A Step in the Modelling, Analysis and Simulations of Coagulant Fluids

Joaquim M.C. Correia
DMat, ECT & CIMA, IIFA, University of Évora, Évora, Portugal
CAMGSD, IST, Lisbon, Portugal

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Abstract

In the mathematical modelling and simulation of coagulating fluids from real life applications in various fields such as biology (populations evolution), chemistry (polymerization) or medicine (blood flows) the effects of viscosity, damping, diffusion or capillarity relative to the transport mechanisms are of the most importance. We are interested in getting a better understanding of the coagulation and fragmentation phenomena in fluids. Here we will focus on the balance of ε-dissipative/δ-dispersive effects and we will analyse the well-posedness and the limit behaviour of some scalar equations of Korteweg-de Vries-Burgers type.

$$
\begin{aligned}
&\partial_t u + \operatorname{div}\left(f(x,t,u) - \varepsilon\,\mathcal{B}(x,t,u,\nabla u) + \delta\,\mathcal{C}(x,t,u,\nabla u,\nabla^2 u)\right) = 0,\\
&u(x,0) = u_0^{\varepsilon,\delta}(x).
\end{aligned}
$$

$$
\begin{aligned}
&u_t + u u_x = \varepsilon u_{xx} - \delta u_{xxx}, \qquad \text{[KdV-B equation}^1\text{]}\\
&f(u) = u^2/2, \quad \mathcal{B}(\nabla u) = u_x, \quad \mathcal{C}(\nabla^2 u) = u_{xx}.
\end{aligned}
$$

[1] See Kurganov-Rosenau[2, 1997]

As $\varepsilon, \delta \ll 1$, in applications, previous equations are approached by

$$\partial_t u + \operatorname{div} f(u) = 0,$$
$$u(x,0) = u_0(x).$$

From the point of view of 'applications', these equations are simplified by neglecting "spurious terms" of higher order giving rise to hyperbolic (first order) conservation laws which have non-unique solution. Still, we know that the zero-dispersion limit of KdV equation don't converges to the inviscid Burgers' equation [Lax-Levermore]. Thus, the classification (failure, reliability and integrity[2]) of these equations is a practical problem.

[2] Correia [1, 2010] and Correia-Sasportes [2, 2009]

The solutions to the KdV equations

$$u_t + (u^2/2)_x = -\delta\, u_{xxx}$$

do not converge as $\delta \searrow 0$ in a strong topology, Lax-Levermore [4, 1983].

The solutions to the Burgers' equations

$$u_t + (u^2/2)_x = \varepsilon\, u_{xx}$$

converge as $\varepsilon \searrow 0$ in a strong topology ("vanishing viscosity method"), Kružkov [3, 1970].

From the point of view of 'applied mathematics', we ask to study the singular limits of the approximated solutions of hyperbolic conservation laws given by the genKdV-B equations, as perturbations (dissipation and dispersion) vanish.

Conclusion

So, we are concerned with a proof of a "vanishing viscosity-capillarity method" relying on the well-posedness of the generalized KdV-B equations (by dispersive techniques) and the convergence of their solutions (by DiPerna's measure-valued solution techniques), an applied problem, or with the (applications:) behaviour and selection of the right models and right solutions.

Slemrod's PDEs Seminar, IST, September 16, 2014

PARTIAL DIFFERENTIAL EQUATIONS SEMINAR [5]

16/09/2014, 15:00 -- Room P4.35, Mathematics Building
Marshall Slemrod, _Department of Mathematics, University of Wisconsin, Madison_

HILBERT'S 6TH PROBLEM REVISITED

In this talk I will outline some thoughts on Hilbert's 6th problem, namely the passage from Boltzmann equation to the classical Euler equations of mass, momentum, and energy for an ideal gas as a small parameter (Knudsen number) tends to zero. The main idea is that via exact summation of the Chapman-Enskog expansion due to Karlin & Gorban the problem can be turned into a limiting problem for PDEs analogous to the KdV-Burgers' equation and for this problem the limit cannot be achieved. Hence, it appears that Hilbert's goal, while admirable, is not attainable.

