T^c_{\text{Gr}(F)}(x_0, y_0).$$

In other words, $y \in CF(x_0, y_0)(x)$ if and only if $(x, y) \in T_{\text{Gr}(F)}(x_0, y_0)$ and $y \in DF(x_0, y_0)(x)$ if and only if $(x, y) \in T_S^c(x_0, y_0)$.

Using these notions, in [7] necessary and sufficient conditions for maximality of problems $\max\limits_{x \in X} F(x)$ are given, where $F\colon X \rightrightarrows \mathcal{P}(\mathbb{R})$ is an internal set-valued mapping.

The two theorems below are modified versions of results in [7] which are stated for the case $F(x) \subset \mathbb{R}, x \in X$.

Theorem 8.5.3. *Let $F\colon A \rightrightarrows \mathcal{P}(\mathbb{R})$ be a set-valued mapping. If x_0 is a maximal point of F at y_0, for all $x \in A$*

$$CF_A(x_0, y_0)(x) \leq 0,$$

and hence

$$DF_A(x_0, y_0)(x) \leq 0.$$

Theorem 8.5.4. *Let F be concave on a convex set $A \subset Dom(F)$. If $DF(x_0, y_0)(x - x_0) \cap [0, +\infty) = \{0\}$ for all $x \in A$, then x_0 is a maximal point at y_0, i.e. $y_0 \in F(x_0)$ and for all $y \in \text{Im}F$ we have $y \leq y_0$.*

As for F a flexible function, in general, $F(A)$ is an external set. We can not use formulas (8.29)-(8.31) since the distance from a point to an external set does not exists. Because the notion of radical cone 8.28 does not use the notion of metric, we will modify this notion and then apply it to construct necessary and sufficient conditions for optimality problems with flexible objectives.

Definition 8.5.5. Let $F\colon X \to \mathbb{E}$ be a flexible function defined on $X \subseteq \mathbb{R}^n$. Let $(x_0, y_0) = \text{Gr}(F)$. We define the *radical derivative* of F at $(x_0, y_0) \in \text{Gr}(F)$

$$DF(x_0, y_0) = \Big\{(x, y) \in \mathbb{R}^{n+1} \big| (x_0, y_0) + t(x, y) \in \text{epi}(F), \forall t \in [0, 1]\Big\}.$$

We write $y \in DF(x_0, y_0)(x)$ if and only if $(x, y) \in DF(x_0, y_0)$

Example 8.5.6. Consider the flexible function $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ given by $F(x) = x^2 + \oslash$. It is easy to verify that F is convex on $\mathbb{R}$. We have

$$DF(0,0) = \left\{ (x,y) \in \mathbb{R}^2 \middle| y \geq x^2 + \oslash \right\}$$

and

$$DF(-1,1) = \left\{ (x,y) \in \mathbb{R}^2 \middle| y \geq (x-1)^2 + \oslash - 1 \right\}.$$

Indeed, let $(x,y) \in DF(-1,1)$. Then $(-1,1) + (x,y) \in \text{epi}(F)$. So $y + 1 \geq (x-1)^2 + \oslash$.

Generalizing Example (8.5.6), let $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ be a flexible convex function on $\mathbb{R}$. Then

$$DF(x_0, y_0)(x - x_0) = \left\{ y - y_0,\, y \geq F(x) \right\}$$

for all $x \in \mathbb{R}$. Indeed, let $z \in DF(x_0, y_0)(x - x_0)$. Then $(x_0, y_0) + (x - x_0, z) \in \text{epi}(F)$. This implies that $z + y_0 \geq F(x)$. Put $y = z + y_0$. Then $DF(x_0, y_0)(x - x_0) = \{y - y_0\, y \geq F(x)\}$.

