Social Infrastructure and the Preservation of Physical Capital: Equilibria and Transitional Dynamics

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Abstract

We study the mechanisms according to which social infrastructure influences the preservation of physical capital and, consequently, economic growth. The model considers that social infrastructure is a specific type of human capital, which acts in order to preserve already existing physical capital, by, e.g., reducing the incentive for rent seeking or corruption. Using an innovative methodology in economics, the Gröbner bases, we study the equilibrium of our model and conclude for the existence of two feasible steady-states or of unicity according to different combinations of parameters, highlighting a trade-off between consumption and production on one hand and social infrastructure and physical capital accumulation, on the other. We also present sufficient conditions for saddle-path stability. Finally, we describe transitional dynamics and calculate welfare effects from which we show that strengthening social infrastructure increases welfare.

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Keywords: Social Infrastructure, Physical Capital Depreciation, Endogenous Growth, Multiple Equilibria, Gröbner Bases.

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1 Introduction

We explore the effects of social infrastructure on the preservation of physical capital and, consequently, on economic growth. This is an unexplored link in the theory of economic growth, even within the literature that relates institutions to growth. In fact, social infrastructure can be associated with the existence of institutions, formal and/or informal in nature, that may help to decrease corruption, rent seeking, and cheating while improving transparency and trust in the economic environment of a country, facilitating the preservation of the existing physical capital stock, and enhancing economic growth.

The role of institutions on the economic performance of countries acquired such a relevance that it gave rise to a new branch in economics, designated by "institutional economics", which was born with the seminal work of North (1990), among others. Empirical work has emphasized the important contribution of good institutions to economic growth and development, and there is an important consensus on this conclusion, as we can see in the work of Hall and Jones (1999), Acemoglu *et al.* (2001, 2002), Easterly and Levine (2003), Dollar and Kray (2003), and Rodrik *et al.* (2004). In this study we follow this consensual view and assume that good institutions contribute to economic growth. However, we go further and consider that the channel is through the protection of physical capital or investment. In fact, empirical literature has found a negative relation between corruption levels and capital accumulation (Campos and Lien, 1999), corruption and productivity (Salinas-Jiménez and Salinas-Jiménez, 2007), social barriers and capital accumulation (Grafton *et al.*, 2007), and social capital and corruption (Bjørnskov, 2003a); it also found a positive relationship between governance institutions and investment (Aysan *et al.*, 2007), responsibility and capital accumulation (Breuer and McDermott, 2009), and trust and capital accumulation (Yamamura and Inyong, 2010). Closer to our work, Bu (2006) presented evidence according to which depreciation rates are higher in developing countries than in developed ones. According to the author and references therein, some of the explanations may be related to greater risk of expropriation, higher uncertainty on future returns from investments, lower maintenance expenditures in those countries, associated with greater corruption, e.g. factors linked with institutions. For instance, Tanzi and Davoodi (1997) showed that higher corruption is associated with lower expenditures on operations and maintenance of physical capital, which calls for a relationship between institutions and the depreciation of physical capital, exactly the link that we uncover.

There is a related extensive literature on the importance of institutions in the realm of evolutionary game theory. A very detailed and recent survey is Perc *et al*. (2017). In this literature the interaction behaviour of different individuals is analyzed within game theoretical framework (other examples are Hilbe and Traulsen, 2012 and Szolnoki and Perc, 2015). Because different possibilities emerge from different behaviours (e.g. free-riding, cooperative), then multiple states may arise. In this sense there is a parallel between that literature and our contribution, although we place ours in more aggregated terms.

We define institutions as being associated with the concept of social infrastructure as in the work of Hall and Jones (1999, pp.84). For these authors social infrastructure is composed by "...institutions and government policies that determine the economic environment within which individuals accumulate skills, and firms accumulate capital and produce output". We use this definition of institutions in a broad sense, including both formal and informal institutions. While formal institutions include constitutional constraints, statutory rules, property rights, rule of law, and other political and legal constraints; informal institutions arise from norms, culture, and customs, emerging spontaneously (Williamson, 2009). But formal institutions can contribute to economic growth only if they incorporate some of the principles established and agreed upon by informal institutions. This definition of informal institutions proposed by Williamson (2009) is closely related to the concept of social capital, as well as the notions of social infrastructure and trustworthy institutions.[1] The notions of social capital and its most commonly used empirical proxy, trust, are related, and work as a substitute for the notion of property rights (Aharonovitz *et al.*, 2009). There is a growing empirical literature relating institutions, social capital, and economic growth, namely Knack and Keefer (1997), Cuesta (2004), Beugelsdjik and van Schaik (2005), and Bjørnskov (2010), among others, pointing to a positive association between the mentioned variables, but still presenting diffuse results. In a model of endogenous growth, Strulik (2008) studies how social fractionalization and aggressiveness affect economic growth and show that civil conflict deters it.

In our work we focus on the positive role of institutions (social infrastructure) in preventing the depreciation of physical capital, a role that earlier empirical studies have uncovered, but that theory has so far neglected. We build an endogenous growth model with both physical and human capital accumulation in which we incorporate the important role of social infrastructure in facilitating physical capital preservation. Our main goal is to study an economic environment in which this feature is incorporated, focusing on the steady-state features and the transition path of the economy to the steady-state. The model will also allow us to access the consequences of increasing this preservation effect both in transition and in equilibrium. The precise mechanisms according to which social infrastructure influences output (and hence economic growth) are underexplored in the literature.[2]

We fill this gap, proposing specific mechanisms according to which social infrastructure influences output by its direct effect on physical capital preservation. In the model, social infrastructure is modelled as a particular type of human capital allocation consisting of hours spent in several activities such as: petitions, influence groups, participation in informal networks that spread information, etc., i.e., activities of civic and community participation, which help to im-

[1] North (1990) and Knowles (2006) also emphasized the importance of informal institutions. Knowles (2006) relates the concepts of informal institutions and social capital, claiming that they are very similar. Berggren and Jordahl (2008) find an empirical positive relationship between the existence of a good legal structure and property rights (formal institutions in our definition) and the level of trust in economies (informal institutions in our definition).

[2] Chin and Chou (2004) also model social infrastructure in a growth model, but in their model this variable affects the division of time between productive and non-productive activities. In our model it affects physical capital accumulation.

prove the level of civic rights, property rights, law and order, and ultimately the social infrastructure of a country. Through these effects social infrastructure reduces the incentive for rent seeking, corruption, predation, and cheating, and thus helps to preserve the existing physical capital stock of the economy. We analyze the economic consequences of such mechanisms.

To this end and given the structure of the model, involving four variables, four equations and seven parameters, we use an innovative method of algebraic geometry in the economics field, recently proposed by Kubler and Schmedders (2010a, 2010b), to study the existence and multiplicity of steady-states' solutions and equilibria - the Gröbner bases. The solution of economic growth models is often characterized as a set of multivariate parameterized polynomial equations, resulting from setting growth rates of stationary variables to zero. Finding all steady-states of the model is thus equivalent to being able to solve the corresponding polynomial system. In many cases, as referred by Kubler and Schmedders (2010a), standard numerical methods only search for a single equilibrium. Moreover, determining all solutions of a parameterized system of polynomial equations is sometimes hard to compute and the usual techniques either use numerical approximations or give us a general solution too complex to handle and analyze. In the last 30 years, computational algebraic geometry has seen considerable advances in methods that solve polynomial systems. The method of Gröbner bases is a powerful example of this progress. Kubler and Schmedders use them to study the multiplicity of equilibria (see Kubler and Schmedders, 2010a), and to compute the equilibrium correspondence for exchange economies with semi-algebraic preferences (see Kubler and Schmedders, 2010b). Gröbner bases' algorithm allow us to find all solutions of a polynomial system of equilibrium equations. But, more significantly, all computations are exact without rounding errors provided all coefficients in the equations are rational numbers or parameters. This will give us the possibility of proving the exact number of equilibria of the given economic model. Thus our contribution is twofold. First we consider a neglected channel through which institutions contribute to growth in an endogenous growth model and analyze its consequences. Second, we use a novel computational algebraic method to characterize the steady-states in endogenous growth literature.

The structure of the paper is as follows. In this Section, Subsection 1.1 presents some empirical evidence that motivates the paper. Section 2 presents the model. Section 3 characterizes the main results concerning steady-state equilibrium and its (local) stability. Section 4 presents simulation results for the transitional dynamics of the model when the effect of social infrastructure in investment is increased. In Section 5 we conclude.

1.1 Some Empirical Motivation

We present empirical motivation for the relationship between social infrastructure and the accumulation of physical capital (investment). For that purpose we found two proxies that could be interpreted as social infrastructure: The Social Capital Index of the Prosperity Index from the Legatum Institute and the Social Capital Index from Hall and Jones (1999).

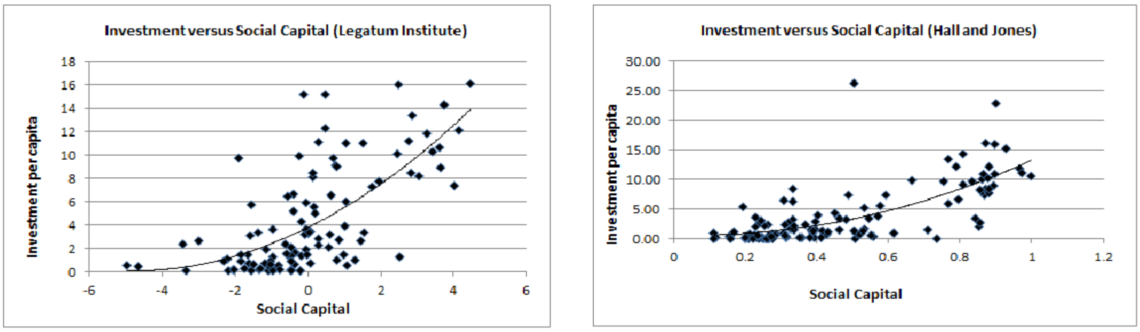

Figure 1: Relationship between Social Capital and Investment

Figure 1 shows the relationship between the Social Capital Index 2010 from the Prosperity Index from the Legatum Institute and the Social Capital Index from Hall and Jones (1999) and Investment *per capita*, for about 120 countries.[3] Both panels in the figure show a positive relationship between Investment and Social Capital (with partial correlations above 0.7[4]), empirically supporting the theoretical modelling followed in this paper, i.e., modelling social infrastructure as a positive effect over physical capital investment.

