To my parents and my brother

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I can not forget all my friends for their enthusiasm and motivation. They were the lower and upper solutions that bound all my problems.

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ACRONYMS AND NOTATIONS

BVP Boundary Value Problem

BVPs Boundary Value Problems

ODE Ordinary Differential Equation

ODEs Ordinary Differential Equations

FDE Functional Differential Equation

FDEs Functional Differential Equations

$$\begin{aligned} \mathbb{R}_0^+ &:= [0, +\infty[\\ \mathbb{R}^+ &:=]0, +\infty[\end{aligned}$$

$$\begin{aligned} AC(\mathbb{R}_0^+) &: \text{Space of absolutely continuous functions } u : \mathbb{R}_0^+ \to \mathbb{R} \\ W^{m,p}(I) &:= \{u \in L^p(I) : u^{(i)} \in L^p(I), i = 1, ..., m\} \end{aligned}$$

ABSTRACT

The relative scarcity of results that guarantee the existence of solutions for BVP on unbounded domains, contrasts with the high applicability on real problems of differential equations defined on the half-line or on the whole real line. It is this gap the main reason that led to this work.

The differential equations studied vary from second order to higher orders and they can be discontinuous on time. Different types of boundary conditions will be discussed herein, for example, Sturm-Liouville, homoclinic, Lidstone and functional conditions.

The non-compactness of the time interval and the possibility of study unbounded functions will require the redefinition of the admissible Banach spaces. In fact the space considered and the functional framework assumed define the set of admissible solutions for each problem under a main goal: the functions must remain bounded for the space and the norm in consideration. This is achieved by defining some weight functions (polynomial or exponential) in the space or assuming some asymptotic behavior.

In addition to the existence, solutions will be localized in a strip. The lower and upper solutions method will play an important role, and combined with other tools like the one-sided Nagumo growth conditions, Green's functions or Schauder's fixed point theorem, provide the existence and location results for differential equations with various boundary conditions.

Different applications to real phenomena will be presented, most of them translated into classical equations as Duffing, Bernoulli-Euler-v.Karman, Fisher-Kolmogorov, Swift-Hohenberg, Emden-Fowler or Falkner-Skan-type equations.

All these applications have a common denominator: they are defined in unbounded intervals and the existing results in the literature are scarce or proven only numerically in discrete problems.

Keywords: Unbounded intervals, Lower and upper solutions, Nagumo condition, Green's function, Fixed point theory.

RESUMO

Problemas de valor na fronteira de ordem superior em intervalos não limitados

A relativa escassez de resultados que garantam a existência de soluções para problemas de valor na fronteira, em domínios ilimitados, contrasta com a alta aplicabilidade em problemas reais de equações diferenciais definidas na semi reta ou em toda a reta real. É esta lacuna o principal motivo que conduziu a este trabalho. As equações diferenciais estudadas variam da segunda ordem a ordens superiores e podem ser descontínuas no tempo. As condições de fronteira aqui analisadas são de diferentes tipos, nomeadamente, Sturm - Liouville, homoclínicas, Lidstone e condições funcionais.

A não compacidade do intervalo de tempo e a possibilidade de estudar funções ilimitadas, exigirá a redefinição dos espaços de Banach admissíveis. Na verdade, o espaço considerado e o quadro funcional assumido define o conjunto de soluções admissíveis para cada problema sob um objetivo principal: as funções devem permanecer limitadas para o espaço e norma considerados. Isto é conseguido através da definição de algumas "funções de peso" (polinomiais ou exponenciais) no espaço considerado ou assumindo um comportamento assintótico. Além da existência, as soluções serão localizadas numa faixa. O método da sub e sobre-soluções irá desempenhar aqui um papel importante e, combinado com outras ferramentas como a condição unilateral de Nagumo, as funções de Green ou o teorema de ponto fixo de Schauder, fornecem a existência e localização de soluções para equações diferenciais com diversas condições de fronteira.

Apresentam-se também diferentes aplicações a fenómenos reais, a maioria deles traduzidos para equações clássicas como as equações de Duffing, Bernoulli-Euler-v.Karman, Fisher-Kolmogorov, Swift - Hohenberg, Emden-Fowler ou ainda Falkner-Skan. Todas estas aplicações têm um denominador comum: são definidas em intervalos ilimitados e os resultados existentes na literatura são raros ou estão provados apenas numericamente em problemas discretos.

Palavras-chave: Intervalos ilimitados, Sub e sobre-soluções, Condição de Nagumo, Funções de Green, Teoria do ponto fixo.

INTRODUCTION

The *leitmotiv* of this work is related with higher order boundary value problems (BVPs) defined on unbounded domains, more precisely on the half-line or on the whole real line.

Roughly speaking, we can say that BVPs are rather different of initial (or final) value problems as they have not a continuous dependence of the boundary data. In fact, small perturbations on boundary values may cause vital changes on the qualitative properties of the corresponding solutions, and even on existence, non-existence or multiplicity of solutions. The following example will illustrate this fact:

Consider the second order homogeneous differential equation

$$y'' + y = 0. \tag{0.0.1}$$

The initial value problem, known as Cauchy problem, composed by (0.0.1) and the initial values

$$y(0) = k_1, \quad y'(0) = k_2$$

has a unique solution given by $y(x) = k_1 \cos x + k_2 \ \sin x$, for every real k_1, k_2.