Hilbert's 6th problem (from Boltzmann to Euler)

4. Implications of Gorban and Karlin's summation for Hilbert's 6th problem

The implication of the exact summation of C–E by Gorban and Karlin now becomes clear. The whole issue may be seen in Eq. (11), the energy balance. If we put the Knudsen number scaling into (11), the coefficient α is actually a term $\alpha_0\varepsilon^2$ and to recover the classical balance of energy of the Euler equation would require the sequence

$$\varepsilon^2\rho^\varepsilon\partial_i\rho^\varepsilon\partial_i\rho^\varepsilon \to 0$$

in the sense of distributions as $\varepsilon \to 0$. This would require a strong interaction with viscous dissipation. The natural analogy is given by the use of the KdV–Burgers equation:

$$u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx} \tag{12}$$

where at a more elementary level we see the competition between viscosity and capillarity. The result in (12) is known but far from trivial. Specifically in the absence of viscosity we have the KdV equation

$$u_t + uu_x = -K\varepsilon^2 u_{xxx} \tag{13}$$

and we know from the results of Lax and Levermore [9] that as $\varepsilon \to 0$ the solution of (13) will not approach the solution of the conservation law

$$u_t + uu_x = 0 \tag{14}$$

after the breakdown time of smooth solutions of (14). On the other hand, addition of viscosity which is sufficiently strong, i.e. K sufficiently small in (12), will allow passage as $\varepsilon \to 0$ to a solution of (14). This has been proven in the paper of Schonbek [10]. So, the next question is whether we are in the Lax–Levermore case (13) or the Schonbek case (12) with K sufficiently small. In my paper [11] I noted the C–E summation of Gorban and Karlin for the Grad 10-moment system leads to a rather weak viscous dissipation, i.e. Eqs. (5.10), (5.11) of [11]. At the moment, this is all we have to go on and I can only conclude that things are not looking too promising for a possible resolution of Hilbert's 6th problem. It appears that in the competition between viscosity and capillarity (mathematically, dissipation of oscillation versus generation of oscillation), capillarity has become a very dogged opponent, and the capillarity energy will not vanish in the limit as $\varepsilon \to 0$. Hilbert's hope may have been justified in 1900, but as a result of the work of Gorban, Karlin, Lax, Levermore, and Schonbek, I think that serious doubts are now apparent.

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Nonlinear hyperbolic conservation laws

- Cauchy Problem (1st order nonlin. pde's) $\Rightarrow$ hyperb. (real eigenvalues $\equiv$ finite velocity) $\Rightarrow$ discontinuities (characteristic lines meet) $\Rightarrow$ weak sol. (global in time) $\Rightarrow$ non uniqueness

- Entropy Methods from Gas Dynamics and 2nd Law of Thermodynamics (for Euler Equations $\equiv$ inviscid and compressible Navier-Stokes Equations)

- Equivalence to the Vanishing Viscosity Method selection: "classical" entropy weak solutions or Kružkov solutions

Traffic Burgers' Inviscid Equation or Arnold's particle/wave duality

In a straight line particles move freely and $u(x, t)$ is the velocity of the particle which is in position x at time t.

Let $x = x(t; 0, x_0)$ be the position at time t of the particle in x_0 at initial time $t_0 = 0$, which we abbreviate as $x = x(t)$.

By Newton's law (particles are moving freely) $x''(t) = 0$, then $x(t) = x_0 + \vec{v}t$ where $\vec{v} = u(x_0, 0)$.

'Particle description': the physical system is described by an infinite set of ODEs, one for each $x_0 \in \mathbb{R}$,

$$\begin{cases} x'(t) = u(x_0, 0), & t \geq 0 \\ x(0) = x_0. \end{cases}$$

Now, $\vec{v} = x'(t) = u(x(t), t)$, then

$$0 = x''(t) = u_t(x(t), t) + x'(t)u_x(x(t), t) = u_t + uu_x.$$

'Wave description': the physical system is described by a single PDE

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = u_0(x). \end{cases}$$

Rk. if we reverse that computation, we are solving the PDE by the 'characteristics method'.