2 Model

We build an endogenous model of economic growth with both physical and human capital accumulation in which we incorporate the important role of social infrastructure in facilitating physical capital preservation. Human capital has different uses: it is employed in the production of the final good, in school attendance, which is the main input to the accumulation of new human capital, and it is also employed in the formation of social infrastructure. Physical capital is used in the production of the final good and social infrastructure facilitates the preservation of physical capital by decreasing its depreciation.

A crucial feature of the model is that there is no market for social infrastructure. Social infrastructure arises from the civic engagement of people and as a result provides utility. Also, households can help build and improve social infrastructure through allocating time to activities of civic and community participation. This follows the notion of *bonding* social capital in Beugelsdijk and Smulders (2009).

[3]The Social Capital Index 2010 was taken from the Legatum Institute website (http://www.prosperity.com/) and data for Investment (*per capita* and share of GDP in constant 2005 prices) were taken from the Penn World Tables, version 7.0.

[4]For both linear and polynomial adjustments.

2.1 Production Factors and Final Goods

2.1.1 Capital Accumulation

Individual human capital can be divided into skills allocated to different activities (as in Lucas, 1988). Thus, skills can be allocated to the final good production (H_Y), to school attendance (H_H), and to the building and improving of social infrastructure (H_S). Assuming that the different human capital activities are not done cumulatively, we have:

$$K_H = H_Y + H_H + H_S. \tag{2.1}$$

This restriction can be written in shares of human capital utilization as $1 = u_Y + u_H + u_S$, with $u_Y = H_Y/K_H$, $u_H = H_H/K_H$ and $u_S = H_S/H_Y$.

As in the literature that began with Arnold (1998), in this model human capital is the "ultimate" source of growth. To have endogenous growth, one should have non-decreasing returns in the human capital production function, regardless of the inputs to human capital that are considered. Human capital K_H is accumulated using human capital allocated to school attendance according to:

$$\dot{K}_H = \xi H_H \tag{2.2}$$

where $\xi > 0$ is a parameter that measures productivity in school attendance.

The accumulation of physical capital (K_P) arises through production that is not consumed, and is subject to depreciation:

$$\dot{K}_P = Y - C - \delta_P \left(1 - \sigma \frac{H_S}{K_H}\right) K_P \tag{2.3}$$

where Y denotes production of final goods, C is consumption, δ_P represents depreciation of physical capital, σ is the effect of social infrastructure in decreasing physical capital depreciation, and $\frac{H_S}{K_H} = u_S$ is the share of human capital in building and improving social infrastructure. We support our formalization on the work of Tanzi and Davoodi (1997) and Bu (2006). Note that the constraint $\sigma u_S < 1$ must be satisfied to allow for a positive depreciation of physical capital.[5]

2.1.2 Final Good Production

The final good is a homogeneous one, produced with a Cobb-Douglas technology:

$$Y = K_P^{\beta} H_Y^{1-\beta}, \; 0 < \beta < 1 \tag{2.4}$$

where β is the share of physical capital in the final good production. If we substitute this equation into (2.3) physical capital is accumulated according to $\dot{K}_P = K_P^{\beta} H_Y^{1-\beta} - C - \delta_P(1-\sigma u_S)K_P$. This

[5] As we discuss above, we consider that social infrastructure is acting in order to preserve physical capital, decreasing its net depreciation rate. However, we would obtain similar results if we considered a direct and positive effect of social infrastructure on investment.

means that the output-capital ratio can be written as $\frac{Y}{K_P} = \left(\frac{H_Y}{K_P}\right)^{1-\beta} = \left(\frac{K_H}{K_P}\right)^{1-\beta} u_Y^{1-\beta}$. Renaming $v_H = \frac{K_H}{K_P}$, we obtain:

$$\frac{Y}{K_P} = (v_H u_Y)^{1-\beta} \tag{2.5}$$

Similarly, we define $u_C = \frac{C}{K_P}$.

The markets for purchased production factors are assumed to be competitive. However, we assume that the firm cannot buy social infrastructure, as there is, in effect, no market for it. Social infrastructure is treated here as exogenous for the firm, although it affects the accumulation of physical capital.

From this problem we know that returns on production are as follows:

$$W_H = \frac{(1-\beta)Y}{H_Y} \tag{2.6}$$

$$r = \frac{\beta Y}{K_P} \tag{2.7}$$

where W_H is the market wage of workers and r is the rate of return of physical capital.

2.2 Consumers

We assume that households benefit directly from socializing, specifically engaging in civic activities. This follows the concept of bonding (as, for example, in Beugelsdijk and Smulders, 2009). Hence, household preferences specify time spent in building and improving social infrastructure, along with consumption, as arguments of the intertemporal utility function:

$$U(C_t, H_{S_t}) = \frac{\tau}{\tau-1} \int_0^\infty \left(C_t H_{St}^{\psi}\right)^{\frac{\tau-1}{\tau}} e^{-\rho t} dt \tag{2.8}$$

where ψ represents the preference for social infrastructure and ρ is the utility discount rate.[6]

In the market economy both consumers and firms make choices that maximize, respectively, their own utility or profits.[7] Consumers maximize their intertemporal utility function subject to the budget constraint:

$$\dot{a} = (r - \delta_p(1-\sigma u_s))a + W_H (K_H - H_H - H_S) - C \tag{2.9}$$

where a represents the household's physical assets. The market price for the consumption good is normalized to 1. Since it is making an intertemporal choice, the household also takes into account equation (2.2), i.e., human capital accumulation.

[6]The t subscripts are dropped hereinafter for ease of notation.

[7]In this section we are working with variables for individual consumers.

The choice variables for the consumers are C, H_H, and H_S, so the first-order conditions for the consumer problem yield:

$$\begin{aligned} \frac{\partial U}{\partial C} &= \lambda_a && (2.10)\\ \xi \lambda_H &= \lambda_a W_H && (2.11)\\ \frac{\partial U}{\partial H_S} &= \lambda_a W_H && (2.12) \end{aligned}$$

as well as:

$$\begin{aligned} \frac{\dot{\lambda}_a}{\lambda_a} &= \rho + \delta_P(1-\sigma u_s) - r && (2.13)\\ \frac{\dot{\lambda}_H}{\lambda_H} &= \rho - \xi && (2.14) \end{aligned}$$

where λ_a is the co-state variable for the budget constraint and λ_H is the co-state variable for the stocks of human capital. Finally $\frac{\partial U}{\partial C} = C^{-1/\tau} H_S^{\psi \frac{\tau-1}{\tau}}$ and $\frac{\partial U}{\partial H_S} = \psi C^{\frac{\tau-1}{\tau}} H_S^{\psi \frac{\tau-1}{\tau}-1}$.

The transversality conditions are: $\lim_{t\to\infty} \lambda_a a e^{-\rho t} = 0$ and $\lim_{t\to\infty} \lambda_H K_H e^{-\rho t} = 0$.

2.3 The Economy Dynamics

Using (2.10), (2.13), (2.5), and (2.3), we obtain g_{u_C}:

$$g_{u_C} = (\tau-1)\psi g_{u_S} + (\tau-1)\psi\xi(1-u_Y) - (1-\tau\beta)(u_Y v_H)^{1-\beta} + \\ +(1-\tau)\delta_P + u_C - ((\tau-1)\psi\xi + (1-\tau)\sigma\delta_P)u_S - \tau\rho. \qquad (2.15)$$

Resorting to (2.2), (2.1), and (2.3), the expression for g_{v_H} becomes:

$$g_{v_H} = \xi(1-u_Y) - (u_Y v_H)^{1-\beta} + u_C + \delta_P - (\xi + \sigma\delta_P)u_S \qquad (2.16)$$

From (2.11) and (2.6), we obtain the growth rate of u_Y:

$$g_{u_Y} = 1/\beta \frac{\dot{\lambda}_a}{\lambda_a} + g_{K_P} - 1/\beta \frac{\dot{\lambda}_H}{\lambda_H} - \xi(1-u_Y-u_S) \qquad (2.17)$$

and from (2.13) and (2.14) we reach:

$$g_{u_Y} = \frac{\delta_P}{\beta}(1-\sigma u_S) - (u_Y v_H)^{1-\beta} + \frac{\xi}{\beta} + g_{K_P} - \xi(1-u_Y-u_S). \qquad (2.18)$$

Replacing g_{K_P} by its expression (2.3), we then obtain:

$$g_{u_Y} = \left(\frac{1}{\beta}-1\right)\delta_P(1-\sigma u_S) - \xi(1-u_Y-u_S) - u_C + \frac{\xi}{\beta}. \qquad (2.19)$$