Theorem 8.5.7. *Let $F\colon X \subseteq \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function and N_F be the neutrix part of F. Assume that x_0 is a minimizer of F on X. Then*

$$DF(x_0, y_0)(x) \cap \mathbb{R}_- \subseteq N_F(x_0) \quad \textit{for all} \quad x \in X.$$

Proof. Let $y_0 \in F(x_0)$. We suppose to contrary that there exists $\hat{x}, \hat{y}$ such that $\hat{y} \in DF(x_0, y_0)(\hat{x}) \cap \mathbb{R}_-, \hat{y} \notin N(x_0)$. Because $\hat{y} \in \mathbb{R}_-$, the fact $\hat{y} \notin N(x_0)$ implies that $\hat{y} < N(x_0)$. Since, $\hat{y} \in DF(x_0, y_0)(\hat{x})$, by the definition, there exists $(x,y) \in \text{epi}(F)$ such that

$$\begin{cases} x_0 + \hat{x} = x \\ y_0 + \hat{y} = y. \end{cases}$$

One has $(x,y) \in \text{epi}(F)$ so that

$$y \geq F(x) \geq F(x_0) = y_0 + N_F(x_0). \tag{8.32}$$

Also $\hat{y} < N_F(x_0)$. It follows that

$$y = y_0 + \hat{y} < y_0 + N_F(x_0),$$

which is a contradiction to (8.32). $\square$

To illustrate this theorem we consider the simple following examples.

Example 8.5.8. Consider the flexible function $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ given $F(x) = x^2 + \oslash$. We knew that $x = 0$ is an minimizer of F on $\mathbb{R}$ and $N_F(0) = \oslash$. Also

$$DF(0,0) = \left\{ (x,y) \in \mathbb{R}^2 \middle| y \geq x^2 + \oslash \right\}.$$

For $x \notin \oslash$, we have $DF(0,0)(x) = \{y | y \geq x^2 + \oslash > \oslash\}$. It follows that $DF(0,0)(x) \cap \mathbb{R}_- = \emptyset \subseteq \oslash = N_F(0)$. For $x \in \oslash$ we have $DF(0,0)(x) = \{y | y \geq \oslash\}$. It follows that $DF(0,0)(x) \cap \mathbb{R}_- = \oslash$. We see that the conclusion is true in this case.

Using this theorem we also know that the point $x = -1$ is not a minimizer of F on $\mathbb{R}$. Indeed, we have $C(-1,1) \in \text{Gr}(F)$ and $DF(-1,1)(1) = \{y \in \mathbb{R} \,|\, y \geq -1 + \oslash\}$. It follows that $DF(-1,1)(1) \cap \mathbb{R}_- \not\subseteq N_F(-1) = \oslash$. By Theorem 8.5.7 we conclude that $C(-1,1)$ is not a minimizer of F on $\mathbb{R}$.

Example 8.5.9. Let $\epsilon > 0$ be infinitesimal. Consider the flexible function $F\colon \mathbb{R} \longrightarrow \mathbb{E}$ given by $F(x) = |x| + \epsilon£$ for all $x \in \mathbb{R}$. Clearly, $x = 0$ is a minimizer of F on $\mathbb{R}$ and $N_F(0) = \epsilon£$. We also have that

$$DF(0,0) = \{(x,y) \in \mathbb{R}^n \,|\, y \geq |x| + \epsilon£\}.$$

Then for $x \notin \epsilon£$ we have $DF(0,0)(x) \cap \mathbb{R}_- = \emptyset \subset \epsilon£ = N_F(0)$, for $x \in \epsilon£$ we have $DF(0,0)(x) \cap \mathbb{R}_- = \epsilon£ = N_F(0)$. So the conclusion is true in this case.

In addition, using this theorem we can verify that $x = 1$ is not a minimizer of F on R. Indeed, we have $C = (1,1) \in \text{Gr}(F)$ and $DF(1,1) = \{(x,y) \in \mathbb{R}^2 \,|\, y \geq (|x+1| + \epsilon£ - 1)$. It follows that $-1 \in DF(1,1)(-1) \cap \mathbb{R}_-$ and $-1 \notin \epsilon£$. So $DF(1,1)(-1) \cap \mathbb{R}_- \not\subseteq \epsilon£ = N_F(1)$.