A convergence result

Correia [2, 2017] "Zero Limit for Multi-D Conservation Laws with Nonlinear Dissipation and Dispersion": we have (formal)[3] convergence, if $r \geq \rho + 1 + \vartheta$ and $\delta = o(\varepsilon^{\gamma})$ with $\gamma = \frac{\rho+2}{r+1-\vartheta} (\leq 1)$, when

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}\left(\varepsilon\, b_j\big(u, \nabla u\big) + \delta\, g(u) \sum_{k=1}^{d} \partial_{x_k} c_{jk}\big(g(u)\, \nabla u\big)\right)_{1 \leq j \leq d}$$

(A_1) for some $m > 1$, $|f'(u)| = \mathcal{O}\left(|u|^{m-1}\right)$ as $|u| \to \infty$,

(A_2) for some $\mu \geq 0$, $r > 2$, $|b(u, \lambda)| = \mathcal{O}\left(|u|^{\mu}\right) \mathcal{O}\left(|\lambda|^{r}\right)$ as $|u|, |\lambda| \to \infty$,

(A_3) for some $\varphi \geq 0$, $\vartheta < r$, $D > 0$, $\lambda \cdot b(u, \lambda) \geq D\, |u|^{\mu\varphi}\, |\lambda|^{r+1-\vartheta}$ $\forall u \in \mathbb{R}$, $\lambda \in \mathbb{R}^d$.

(A_4) for some $\rho > 0$, $\|[c_{jk}(\lambda)]\| = \mathcal{O}(|\lambda|^{\rho})$ as $|\lambda| \to \infty$.

[3] Cf. Bedjaoui-Correia-Mammeri [3, 2015] "Well-Posedness of the Generalized Korteweg-de Vries-Burgers Equation with Nonlinear Dispersion and Nonlinear Dissipation".

Unexpected regime [4]

with $r = 1$ and $\rho = 2$ ($\delta = o(\varepsilon^{5/2})$), we proved the well-posedness of the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta(u_{xx}^2)_x,$$
$$u(x,0) = u_0^{\varepsilon,\delta}(x),$$

and as $\varepsilon, \delta \searrow 0$ the convergence of the previous solutions to the entropy weak solution of the initial value problem

$$u_t + f(u)_x = 0,$$
$$u(x,0) = u_0(x).$$

[4] Bedjaoui-Correia-Mammeri [1, 2016] and [4, 2014]: "On a Limit of Perturbed Conservation Laws with Diffusion and Non-positive Dispersion" and "On vanishing dissipative-dispersive perturbations of hyperbolic conservation laws".

Shocks

Breaking paradigmas

- Truskinovsky [4, 1993]: physical nonclassical solutions (considering dispersive terms; phase transition problems)
- Brenier-Levy [1, 1999]: dissipative KdV-type equations (3rd order equations without the 2nd order viscosity term; conjecture)
- Perthame-Ryzhik [1, 2007]: δ/ε balance in KdV-B equation $\delta = o(\varepsilon^1)$ (Riemann problem; travelling waves ε, δ-limit)

In use...

Modelling on continuum physics, chemistry, biology, environment, etc.

Areas as gas dynamics, nonlinear elasticity, shallow water theory, geometric optics, magneto-fluid dynamics, kinetic theory, combustion theory, cancer medicine, petroleum engineering, irrigation systems, etc.

Applications as optimal shape design (aeronautics, automobiles), noise reduction in cavities and vehicles, flexible structures, seismic waves (earthquakes, tsunamis), laser control in quantum mechanical and molecular systems, chromatography, chemostasis, oil prospection and recovery, cardiovascular system, traffic flow, the Thames barrier, etc.

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Robotics: Comp. Sci.-UE [UÉvora] (Nonsmooth Analysis)

Financial Math.: Economy-UE (Nonsmooth Analysis);
CSU [Caraga State University] (Stochastic; Dissip-Dispers in Black-Scholes with Jumps)

Biol./Environm.: LAMFA-UPJV [UPicardie Jules Verne] (Dissip-Dispers; Tsunamis);
UStrathclyde (Modelling & 6th Hilbert Probl.);
Physics-UE, CAMGSD-IST, UStrathclyde (Fog, Porous Media, Aggregation, Coagulation-Fragmentation)

Biol./Chemistry: LAMFA-UPJV (Lithium-ion Batteries), CAMGSD-IST, IIT Bhubaneswar (Aggregation, Coagulation-Fragmentation);
CAMGSD-IST, Mongolian Nat. Univ. Edu. (Wavelets, Coagulation-Fragmentation)

Health Care: UStrathclyde, UPJV, CAMGSD-IST, UE, NUOL, CSU, IIT Bhubaneswar (+ Calcuta University!?)

Thank you very much!