Finally, from (2.11) and (2.12), we compute $\psi C^{\frac{\tau-1}{\tau}} H_S^{\psi\frac{\tau-1}{\tau}-1} = \lambda_a W_H$. Using (2.10) and (2.6) we obtain $u_S = \frac{\psi}{1-\beta}\frac{C}{Y}u_Y$, which is easily converted into the static equation:

$$u_S = \frac{\psi}{1-\beta}\frac{u_C}{(u_Y v_H)^{1-\beta}}u_Y \tag{2.20}$$

We now have a system of three differential equations on u_C, u_Y, and v_H with a static equation on u_S, which, using $\bar{z} = v_H^{1-\beta}u_Y^{1-\beta}$, can be written as:

$$\begin{aligned}
g_{u_C} &= (\tau-1)\psi g_{u_S} + (\tau-1)\psi\xi(1-u_Y) - (1-\tau\beta)\bar{z} + \\
&\quad +(1-\tau)\delta_P + u_C - ((\tau-1)\psi\xi + (1-\tau)\sigma\delta_P)u_S - \tau\rho \\
g_{v_H} &= \xi(1-u_Y) - \bar{z} + u_C + \delta_P - (\xi+\sigma\delta_P)u_S \\
g_{u_Y} &= \left(\frac{1}{\beta}-1\right)\delta_P(1-\sigma u_S) - \xi(1-u_Y-u_S) - u_C + \frac{\xi}{\beta} \\
u_S &= \frac{\psi u_C}{(1-\beta)\bar{z}/u_Y}
\end{aligned} \tag{2.21}$$

3 Steady-State

In this section, we study the long term properties of the growth model conceptualized above. To approach the main steady-state features, an incursion into the Gröbner bases technique is required. Let $\mathbb{R}[x_1,\ldots,x_n]$ be the ring of polynomials in n variables x_i with coefficients in the field of real numbers $\mathbb{R}$. The main idea behind the Gröbner bases technique is the following: given a set Υ of polynomials in $\mathbb{R}[x_1,\ldots,x_n]$ that describes the problem in hand, one transforms Υ into another set Φ of polynomials of much simpler form, called a Gröbner basis, such that Υ and Φ are "equivalent", i.e., they have the same set of solutions. Thus, difficult problems for general Υ become "easier" for Gröbner basis Φ. For linear polynomials, the Gröbner bases algorithm specializes to Gauss' algorithm, whereas for univariate polynomials it specializes to Euclid's algorithm.[8]

One of the main advantages of this algorithm is that we can compute Gröbner bases for parameterized polynomials. Furthermore, all computations are exact provided all coefficients in the equations are rational numbers or parameters. In particular, one can compute the number of equilibria for entire classes of economic models (or bounds for this number), search for specific parameter values for which there are multiple equilibria, or prove that equilibria are unique for all parameter values in a given set. However, one must be aware that there may exist some parameters for which the corresponding Gröbner basis obtained is not the correct one. More precisely, Gröbner bases behave nicely for most (but not all) values of the parameters in the

[8]For the basic definitions and concepts on algebraic geometry and Gröbner bases Kubler and Schmedders (2010a,b) provide an introduction to this subject. We refer the reader to the textbook Cox *et al.* (1997) for more profound reading on this topic.

following sense: there is a proper subvariety $W \subset \mathbb{R}^m$ (where m is the number of parameters of the system) such that the Gröbner basis obtained is the same when the parameters take values in $\mathbb{R}^m - W$ (see Cox *et al.*, 1997, Chapter 6, §3). In the present work we compute this subvariety W in order to give a complete study of our model. We thus use a different approach from the one in Kubler and Schmedders (2010a) (see section 2.5).

To obtain a Gröbner basis for the system that defines our model we have used the free computer algebra system SINGULAR (Decker *et al.*, 2012), considered as one of the best software for Gröbner bases computations. We note that one can also find Gröbner bases' packages in other computer systems.

The system characterizing the decentralized equilibrium is a parameterized system of *four* variables, *four* equations, and *seven* parameters. As explained above, the Gröbner bases' method allows us to simplify the system. Even so, the analysis of this simpler system still involves *seven* parameters and it is obvious that any general solution will be too complex to analyze and will lead us to inconclusive results.

In order to obtain a sensible analysis of the steady-state, we must calibrate our model with sensible values for the parameters, usually used in endogenous growth theory and keep free the most important parameters linked with social infrastructure, the focus of our paper. Some parameters in our model are quite standard in the literature: the intertemporal substitution parameter ($\tau = 0.5$), the intertemporal discount factor ($\rho = 0.02$), and the share of physical capital in income ($\beta = 0.36$), so we shall not discuss them. For other parameters there are a range of plausible values, although most of them present typical values that are most used in the literature: the depreciation rate (δ_P), which we set to be 0.05 and the productivity of school attendance (ξ), which we set to be 0.05. We begin by studying steady-state solutions in which we calibrate all the parameters except those directly related with social infrastructure, ψ and σ, which we keep free. We then move from this general approach to more specific solutions in which we calibrate ψ and allow the parameter that governs the impact of social infrastructure on investment $-\ \sigma -$, the main mechanism analyzed in this paper, to be free. We assume that the weight the consumer attributes to social infrastructure is lower that the weight attributed to consumption, thus we implement solutions with ψ equal to 0.1, 0.5, and 0.9. For example, with $\psi = 0.1$, the consumer weights social capital just at 10% the weight he attributes to consumption; with $\psi = 0.9$, the consumer weights social capital at 90% the weight he attributes to consumption. This parameter should be regarded as the theorectical counterpart of the estimated elasticity of life satisfaction towards social capital. For example, Bjørnskov (2003b) estimated values around 0.5 and Elgar et al. (2011) estimated values around 0.1. We base on these estimates to choose our values. We add a higher value for comparison.

At this point we would like to stress that if one seeks to solve the resulting 2-parameter system with standard techniques with the help of a conventional system, the general solution obtained is too complex to handle and study. Nevertheless, the use of the Gröbner bases algorithm will allow us to determine all equilibria of our model.[9]

3.1 Steady-State for Free Social Infrastructure Parameters (ψ and σ)

The system of equations describing the decentralized equilibrium (2.21) when all parameters but ψ and σ are calibrated is:

$$\begin{cases} u_C + (0.025\psi - 0.025\sigma)u_S + (0.025\psi)u_Y - 0.82\bar{z} - 0.025\psi + 0.015 = 0 \\ u_C - (0.05 + 0.05\sigma)u_S - 0.05u_Y - \bar{z} + 0.1 = 0 \\ u_C + (4/45\sigma - 0.05)u_S - 0.05u_Y - 8/45 = 0 \\ \psi u_C u_Y - 0.64u_S\bar{z} = 0 \end{cases} \tag{3.1}$$

SINGULAR gives us the following Gröbner basis for the above system of polynomial equations (see Appendix):

$$\begin{aligned} g_1(\bar{z}) &= (2304\psi^3 + 11520\psi^2 + 18432\psi + 9216)\bar{z}^2 + \\ &\quad + (320\psi^3\sigma - 460\psi^3 + 1088\psi^2\sigma - 2588\psi^2 + 896\psi\sigma - 4616\psi - 2560)\bar{z} + \\ &\quad + 25\sigma\psi^3 + 70\sigma\psi^2 + 49\sigma\psi - 50\psi^3 - 170\psi^2 - 140\psi; \\ g_2(u_Y,\bar{z}) &= (5\psi\sigma + 10\sigma)u_Y + (-36\psi - 72)\bar{z} - 5\psi\sigma + 10\psi - 7\sigma + 20; \\ g_3(u_S,\bar{z}) &= 5\sigma u_S + 36\bar{z} - 10; \\ g_4(u_C,\bar{z}) &= (100\psi + 200)u_C + (-64\psi - 128)\bar{z} - 5\psi - 7. \end{aligned} \tag{3.2}$$

This means that the system (3.1) is equivalent to the simplified system:

$$g_1(\bar{z}) = g_2(u_Y,\bar{z}) = g_3(u_S,\bar{z}) = g_4(u_C,\bar{z}) = 0.$$

We note that $g_1(\bar{z})$ is now a one-variable polynomial of degree two whose coefficients are functions with variables ψ and σ. For the sake of simplicity, let us write $g_1(\bar{z}) = A\bar{z}^2 + B\bar{z} + C$, $g_2(u_Y,\bar{z}) = Du_Y + E\bar{z} + F$ and observe that $A, D \neq 0$. Solving $g_1(\bar{z}) = 0$ the system has the following recursive form solution:

$$\begin{cases} \bar{z} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ u_Y = -\frac{E\bar{z} + F}{D} \\ u_S = -\frac{36}{5\sigma}\bar{z} + \frac{2}{\sigma} \\ u_C = \frac{16}{25}\bar{z} + \frac{5\psi + 7}{100\psi + 200}. \end{cases}$$

[9] In 2010a, Kubler and Schmedders provide three examples of applications of Gröbner bases that prove the great advantage of using this algorithm. The corresponding polynomial systems of the three models cannot be easily analyzed with standard techniques. However, the computation of a Gröbner basis gives an equivalent system from which one can get information.

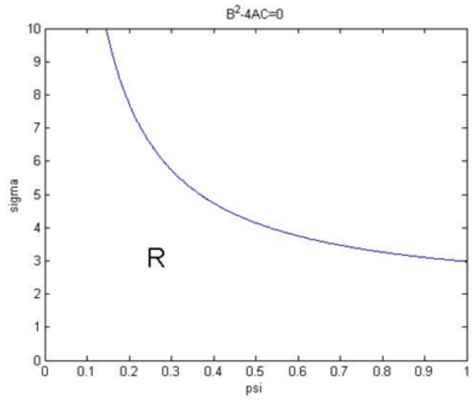

Figure 2: Region R for which $\overline{z}$ is real and $\overline{z_1}$ is real and positive

We can rewrite the system in the following way (after substituting the value of $\bar{z}$ in all equations):

$$\begin{cases} \bar{z} = \frac{-B\pm\sqrt{B^2-4AC}}{2A} \\ u_Y = \frac{-2AF+BE\mp E\sqrt{B^2-4AC}}{2AD} \\ u_S = \frac{10A+18B\mp 18\sqrt{B^2-4AC}}{5\sigma A} \\ u_C = \frac{5\psi A-32\psi B+7A-64B\pm 32(\psi+2)\sqrt{B^2-4AC}}{100\psi A+200A} \end{cases} \tag{3.3}$$

Our goal is to determine, for each ψ and σ, the number of real positive solutions for this system.

We first analyze when $\bar{z}$ is real and positive. We need to study the sign of $B^2 - 4AC$, a polynomial of degree 6 whose variables are the parameters ψ and σ.