Theorem 8.5.10. *Let $F(x)$ be a convex flexible function on X. If $DF(x_0,y_0)(x - x_0) \cap \mathbb{R}_- \subseteq N_F(x_0)$, the point x_0 is a minimizer of $F(x)$ on X.*

Proof. Suppose on contrary that x_0 is not a minimizer of F on X. Then there exists $x_1 \in X$ and $y_1 \in F(x_1)$ such that $y_1 < F(x_0) = y_0 + N_F(x_0)$. This implies that $y_1 - y_0 < N_F(x_0)$. Moreover, $(x_0,y_0) + (x_1 - x_0, y_1 - y_0) = (x_1,y_1) \in \text{Gr}(F) \subseteq \text{epi}(F)$. Because epi$(F)$ is convex, one has $(x_0,y_0) + t(x_1 - x_0, y_1 - y_0) = \big(x_0(1-t) + tx_1, y_0(1-t) + ty_1\big) \in \text{epi}(F)$ for all $t \in [0,1]$. It follows that $y_1 - y_0 \in DF(x_0,y_0)(x_1 - x_0) \cap \mathbb{R}_-$ and $y_1 - y_0 < N_F(x_0)$, which is a contradiction to the assumption. Hence x_0 is a minimizer of F on X. □

8.6 Lagrange multiplier

In this section we develop a modified version of the Lagrange multiplier method for optimization problem with flexible objective function. Recall that a conventional optimization problem has a minimizer x_0, the Lagrange multiplier method confirms that there exist multipliers $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x_j}(x_0) + \sum\limits_{i=1}^{m} \lambda_i \dfrac{\partial g_i}{\partial x_j}(x_0) = 0 (j = 1, \ldots, n) \\ g_i(x_0) = 0 \end{cases}$$

In this context we will show that there are Lagrange multipliers and a neutrix K such that

$$\begin{cases} \dfrac{\partial_N F}{\partial_M x_j}(x_0) + \sum\limits_{i=1}^{m} \lambda_i \dfrac{\partial g_i}{\partial x_j}(x_0) \subseteq K (j = 1, \ldots, n) \\ g_i(x_0) = 0. \end{cases}$$

Let $F\colon \mathbb{R}^n \longrightarrow \mathbb{E}$ be a flexible function. Consider the optimization problem

$$\min_{x\in\mathbb{R}^n} F(x) \tag{8.33a}$$

subject to the constraints

$$\begin{cases} g_1(x_1,\ldots,x_n) &= 0 \\ \quad\vdots & \vdots \\ g_m(x_1,\ldots,x_n) &= 0, \end{cases} \tag{8.33b}$$

where $m < n$.

Theorem 8.6.1. *Let N_1, N_2, M_1, M_2 be neutrices such that $(N_1)_A \subseteq M_1, (N_2)_A \subseteq N_2$. Consider the problem* (8.33a)-(8.33b). *Let $h = (h_1,\ldots,h_m)$ be a vector implicit function determined by the constraints* (8.33b). *Assume that F is $M_2 \times N_2$- totally differentiable, and g_i is $M_1 \times N_1$-totally differentiable for all $i = 1,\ldots,m$. Let $H(x) = \sum\limits_{i=1}^{n} H_i(x)$ where $H_i(x) = N_1 + N_2 + N_1^2 + N_2^2 + \dfrac{\partial_{N_1} h_i}{\partial_{M_1} x_i}(x)N_2 + \dfrac{\partial_{N_2} F}{\partial_{M_2} x_i}(x)N_1$. We assume that x_0 is an $H(x_0)$-minimizer of the problem* (8.33a)-(8.33b)*, and that*

(i) The function g_i is continuously differentiable on $\mathbb{R}^n$.

(ii) The derivative g' is $M_1 \times N_1$-continuous at x_0 and $g'(x) \in {}^{N}_{M}Dg(x)$ for all $x \in Z$, here Z is an M_1 neighbourhood of x_0,

(iii) $\|g(x+h) - g(x)\| \le r\|h\|$ for all $x, x+h \in Z$ where Z is an M_1-neighbourhood of x_0 and r^{-1} is not an absorber of M_1.

(iv) Let $A(x) = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & a_{1(n-m)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(n-m)} & \cdots & a_{mn} \end{pmatrix}$*, where $a_{ij} = \dfrac{\partial g_i}{\partial x_j}(x), 1 \le i \le m; 1 \le j \le n$. Then $\|A^{-1}(x_0)\|^{-1} > N_1$ and $\left\|\big(A(x)\big)^{-1}\right\|$ is not an absorber of N_1 on an M_1-neighbourhood of x_0.*

(v) The flexible function F is M_2-outer N_2-inner continuous at x_0.