The line in Figure 2, $B^2 - 4AC = 0$, divides the plane into two regions. The one labeled by R represents the set of (almost) all values of ψ and σ for which $B^2 - 4AC > 0$, i.e. the region where $\bar{z}$ is real, when $0 < \psi < 1$ and $0 < \sigma < 10$. These intervals for the social infrastructure parameters are based on quite weak assumptions. The first one ($0 < \psi < 1$) means that social infrastructure contributes (positively) to utility but weights less than consumption (which weights 1); thus ψ measures the relative welfare-substitutability between social infrastructure and consumption. The second interval ($0 < \sigma < 10$) means that social infrastructure preserves physical capital (the main assumption of this article) – as $0 > \sigma$ would clearly be dismissed by data – and $\sigma < 10$ prevents the overall effect of social infrastructure share in the growth rate of capital from exceeding one, i.e., $1 > \partial g_K / \partial u_S > 0$. This restriction also keeps the value of the overall effect of social infrastructure in preserving physical capital within a reasonable interval, even though for higher values of that interval it would be possible that the strength of the social infrastructure

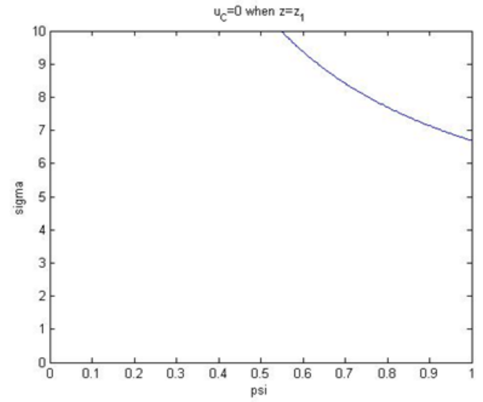

Figure 3: Region for which u_C is real and positive, when $\bar{z} = \bar{z}_1$

effect offsets the negative effect of depreciation.[10] We will thus focus only in this "admissible" region, where $\bar{z}$ takes real values.

Note that given any $\bar{z} > 0$ (i.e. given general values of $\psi, \sigma \in R$), the system has at least one real solution $(\bar{z}, u_Y, u_S, u_C)$. We next examine the situation when this solution is positive and determine how many positive solutions the system has.

One can easily check that $A > 0$ for all ψ and σ, and $B < 0$ in the region R. So, we conclude that $\bar{z}_1 = \frac{-B+\sqrt{B^2-4AC}}{2A}$ is always positive for general values of ψ and σ in R.

If $\bar{z}_1 \in R$ then from (3.3) it is easy to see that $u_Y > 0$ if and only if the numerator $-2AF + BE - E\sqrt{B^2-4AC} > 0$. The study of this function allows us to conclude that it is positive for general values of ψ and σ in R.

The following step is to evaluate the sign of the variable u_S when $\bar{z} = \bar{z}_1$. As before, u_S is positive if and only if its numerator is positive. This holds in R and hence, there is a positive solution $u_S > 0$ in R.

Finally, from the expression obtained for u_C, we see that $u_C > 0$ if and only if $5\psi A - 32\psi B + 7A - 64B \pm 32(\psi+2)\sqrt{B^2-4AC} > 0$. Studying this two-variable function, we see that $u_C > 0$ for general values of σ and ψ in the region R when $z = z_1$ (Figure 3 shows the region where $u_C > 0$, clearly containing region R shown in Figure 2).

Now, let us study the case when $\bar{z}_2 = \frac{-B-\sqrt{B^2-4AC}}{2A}$. Figure 4 shows us how the sign of $-B - \sqrt{B^2-4AC}$ changes inside R.

We see that $-B - \sqrt{B^2-4AC} = 0$ divides R into two smaller open regions. More precisely,

$$R_1 = R \cap \{(\sigma, \psi) \in]0,10[\times]0,1[: -B - \sqrt{B^2-4AC} < 0\}$$

[10]Below, we will note that u_S would be around 0.3 even for values of σ approaching 10. This means that $(1-\sigma u_S)$ would reach -2 with $\sigma = 10$ As $g_K = \frac{Y}{K_P} - \frac{C}{K_P} - \delta_P(1-\sigma u_S)$, with the mentioned values and $\delta_P = 0.05$, then $\delta_P(1-\sigma u_S) = -0.1$. This corresponds to add 10% (due to the effect of social infrastructure) to the g_K, a quite high and unreasonable value. With $\sigma < 10$ we limit the analysis to effects that are always lower than that.

and

$$R_2 = R \cap \{(\sigma,\psi) \in]0,10[\times]0,1[: -B - \sqrt{B^2-4AC} > 0\}.$$

Therefore, $\bar{z}_2 > 0$ if and only if ψ and σ belong to the region R_2. In this case, when we study the sign of the corresponding u_Y (i.e., when $\bar{z} = \bar{z}_2$) we have $u_Y > 0$ in R_2 in Figure 4.[11] Therefore, there is another positive solution for u_Y when $\sigma, \psi \in R_2$ (the line dividing R_1 and R_2 describes the set of points where $u_Y = 0$ in R).

Studying the functions defining u_S and u_C in the case when $\bar{z} = \bar{z}_2$, we conclude that both are positive for general values of $\sigma, \psi \in R_2$ (in fact, they are positive in R).

We can now conclude our study. For almost all $\psi, \sigma \in R_1$, the system (3.1) has a unique positive solution:

$$\begin{cases} \bar{z} = \frac{-B+\sqrt{B^2-4AC}}{2A} \\ u_Y = \frac{-2AF+BE-E\sqrt{B^2-4AC}}{2AD} \\ u_S = \frac{10A+18B-18\sqrt{B^2-4AC}}{5\sigma A} \\ u_C = \frac{5\psi A-32\psi B+7A-64B+32(\psi+2)\sqrt{B^2-4AC}}{100\psi A+200A} \end{cases}$$

For generic values of ψ and σ in the region R_2, the system has two positive solutions:

$$\begin{cases} \bar{z} = \frac{-B+\sqrt{B^2-4AC}}{2A} \\ u_Y = \frac{-2AF+BE-E\sqrt{B^2-4AC}}{2AD} \\ u_S = \frac{10A+18B-18\sqrt{B^2-4AC}}{5\sigma A} \\ u_C = \frac{5\psi A-32\psi B+7A-64B+32(\psi+2)\sqrt{B^2-4AC}}{100\psi A+200A} \end{cases} \vee \begin{cases} \bar{z} = \frac{-B-\sqrt{B^2-4AC}}{2A} \\ u_Y = \frac{-2AF+BE+E\sqrt{B^2-4AC}}{2AD} \\ u_S = \frac{10A+18B+18\sqrt{B^2-4AC}}{5\sigma A} \\ u_C = \frac{5\psi A-32\psi B+7A-64B-32(\psi+2)\sqrt{B^2-4AC}}{100\psi A+200A} \end{cases}$$

The most interesting result in this subsection is that we can define the regions in the space (ψ, σ) in which the equilibrium is unique and the regions in which there are two different feasible equilibria. Unicity of equilibria is obtained for low values of σ ($\lesssim 2.8$) and for almost all values of ψ. We can observe this in Figure 4, where R_1 is the region in which there is only a single positive steady-state and R_2 is the region in which there are two positive steady-states.

3.1.1 Finding the proper Subvariety $\mathcal{F}$

As mentioned in the introduction to this section, there is a proper subvariety $\mathcal{F} \subset \mathbb{R}^2$ such that when parameters ψ and σ take values outside $\mathcal{F}$, Gröbner basis behave nicely, i.e., the polynomials obtained from $g_1, \ldots, g_4$ by choosing values for ψ and σ are still a Gröbner basis for the ideal generated by the polynomials obtained from the original polynomials in equations (3.1). We will determine $\mathcal{F}$ in order to ensure that the Gröbner basis defined above is the correct one for this

[11] It is sufficient to look at the numerator since the denominator is always positive. The same holds when we study u_S and u_C.

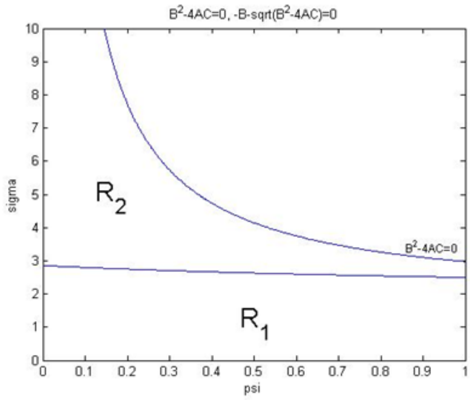

Figure 4: Region R_2 for which $\bar{z}_2$ is real and positive

problem. This calculation is not straightforward, as the literature on the subject mentioned above points out.

In Cox *et al.*, 1997, Chapter 6, §3, exercises 7–9, we have a set of guidelines to compute $\mathcal{F}$, which we will follow here. In appendix, we present the code used in SINGULAR for these computations.