(vi) The flexible function F is $M_2 \times H(x_0)$-totally differentiable at x_0.

Then there exist a neutrix K and $\lambda^{(0)} = (\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m$ such that

$$\begin{cases} \dfrac{\partial_{N_2} F}{\partial_{M_2} x_j}(x_0) + \sum\limits_{i=1}^{m} \lambda_i \dfrac{\partial g_i}{\partial x_j}(x_0) \subseteq K (j = 1,\ldots,n) \\ g_i(x_0) = 0 \end{cases}$$

where K is determined through N, the partial derivatives of $g_i(x_0)$ and N-partial derivatives of $h_i(u_0)$ with $u_0 = (x_1^{(0)},\ldots,x_{n-m}^{(0)}), h(u_0) = (x_{n-m+1}^{(0)},\ldots,x_n^{(0)}), x_0 = (u_0, h(u_0))$.

Proof. Let $x_{n-m+j} = y_j, 1 \leq j \leq m$. Because the vector function $g = (g_1, \ldots, g_m)$ satisfies all conditions in Theorem 6.11.1, the system (8.33b) determines the implicit function $h\colon U \subseteq \mathbb{R}^{n-m} \to \mathbb{R}^m$ such that for all $u = (x_1, \ldots, x_{n-m}) \in U,\ h(u) = \big(h_1(u), \ldots, h_m(u)\big)$.

On the other hand, $F\left(x_1, \ldots, x_{n-m}, h_1(x_{x_1}, \ldots, x_{n-m}), \ldots, h_m(x_1, \ldots, x_{n-m})\right)$ has the $H(x_0)$-minimum at x_0, due to Theorem 8.2.4 it holds that

$$\frac{\partial_{H_0} F}{\partial_{M_2} x_j}(x_0) = H(x_0) \equiv H_0, \quad \text{for all} \quad i = 1, \ldots, n-m.$$

By formula (6.12) we have

$$\frac{\partial_{N_2} F}{\partial_{M_1} x_j}(x_0)(1+N_2) + \sum_{i=1}^{m} \frac{\partial_{N_2} F}{\partial_{M_2} y_i}(x_0) \frac{\partial_{N_1} h_i}{\partial_{M_1} x_j}(u_0) = H_0, \quad \text{for all} \quad j = 1, \ldots, n-m.$$

Hence there exist a neutrix G and representatives $\dfrac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0)$ of $\left(\dfrac{\partial_{N_2} F}{\partial_{M_2} x_j}(x_0)\right)$ for all $1 \leq j \leq n-m$ such that

$$\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) + \frac{\partial_{N_2} f}{\partial_{M_2} y_1}(x_0) \frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \frac{\partial_{N_2} f}{\partial_{M_2} y_m}(x_0) \frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) = G \subseteq H_0. \tag{8.34}$$

In addition, by assumption (iv) it holds that $|J| = \det \begin{pmatrix} \dfrac{\partial g_1}{\partial y_1}(x_0) & \cdots & \dfrac{\partial g_1}{\partial y_m}(x_0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m}{\partial y_1}(x_0) & \cdots & \dfrac{\partial g_m}{\partial y_m}(x_0) \end{pmatrix} \neq 0$, the system

$$\begin{cases} \lambda_1 \dfrac{\partial g_1}{\partial y_1}(x_0) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial y_1}(x_0) = \dfrac{\partial_N f}{\partial x_1}(x_0) \\ \vdots \\ \lambda_1 \dfrac{\partial g_1}{\partial y_m}(x_0) + \cdots + \lambda_m \dfrac{\partial g_m}{\partial y_m}(x_0) = \dfrac{\partial_N f}{\partial y_m}(x_0) \end{cases} \tag{8.35}$$

has the unique solution $\lambda^{(0)}$ corresponding to unknown variables $\lambda_i, i = 1, \ldots, m$.