Let f_1, f_2, f_3, f_4 be the polynomials

$$
\begin{aligned}
f_1 &= u_C + (0.025\psi - 0.025\sigma)u_S + (0.025\psi)u_Y - 0.82\bar{z} - 0.025\psi + 0.015;\\
f_2 &= u_C - (0.05 + 0.05\sigma)u_S - 0.05u_Y - \bar{z} + 0.1;\\
f_3 &= u_C + (4/45\sigma - 0.05)u_S - 0.05u_Y - 8/45;\\
f_4 &= \psi u_C u_Y - 0.64u_S\bar{z}.
\end{aligned}
$$

Let I be the ideal of $\mathbb{C}(\psi,\sigma)[u_C,u_S,u_Y,\bar{z}]$ generated by the polynomials f_1, f_2, f_3, and f_4. Consider the lexicographical ordering for monomials with

$$u_C > u_S > u_Y > \bar{z}.$$

A reduced Gröbner basis (Cox *et al.*, 1997, Chapter 2, §7, Definition 5) for the ideal I is

$$
\begin{aligned}
\bar{g}_1 &= \bar{z}^2 + \tfrac{80\sigma\psi^2+112\sigma\psi-115\psi^2-417\psi-320}{576\psi^2+1728\psi+1152}\bar{z} + \tfrac{25\sigma\psi^3+70\sigma\psi^2+49\sigma\psi-50\psi^3-170\psi^2-140\psi}{2304\psi^3+11520\psi^2+18432\psi+9216};\\
\bar{g}_2 &= u_Y - \tfrac{36}{5\sigma}\bar{z} + \tfrac{-5\sigma\psi-7\sigma+10\psi+20}{5\sigma\psi+10\sigma};\\
\bar{g}_3 &= u_S + \tfrac{36}{5\sigma}\bar{z} - \tfrac{2}{\sigma};\\
\bar{g}_4 &= u_C - \tfrac{16}{25}\bar{z} - \tfrac{5\psi+7}{100\psi+200}.
\end{aligned}
$$

We can now see that f_1, f_2, and f_3 are monic polynomials for the monomial ordering we considered. If we divide f_4 by ψ, we obtain a monic polynomial, as well. Being a reduced Gröbner basis, polynomials $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, and $\bar{g}_4$ are also monic. Let us consider all denominators present in the coefficients of polynomials f_1, f_2, f_3, $\frac{1}{\psi}f_4$, $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, and $\bar{g}_4$ (coefficients are elements of $\mathbb{C}(\psi, \sigma)$). They are:

$$d_1 = \psi; \qquad d_4 = 5\sigma;$$
$$d_2 = 576\psi^2 + 1728\psi + 1152; \qquad d_5 = 5\sigma\psi + 10\sigma;$$
$$d_3 = 2304\psi 3 + 11520\psi 2 + 18432\psi + 9216; \qquad d_6 = 100\psi + 200.$$

When we consider these polynomials in the ring $\mathbb{C}[\psi, \sigma]$, their least common multiple can be computed using library `poly.lib` (Bachmann *et al.*, 2012) in SINGULAR. It is

$$d = \sigma\psi(\psi^3 + 5\psi^2 + 8\psi + 4).$$

Now let $\tilde{I}$ be the ideal of $\mathbb{C}[u_C, u_S, u_Y, \bar{z}, \phi, \sigma]$ generated by the polynomials f_1, f_2, f_3, and f_4. Now observe that the polynomials we obtain by clearing denominators in $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$, and $\bar{g}_4$ are precisely g_1, g_2, g_3, and g_4, respectively, the Gröbner basis we obtained in (3.2).

By computing a Gröbner basis for $\tilde{I}$, we can easily see that all polynomials g_1, g_2, g_3, and g_4 are in $\tilde{I}$, and we can therefore conclude that if $\mathcal{F}$ is the variety defined by d in $\mathbb{R}^2$, then for all $(\psi, \sigma) \in \mathbb{R}^2 \setminus \mathcal{F}$ the Gröbner basis specializes well.

Note that d vanishes for $\sigma = 0$, $\psi = 0$ or negative values of ψ. All these values are excluded in the present context, so for the values relevant herein, the Gröbner basis computed above will specialize well.

3.2 Steady-State for a Varying Effect of Social Infrastructure on Investment (σ)

The main focus of this paper is to study an endogenous growth model in which we incorporate an effect of social infrastructure in preserving physical capital. Thus, we wish to detail the steady-state solutions for some given values of the effect of social infrastructure in utility (ψ) and only for a varying effect of social infrastructure in investment (σ). We use three values for $\psi : 0.5, 0.1$, and 0.9.

Replacing $\psi = 0.5$ in system (3.1) and computing its Gröbner basis is the same as replacing it in the Gröbner basis above, as we saw in the last section. It yields the following:

$$\begin{aligned} g_1(\bar{z}) &= 172800\bar{z}^2 + (6080\sigma - 44580)\bar{z} + 361\sigma - 950 \\ g_2(u_Y, \bar{z}) &= -25\sigma u_Y + 180\bar{z} + 19\sigma - 50 \\ g_3(u_S, \bar{z}) &= -5\sigma u_S - 36\bar{z} + 10 \\ g_4(u_C, \bar{z}) &= 500u_C - 320\bar{z} - 19 \end{aligned} \tag{3.4}$$

The solution of this system is:

$$\begin{cases} \bar{z} = \frac{-304\sigma+2229\pm\sqrt{92416\sigma^2-1979040\sigma+6610041}}{17280} \\ u_Y = \frac{36}{5\sigma}\bar{z} - \frac{50-19\sigma}{25\sigma} \\ u_S = -\frac{36}{5\sigma}\bar{z} + \frac{2}{\sigma} \\ u_C = \frac{16}{25}\bar{z} + \frac{19}{500} \end{cases}$$

or, equivalently:

$$\begin{cases} \bar{z} = \frac{-304\sigma+2229\pm\sqrt{92416\sigma^2-1979040\sigma+6610041}}{17280} \\ u_Y = \frac{1520\sigma-2571\pm\sqrt{92416\sigma^2-1979040\sigma+6610041}}{2400\sigma} \\ u_S = \frac{304\sigma+2571\mp\sqrt{92416\sigma^2-1979040\sigma+6610041}}{2400\sigma} \\ u_C = \frac{-304\sigma+3255\pm\sqrt{92416\sigma^2-1979040\sigma+6610041}}{27000} \end{cases}$$

Comparing equilibria in the case in which they both exist, we can see that one is characterized with a higher allocation of human capital to the final good production and high consumption to capital ratio while the economy invests less in social infrastructure, while the other is characterized by lower allocation to the final good production and consumption and better institutional environment. There is thus a trade-off between present and future, determined by the allocation of human resources to build social infrastructure. In Figure 5, we see that for $\sigma \in R_1 =]0;a[$, where $a \approx 2.6316$, there is exactly one positive solution $\bar{z}$, namely:

$$\bar{z} = \frac{-304\sigma+2229+\sqrt{92416\sigma^2-1979040\sigma+6610041}}{17280},$$

whereas when $\sigma \in R_2 =]a;b[$, where $b \approx 4.1406$, there are two possible positive solutions:

$$\bar{z} = \frac{-304\sigma+2229\pm\sqrt{92416\sigma^2-1979040\sigma+6610041}}{17280}.$$

When $\sigma \in]b;10[$, $\bar{z}$ is a complex solution. Furthermore, we see which values $\bar{z}$ takes when σ varies between 0 and 10. The graphs in Figure 5 show the values for $\bar{z}$, u_Y, u_S, and u_C.

Note that all results obtained are coherent with those obtained in the previous section. Suppose that $\sigma \in R_1 =]0;a[$. In this case, the only $\bar{z} > 0$ determines a unique admissible solution of the system, $(\bar{z}, u_Y, u_S, u_C)$, although the graphs in Figure 5 show us that there are two possible positive solutions for u_S and u_C (recall from the previous section that when $\bar{z} = \bar{z}_1$ all variables are positive for all $(\sigma, \psi) \in R_1$; but when $\bar{z} = \bar{z}_2$ only u_S and u_C are positive for all $(\sigma, \psi) \in R_1$).

On the other hand, if $\sigma \in R_2 =]a;b[$, we are able to check in Figure 5 that a horizontal line above the line $\sigma = a$ and below $\sigma = b$ intersects $\bar{z}$, u_Y, u_S, and u_C at two points. This means that there are two solutions for the system (3.4).

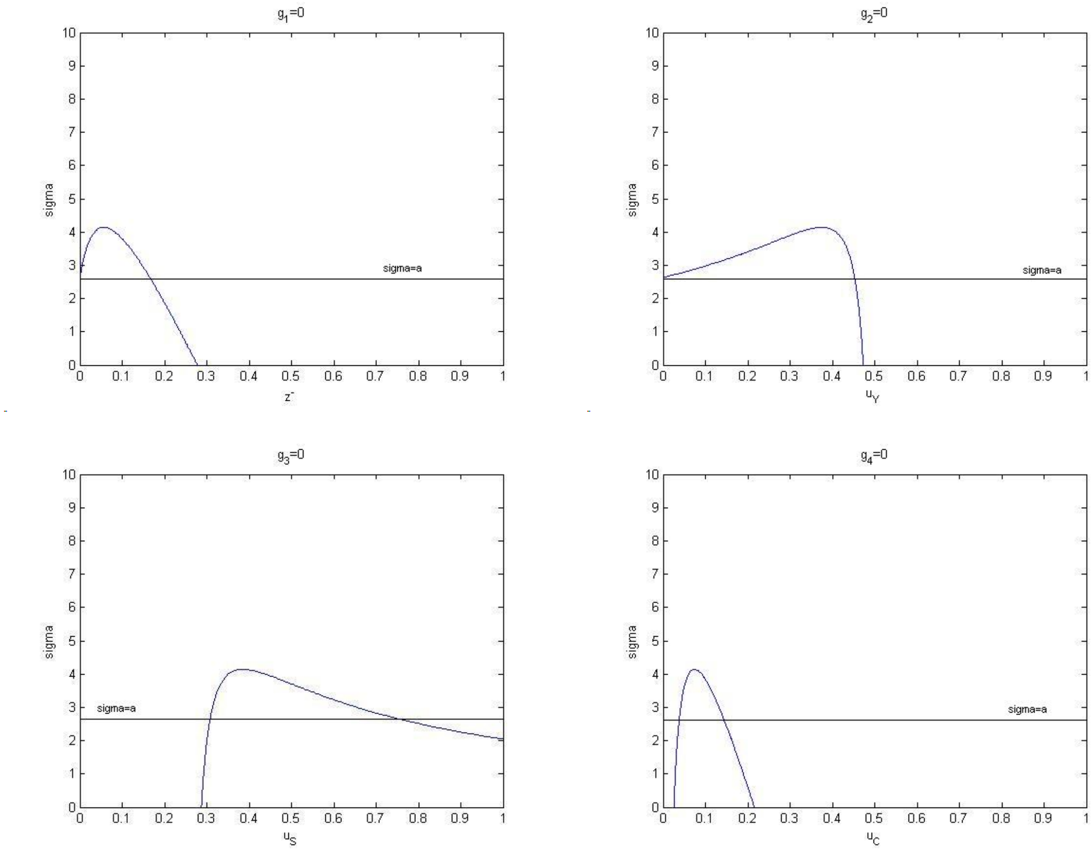

Figure 5: Real and Positive solutions of $\bar{z}$, u_Y, u_S and u_C (respectively) for $\sigma \in]0;10[$ and $\psi = 0.5$.

Looking at Figure 5 gives us an idea about how reasonable this exercise is. In fact, we obtain an allocation of human capital to the final good that can be at most 0.5, an allocation of human capital to social infrastructure that can be around 0.3 (in the unique equilibrium or in one of the equilibria when there are two) and a consumption to capital ratio can be just above zero or nearly 0.2, which are quite reasonable values. When analyzing the implications of the two equilibria solution, we easily reach the conclusion that the country with higher u_S, lower u_Y, u_C, and $\bar{z}$ would also have a lower Y/K_P. Whether the country with higher social infrastructure would have a higher income level than the one with lower infrastructure would depend on the level of K_P. However, this level would depend on, among other things, the efforts countries had made in order to improve σ, since an increase in σ will increase the growth rate of capital above the steady-state level and ultimately determine the income level of the country in each period. This draws attention to the study of transitional dynamics effects, which we present below.

The cases when $\psi = 0.1$ and $\psi = 0.9$ are studied in similar ways and give results that are analogous to the case when $\psi = 0.5$. The Gröbner bases are, respectively:

$$\begin{aligned} g_1(\bar{z}) &= 1241856\bar{z}^2 + (11200\sigma - 338660)\,\bar{z} + 625\sigma - 1750 \\ g_2(u_Y,\bar{z}) &= -35\sigma u_Y + 252\bar{z} + 25\sigma - 70 \\ g_3(u_S,\bar{z}) &= -5\sigma u_S - 36\bar{z} + 10 \\ g_4(u_C,\bar{z}) &= 700u_C - 448\bar{z} - 25 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} g_1(\bar{z}) &= 36815616\bar{z}^2 + (1920960\sigma - 9146020)\,\bar{z} + 119025\sigma - 300150 \\ g_2(u_Y,\bar{z}) &= -145\sigma u_Y + 1044\bar{z} + 115\sigma - 290 \\ g_3(u_S,\bar{z}) &= -5\sigma u_S - 36\bar{z} + 10 \\ g_4(u_C,\bar{z}) &= 2900u_C - 1856\bar{z} - 115 \end{aligned} \tag{3.6}$$

When $\psi = 0.1$, the solution of the system is:

$$\begin{cases} \bar{z} = \frac{-2800\sigma+84665\pm5\sqrt{313600\sigma^2-26726560\sigma+308458969}}{620928} \\ u_Y = \frac{11760\sigma-17563\pm\sqrt{313600\sigma^2-26726560\sigma+308458969}}{17248\sigma} \\ u_S = \frac{560\sigma+17563\mp\sqrt{313600\sigma^2-26726560\sigma+308458969}}{17248\sigma} \\ u_C = \frac{-560\sigma+23863\pm\sqrt{313600\sigma^2-26726560\sigma+308458969}}{194040} \end{cases}$$

For $\sigma \in]0;10[$, $\bar{z}$ is always a real number. In this case, we have $R_1 =]0;c[$, with $c \approx 2.8$, and the system has only one positive solution, and $R_2 =]c;10[$, where we find two positive solutions of the system.

When $\psi = 0.9$, the solution of the system is

$$\begin{cases} \bar{z} = \frac{-480240\sigma+2286505\pm5\sqrt{9225218304\sigma^2-131665479840\sigma+319626276025}}{18407808} \\ u_Y = \frac{309488\sigma-565355\pm\sqrt{9225218304\sigma^2-131665479840\sigma+319626276025}}{511328\sigma} \\ u_S = \frac{96048\sigma+565355\mp\sqrt{9225218304\sigma^2-131665479840\sigma+319626276025}}{511328\sigma} \\ u_C = \frac{-96048\sigma+685415\pm\sqrt{9225218304\sigma^2-131665479840\sigma+319626276025}}{5752440} \end{cases}$$

Now, $R_1 =]0; d[$, where $d \approx 2.5217$, and $R_2 =]d; e[$, where $e \approx 3.1015$. For $\sigma \in R_1$ the system has only one positive solution, while for $\sigma \in R_2$ there are two positive solutions. For $\sigma \in]e; 10[$, $\bar{z}$ is not a real number.

This section divides the space of the effect of social infrastructure on investment according to the existence of steady-state and its unicity. There is unicity of the steady-state when the effect of social infrastructure on investment is relatively low ($0 < \sigma < 3$) and there are two feasible equilibria for values of σ greater than 3. The precise value of σ below which there is a unique equilibrium does not change much when the weight of social infrastructure in utility changes from 0.1 to 0.9.

Given the results obtained so far we can summarize them in the following proposition.

Proposition 1. *For different weights of social infrastructure in utility (0.1; 0.5 and 0.9), there are one steady-state for relatively low effect of social infrastructure in capital accumulation or two steady-states for relatively higher effect of social infrastructure in capital accumulation.*

Proof. In the text above. □

3.3 Stability

In this section we wish to study the stability around the steady-states presented above. This is important in order to know if the system converges to the steady-state, once deviating from it temporarily. To this end we linearize the system (2.21) around the steady-state (v_H^*, u_Y^*, u_C^*) and obtain the following:

$$\begin{pmatrix}\dot{v}_H\\ \dot{u}_Y\\ \dot{u}_C\end{pmatrix} = \begin{pmatrix} u_C + \frac{u_C u_Y(1-\tau)\psi(\xi\psi-\delta_P\sigma)}{\bar{z}(1-\beta)} & J_{12} & J_{13}\\ v_H - \frac{u_Y^\beta v_H^\beta(\xi+\delta_P\sigma)\psi}{(1-\beta)} & J_{22} & J_{23}\\ -u_Y\left(1-\frac{u_Y\psi\Upsilon}{\bar{z}(1-\beta)\beta}\right) & -\frac{u_C u_Y^{1+\beta} v_H^{-2+\beta}\Upsilon\psi}{\beta} & \xi u_Y + \frac{u_C u_Y\Upsilon\psi}{\bar{z}(1-\beta)}\end{pmatrix}\begin{pmatrix} v_H - v_H^*\\ u_Y - u_Y^*\\ u_C - u_C^*\end{pmatrix}, \quad (3.7)$$

$$\begin{aligned}
J_{12} &= -u_C\bar{z}/v_H(1-\beta)(1-\beta\tau) - u_C^2 u_Y^\beta v_H^{-2+\beta}(1-\tau)\psi(\xi\psi-\delta_P\sigma);\\
J_{13} &= u_C\left(-\bar{z}/u_Y(1-\beta)(1-\beta\tau) + \xi(1-\tau)\psi + \frac{u_C\beta(1-\tau)\psi(\xi\psi-\delta_P\sigma)}{(1-\beta)\bar{z}}\right);\\
J_{22} &= -\bar{z}(1-\beta) + u_C u_Y/\bar{z}(\xi+\delta_P\sigma)\psi;\\
J_{23} &= -u_Y^{-\beta} v_H^{2-\beta}(1-\beta) - \xi v_H - \frac{u_C v_H\beta(\xi+\delta_P\sigma)\psi}{\bar{z}(1-\beta)};\\
\bar{z} &= v_H^{1-\beta} u_Y^{1-\beta};\\
\Upsilon &= (\beta\xi - (1-\beta)\delta_P\sigma).
\end{aligned}$$

or $\dot{\mathbf{X}} = \mathbf{J}(\mathbf{X}-\mathbf{X}^*)$, where $\mathbf{J}$ is the Jacobian in (3.7), J_{ij} are the elements of the Jacobian and $\mathbf{X}$ is the vector of variables. To demonstrate the conditions under which the system is stable we use the Routh-Hurwitz theorem.

Using the Routh-Hurwitz theorem, the number of stable roots is equal to the number of variations of sign in the scheme:

$$1 \qquad tr(\bar{J}) \qquad B\bar{J} \equiv \Delta - \det(\bar{J})/tr(\bar{J}) \qquad \det(\bar{J})$$

where $\Delta = J_{11}J_{22} - J_{12}J_{21} + J_{22}J_{33} - J_{32}J_{23} + J_{11}J_{33} - J_{13}J_{31}$.

We now show that a sufficient condition to rule out the case of non-existing stable roots is that $tr(\bar{J}) > 0$ and $\det(\bar{J}) < 0$, noting that if this were to happen we would obtain just one variation in sign independent of the sign of $B\bar{J}$. Thus, the determinant and trace are respectively:

$$\begin{aligned}
\det(\bar{J}) &= -\frac{\xi}{\beta(1+(1-\tau)\psi)}\left((1-\beta)\beta u_C\bar{z}u_Y + \psi\beta u_C^2 u_Y + \psi\delta_P\sigma u_C u_Y^2\left((1-\beta)-u_C/\bar{z}\right)\right):\\
tr(\bar{J}) &= u_C + (\xi u_Y - (1-\beta)\bar{z}) + \frac{u_C u_Y^\beta v_H^{-1+\beta}\psi(\xi+(1-\tau)(\xi\psi-\delta_P\sigma))}{1-\beta}.
\end{aligned}$$

Thus we can write the following Proposition.