Furthermore, because $g_i(x_1, \ldots, x_n) = 0$ for all $i = 1, \ldots, m$, it holds that $\dfrac{\partial g_i}{\partial x_j}(x) = 0, \forall\, i = 1, \ldots, m; \forall\, j = 1, \ldots, n$. By the formula of composition derivatives, for each $j = 1, \ldots, n-m$ we have

$$\begin{cases} \dfrac{\partial g_1}{\partial x_j}(x_0) + \dfrac{\partial g_1}{\partial y_1}(x_0) \dfrac{\partial h_1}{\partial x_j}(u_0) + \cdots + \dfrac{\partial g_1}{\partial y_m}(x_0) \dfrac{\partial h_m}{\partial x_j}(u_0) = 0 \\ \vdots \\ \dfrac{\partial g_m}{\partial x_j}(x_0) + \dfrac{\partial g_m}{\partial y_1}(x_0) \dfrac{\partial h_1}{\partial x_j}(u_0) + \cdots + \dfrac{\partial g_m}{\partial y_m}(x_0) \dfrac{\partial h_m}{\partial x_j}(u_0) = 0. \end{cases} \tag{8.36}$$

Let $A_i = \sum\limits_{j=1}^{m} \frac{\partial g_i}{\partial y_j} N_1$. Due to $\frac{\partial_{N_1} h_k}{\partial_{M_1} x_j}(u_0) = \frac{\partial h_k}{\partial x_j}(u_0) + N_1$, formula 8.36 implies

$$\begin{cases} \frac{\partial g_1}{\partial x_j}(x_0) + \frac{\partial g_1}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \frac{\partial g_1}{\partial y_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) = A_1 \\ \vdots \\ \frac{\partial g_m}{\partial x_j}(x_0) + \frac{\partial g_m}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \frac{\partial g_m}{\partial y_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) = A_m. \end{cases}$$

It follows that

$$\begin{cases} \lambda_1 \left(\frac{\partial g_1}{\partial x_j}(x_0) + \frac{\partial g_1}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \frac{\partial g_1}{\partial y_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \right) = \lambda_1 \cdot A_1 \\ \ddots \\ \lambda_m \left(\frac{\partial g_m}{\partial x_j}(x_0) + \frac{\partial g_m}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \frac{\partial g_m}{\partial y_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \right) = \lambda_m \cdot A_m. \end{cases}$$

Put $\lambda_i A_i = L_i, \forall i = 1, \ldots, m$. As a result,

$$\begin{aligned} &\sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) + \cdots + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_1}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \\ =&L_1 + \cdots + L_m. \end{aligned} \tag{8.37}$$

Subtracting (8.37) from (8.34) we obtain

$$\begin{aligned} &\left(\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \right) + \left(\frac{\partial_{N_2} f}{\partial_{M_2} y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) \right) \\ &\qquad + \cdots + \left(\frac{\partial_{N_2} f}{\partial_{M_2} y_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial y_1}(x_0) \cdot \frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \right) \\ &\qquad = G + L_1 + \cdots + L_m \equiv K. \end{aligned} \tag{8.38}$$

Because of subdistributivity it follows that

$$\begin{aligned} &\left(\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \right) + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_1}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_1}(x_0) \right) \frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) \\ &+ \cdots + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_m}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_1}(x_0) \right) \frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \\ \subseteq &\left(\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \right) + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_1}(x_0)\frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) \right) \\ &+ \cdots + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_m}(x_0)\frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) \right) \subseteq K. \end{aligned} \tag{8.39}$$

By formula (8.35) we obtain

$$\left(\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0)\right) + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_1}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_1}(x_0)\right) \frac{\partial_{N_1} h_1}{\partial_{M_1} x_j}(u_0) +$$
$$\cdots + \left(\frac{\partial_{N_2} f}{\partial_{M_2} x_m}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_m}(x_0)\right) \frac{\partial_{N_1} h_m}{\partial_{M_1} x_j}(u_0) = \frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \quad (8.40)$$

By (8.39) and (8.40) one has $\frac{\partial_{N_2} f}{\partial_{M_2} x_j}(x_0) - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}(x_0) \in K.$ $\square$

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