Proposition 2. *For $\xi u_Y > (1-\beta)\bar{z} > u_C$ and $\xi\psi > \delta_P\sigma$, the steady-states are stable.*

Proof. Note that in the conditions of the Proposition, $tr(\bar{J}) > 0$ and $\det(\bar{J}) < 0$. □

For the calibration values used above, we reach the conditions $0.05u_Y > 0.64\bar{z} > u_C$ and $\psi > \sigma$. These sufficient conditions are stated for their simplicity; however, we must note that, given our experiments, the steady-state is saddle-path for many parameter combinations that do not respect the sufficient conditions stated above. For instance, we ran an exercise in which we analyzed the eigenvalues of that system from $\sigma = 0$ to $\sigma = 10$, with steps of 0.1 between 0 and 1 and steps of 1 between 1 and 10, for the three cases $\psi = 0.1$, $\psi = 0.5$, and $\psi = 0.9$. We always reached one or two eigenvalues with a negative real part which point out to determinate stability or indeterminate stability. Saddle-path determinate stability always occurs for the low effect of social infrastructure in utility ($\psi = 0.1$) and also occurs for $\psi = 0.5$ and for $\psi = 0.9$ for low values of the effect of social infrastructure on investment. An interesting feature of the situation in which social infrastructure is heavily weighted in utility ($\psi = 0.9$) is that convergence to the steady-state tends to be oscillatory for values of $\sigma > 3$, as complex conjugate values for the stable eigenvalues were found for those combinations of parameters.

4 Simulation

In this section we present the results of a simulation for the model economy when the value of our crucial parameter, σ, is changed.[12] We perform two exercises, one in which σ changes from 0.1 to 0.25 and another in which σ changes from 1 to 1.1. These changes fit in the regions obtained for feasible steady-states and are illustrative exercises. However, we conclude that for several combinations of parameters, the transitional dynamics in this model is very similar. We conclude, in particular, that the transitional dynamics obtained have only minor relevance when compared with steady-state differences in this model. This means that convergence speed is quite high and the economy takes at most 25 years to arrive at the new steady-state. This conclusion supports our complete study of the steady-state properties of the model stated above.

We conclude this section by presenting welfare effects of changes in σ for several combinations of parameters σ and ψ. It is important to look at welfare effects to complete the characterization of the model as there is a trade-off between consumption and social infrastructure in this economy. Since an increase in social infrastructure increases utility, it also increases investment. This rise in investment may decrease consumption in the short run. Thus, it is important to measure the relative importance of this short-run negative effect of improving social infrastructure. Figure 6 shows the evolution of the main variables from a steady-state with $\sigma = 0.1$ to a steady-state with $\sigma = 0.25$ for $\psi = 0.5$. In the Figure, we present macroeconomic variables such as growth rates for consumption (g_C), capital (g_{K_P}), and output (g_Y), the shares of human capital allocated to the final good sector, the human capital accumulation sector, and to social capital (u_Y, u_H, and u_S, respectively), and the human capital to physical capital ratio (v_H).

Figure 7 shows the evolution of the main variables from a steady-state in which $\sigma = 1.0$ to a steady-state in which $\sigma = 1.1$ for $\psi = 0.5$.

Once the effect of social infrastructure in preserving physical capital increases, the $v_H = K_H/K_P$ drops, as investment in physical capital begins. This increase in investment is shown in the figure, since g_K increases more than 0.5% in both exercises. The investment growth

[12] We use the Relaxation Algorithm from Trimborn *et al.* (2008).

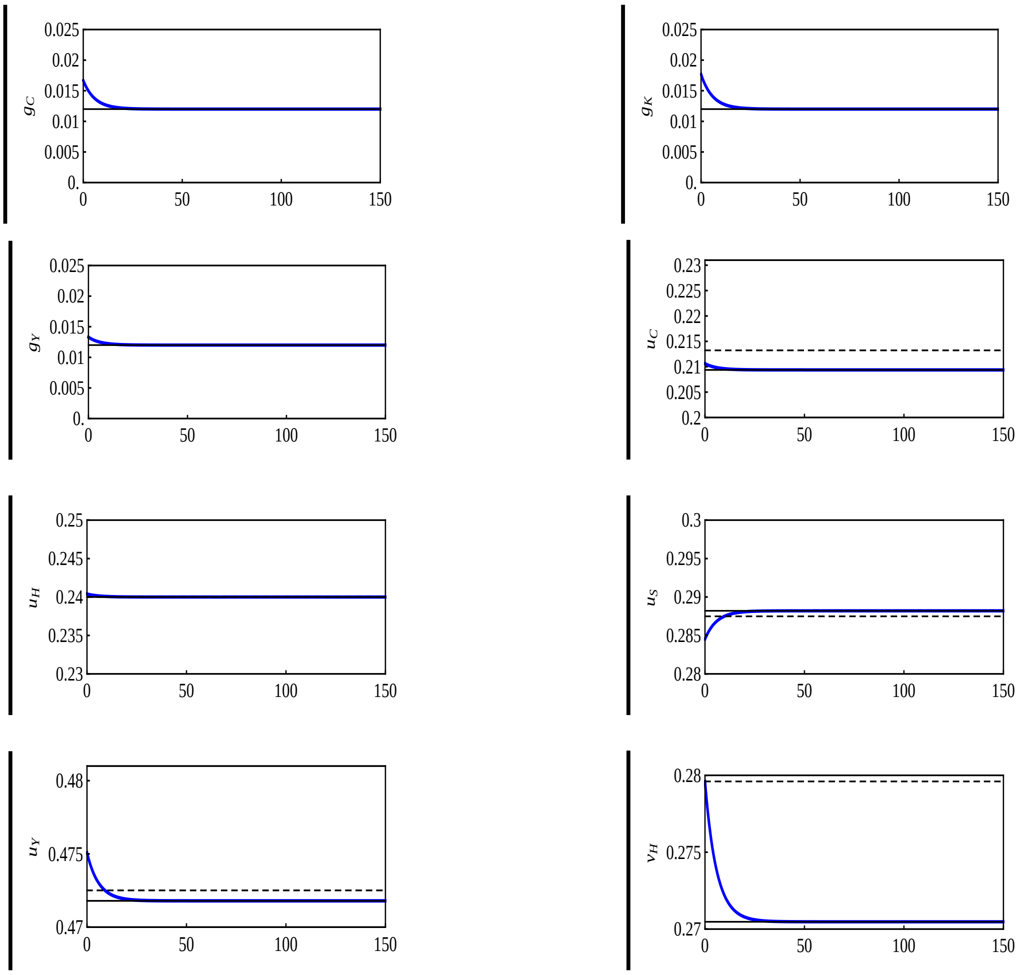

Figure 6: Time paths of representative variables in the model from a steady-state with $\sigma = 0.1$ to a steady-state with $\sigma = 0.25$.

Note: Parameter values are shown at the beginning of the previous section, $\psi = 0.5$. Solid black line refers to the final steady-state and the dashed black line refers to the initial steady-state.

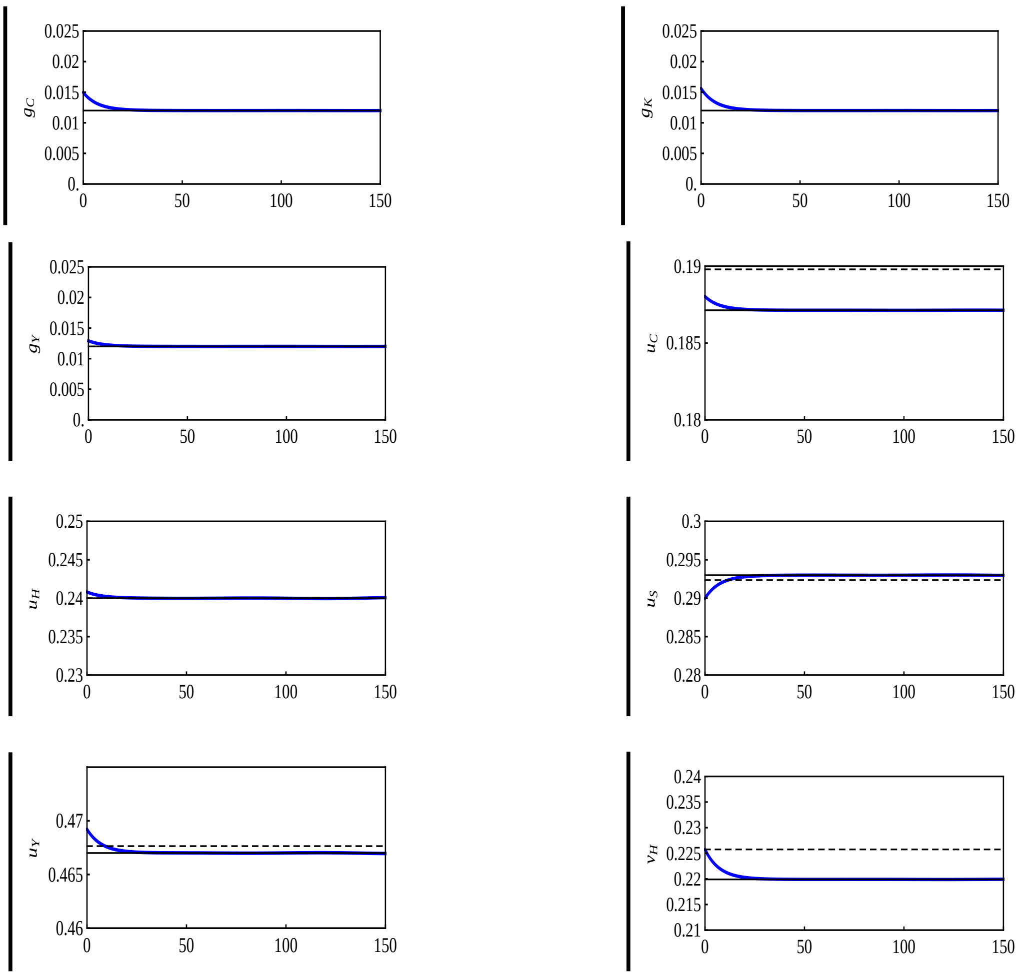

Figure 7: Time paths of representative variables in the model from a steady-state with $\sigma = 1.0$ to a steady-state with $\sigma = 1.1$.

Note: Parameter values are shown at the beginning of the previous section, $\psi = 0.5$. Solid black line refers to the final steady-state and the dashed black line refers to the initial steady-state.

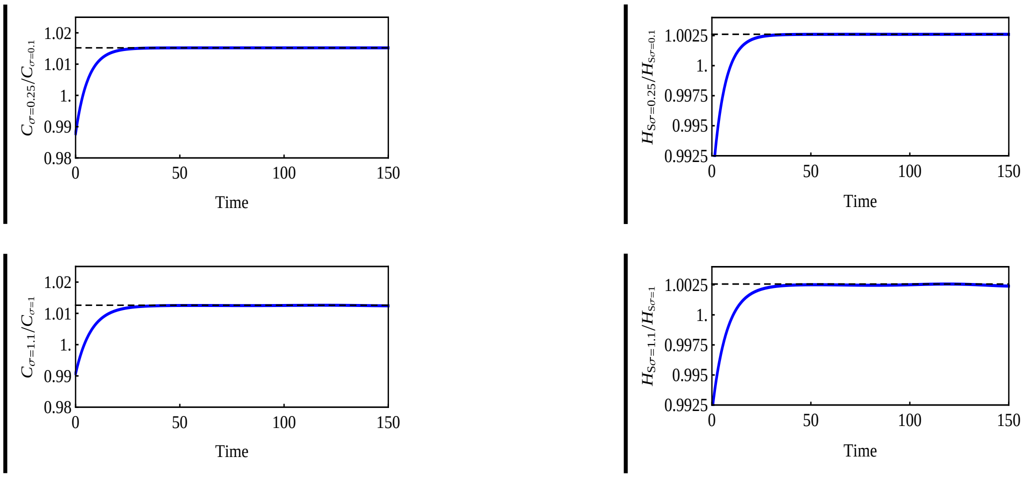

Figure 8: Time paths of consumption and social infrastructure in the model from a steady-state with $\sigma = 0.1$ to a steady-state with $\sigma = 0.25$ and from a steady-state with $\sigma = 1.0$ to a steady-state with $\sigma = 1.1$.
Note: Parameter values are shown at the beginning of the previous section, $\psi = 0.5$. Dashed black lines indicate the final value.

rate stands above its steady-state level for nearly 10 years. This increase in the growth rate of physical capital is followed by the growth rates for consumption and output. However, the increase of the growth rate of consumption stands below the rise in the physical capital growth rate, which is the cause for the drop of the consumption to capital ratio (u_C). In Figure 6, with a lower effect of social infrastructure in preserving physical capital, the share of human capital allocated to social infrastructure activities first decreases (nearly 0.025%) and then increases to a level that is slightly above the initial steady-state value. This corresponds to an initial increase of human capital allocated to the final good (u_Y), with an almost constant share of human capital in education. This transitional higher allocation of human capital to the production of final good matches the higher investment in physical capital. In Figure 7, however, with a greater effect of social infrastructure in preserving physical capital, we observe a higher drop in v_H, and consequently, a higher effect of increasing protection (due to social infrastructure) in investment.

Transitional dynamic analysis also reveals that the compensation to increase allocation to the final good production due to strengthening of social infrastructure effect in the economy comes at the expense of allocations to social infrastructure, with minor effects on education.

The intuition behind the transition path for the variables is maintained for exercises in which $\psi = 0.1$ and $\psi = 0.9$.

We also wish to calculate the effect of this rise in σ on welfare. For that we must first calculate a series for consumption C and for the allocation of human capital to social infrastructure activities, H_S, both of which influence utility. Thus this measure takes all of the transitional dynamics into account. Figure 8 shows the evolution of both variables compared with their initial values (each variable assumes value 1 in the initial steady state).

From Figure 8 we see that there are interesting trade-offs between short and long-run effects that will influence welfare. In both exercises, both consumption and investment in social infrastructure face a short-run negative effect that may be compensated for by a positive effect in the long-run.

Table 1 shows the long-run variations in consumption, in investment in social infrastructure, and in welfare that result from increasing the effect of social infrastructure in protecting investment.

Table 1 - Long-run Effects (%) of Institutional Change in Consumption (C), Social Infrastructure (Hs), and Welfare (W)

	$\sigma = 0.1 \rightarrow \sigma = 0.25$	$\sigma = 1.0 \rightarrow \sigma = 1.10$
$\psi = 0.1$	$\Delta C = 0.39; \Delta H_S = 0.07; \Delta W = 0.48$	$\Delta C = 0.28; \Delta H_S = 0.05; \Delta W = 0.39$
$\psi = 0.5$	$\Delta C = 1.52; \Delta H_S = 0.26; \Delta W = 1.16$	$\Delta C = 1.26; \Delta H_S = 0.26; \Delta W = 0.91$
$\psi = 0.9$	$\Delta C = 2.26; \Delta H_S = 0.40; \Delta W = 1.58$	$\Delta C = 2.05; \Delta H_S = 0.46; \Delta W = 1.28$

These values indicate a considerable effect on welfare of small variations in the parameter that governs the effect of social infrastructure (σ), effects that oscillate from 0.39% to 1.58%. The welfare effects depend positively and monotonically on the weight of social infrastructure in the utility. Interestingly, the effect on consumption of increasing σ is greater than the effect on social infrastructure (H_S).

5 Conclusion

Following the important literature on institutions and growth, the model in this paper considers that social infrastructure is a specific type of human capital, which allows for preserving physical capital.

Due to the polynomial structure and complexity of the model, we use an innovative methodology in economics, the Gröbner basis, to characterize the feasibility of the steady-state. We conclude that for different regions of the crucial parameters space, two feasible or a unique steady-state could emerge. In particular, unicity is ensured when the effect of social infrastructure in preserving investment is particularly low. When this happens, the steady-state always predicts reasonable values for the shares of human capital allocated to the final good production, education, and social infrastructure. When there are two different steady-states, one is characterized by a higher allocation of human capital to the final good production and high consumption to capital ratio while investing less in social infrastructure, and the other is characterized by lower allocation of human capital to the final good production and consumption and better institutional environment. There is thus a trade-off between present and future determined by allocation of human resources to build social infrastructure. For reasonable intervals of the social infrastructure weight in utility and social infrastructure effect in investment, steady-states are stable, saddle-path or indeterminate, and convergence around the steady-state may be monotonic or oscillatory.

Thus, the model that incorporates the role of social infrastructure in preserving physical capital shows a rich set of outcomes.

We also studied transitional dynamics of an economy that strengthens social infrastructure. During the transition path the economy faces a trade-off between the final good production and investment in social infrastructure, inducing a phase of relatively higher transitional growth.

To summarize, our paper presents an alternative modelling of the effect of social infrastructure on economic growth, through linking social infrastructure with human capital effort which acts on physical capital investment. We conclude for a crucial effect of the quality of social infrastructure (measured by the effect of social infrastructure on investment) on determining if the economy has a unique or two feasible steady-states and whether they are or not stable. Finally, we showed that, for a reasonable calibration set of values for parameters, strengthening the effect of social infrastructure in investment is welfare-improving.

Supplementary Material

Let $x = u_C$, $y = u_S$, $z = u_Y$, $w = \bar{z}$, $e = \psi$ and $f = \sigma$. To compute a Gröbner basis for the ideal defined by the polynomials of the system (3.1) in section 3 we introduce the following commands in SINGULAR:

```
ring R=(0,e,f),(x,y,z,w),lp;

option(redSB);

poly f1=x+(25/1000*e-25/1000*f)*y+25/1000*e*z-82/100*w-25/1000*e+15/1000;

poly f2=x-(5/100+5/100*f)*y-5/100*z-w+1/10;

poly f3=x+(4/45*f-5/100)*y-5/100*z-8/45;

poly f4=e*x*z-64/100*y*w;

ideal I=f1,f2,f3,f4;

ideal G=groebner(I);

G;
```

Then, to compute the polynomial d that defines the variety F (section 3.1.1) we introduce the following commands in SINGULAR:

```
poly g1red = G[1]/leadcoef(G[1]);

poly g2red = G[2]/leadcoef(G[2]);

poly g3red = G[3]/leadcoef(G[3]);

poly g4red = G[4]/leadcoef(G[4]);


ring R=0,(x,y,z,w,f,e),dp;

option(redSB);

poly f1 = x + 1/40*(e-f)*y + 1/40ez - 41/50w - 1/40e + 3/200;
```

```
poly f2 = x - 1/20*(1+f)*y - 1/20z - w + 1/10;

poly f3 = x + (4/45f-1/20)*y - 1/20z - 8/45;

poly f4 = exz - 16/25yw;

ideal II = f1, f2, f3, f4;

ideal sII = groebner(II);

poly d1 = e;

poly d2 = 576e2+1728e+1152;

poly d3 = 2304e3+11520e2+18432e+9216;

poly d4 = 5f;

poly d5 = 5fe+10f;

poly d6 = 100e+200;

LIB "poly.lib";

poly d = lcm(d1,d2,d3,d4,d5,d6);

poly d7 = d/leadcoef(d);

poly t1 = d7*((2304e3+11520e2+18432e
+9216)*w2+(320fe3+1088fe2+896fe-460e3-2588e2-4616e-2560)*w
+(25fe3+70fe2+49fe-50e3-170e2-140e));

poly t2 = d7*((5fe+10f)*z+(-36e-72)*w+(-5fe-7f+10e+20));

poly t3 = d7*((5f)*y+36*w-10);

poly t4 = d7*((100e+200)*x+(-64e-128)*w+(-5e-7));
```

```
reduce(t1,sI);

reduce(t2,sI);

reduce(t3,sI);

reduce(t4,sI);
```

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