However the BVP with (0.0.1) and the Dirichlet boundary conditions

$$y(0) = 0, \ \ y(\pi) = \epsilon (\neq 0)$$

has no solution, but the Dirichlet BVP with (0.0.1) and

$$y(0) = 0, \ \ y(\beta) = \epsilon, \ \text{with } 0 < \beta < \pi,$$

has a unique solution, $y(x) = \frac{\epsilon \sin x}{\sin \beta}$, and the BVP composed by (0.0.1) together with the boundary conditions

$$y(0) = 0, \ \ y(\pi) = 0,$$

has infinite solutions, of the type $y(x) = c \sin x$, with arbitrary $c \in \mathbb{R}$.

Last decades the study of BVPs defined on compact intervals has been considered by many authors with application of a huge variety of methods and techniques. However BVPs defined on unbounded intervals are scarce, as they require other type of techniques to overcome the lack of compactness.

Historically, these problems began at the end of 19^{th} century with A. Kneser. This pioneer work described monotone solutions of second order ordinary differential equations. Others followed his results and different techniques have been studied, namely the lower and upper solutions method (see [12] and the references therein).

Several real problems were modeled by these BVPs defined on infinite intervals. As examples, we refer the study of unsteady flow of a gas through a semi-infinite porous medium, the discussion of electrostatic probe measurements in solid-propellant rocket exhausts, the analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, the heat transfer in the radial flow between parallel circular disks, the investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity, as well as numerous problems arising in the study of draining flows, circular membranes, plasma physics, radially symmetric solutions of semilinear elliptic equations, nonlinear mechanics, and non - Newtonian fluid flows, the bending of infinite beams and its applications in the railways and highways. More details and examples can be seen in [5] and the references therein.

This work is divided in three parts, each one related to some type of BVPs on unbounded intervals.

The first part, *Sturm-Liouville boundary value problems on the half-line*, is dedicated to higher order BVPs with the so called Sturm-Liouville boundary conditions defined on the half-line, and it is composed by two chapters:

◇ Chapter 1 - *Third order boundary value problems*. Third order differential equations on infinite intervals can describe the evolution of physical phenomena like draining or coating fluid flow problems. The non-compactness of the time interval and the possibility of studying unbounded functions require the redefinition of the admissible Banach space and its weighted norms. In this chapter it will be proved an existence and localization of, at least, one solution for a BVP with Sturm-Liouville boundary conditions. The tools involved will be the one-sided Nagumo-type growth condition, Green's functions, lower and upper solution method and Schauder's fixed point theorem. An example will finish the chapter.

◇ Chapter 2 - *General n^{th}-order problems*. This chapter arises in the attempt to generalize the previous one to order n. In a particular case, fourth order differential equations can model the bending of an elastic beam. An example is shown to demonstrate the importance of the one-sided Nagumo-type growth condition.

The second part, *Homoclinic solutions and Lidstone problems on the whole real line*, considers BVPs on the whole real line, looking for sufficient conditions on the nonlinearity to guarantee the existence of homoclinic solutions, and its relation to solutions for Lidstone-type problems. It contains three chapters:

◇ Chapter 3 - *Homoclinic solutions for second order problems*. In this chapter it will be used the lower and upper solutions method with unordered functions. An existence and localization result will be settled. Specific applications to Duffing-type equations and beam equations with damping will finish the chapter.

◇ Chapter 4 - *Homoclinic solutions to fourth order problems*. Different problems involving Bernoulli-Euler-v. Karman, Fisher - Kolmogorov or Swift-Hohenberg equations are strongly linked with fourth order differential equations. This chapter will establish existence results and examples for each particular case.

◇ Chapter 5 - *Lidstone boundary value problems*. The Lidstone theory, initially applied to interpolation problems, is considered, in this chapter, in the whole real line with a strong connection to the homoclinic solutions. In this final chapter of this part it will be studied a problem of an infinite beam resting on granular foundations with moving loads.

In the last part, *Functional boundary value problems*, we study BVPs with functional boundary conditions, that is, with boundary data that can depend globally on the correspondent variables. In this way it contains and generalize many types of boundary conditions such as multipoint, advanced or delayed, nonlocal, integro-differential, with maximum or minimum arguments, among others. Part 3 is divided in three chapters, each one with different type of problems:

◇ Chapter 6 - *Second order problems*. BVPs involving functional boundary conditions can model thermal conduction, semiconductor and hydrodynamic problems. An application to a problem composed by an Emden-Fowler-type equation and a infinite multipoint condition will be formulated and solved.

◇ Chapter 7 - *Third order functional problems*. Falkner-Skan equations are obtained from partial differential equations. They can model the behavior of a viscous flow over a plate. Until now, only numerical techniques could deal with this type of problems, however, in this chapter it will be proved an existence and localization result by topological methods.

◇ Chapter 8 - *Phi-Laplacian equations with functional boundary conditions*. This final chapter will deal with weighted norms, namely the Bielecki norm. This will be a fundamental tool to manage unbounded solutions. An important fact is that the homeomorphism ϕ does not need to be surjective.

Throughout this work, the usual Lemma of Arzèla-Ascoli can not be used due to the lack of compactness, and this issue is overcome with some methods, techniques and specific tools. We point out some of them:

- **Weighted spaces** and the corresponding **weighted norms**;
- **Carathéodory functions** admissible for the nonlinearities;
- **Green's functions** on unbounded domains;
- **Equiconvergence at ∞**.

The space considered and the functional framework assumed define the set of admissible solutions for each problem under a main goal: the functions must remain bounded for the space and the norm in consideration. This is achieved by defining some weight functions (polynomial or exponential) in the space or assuming some asymptotic behavior. Therefore, for each problem it is presented the specific space and norm to be used.

The type of nonlinearities in the different problems has a common feature: roughly, they must be measurable in the time variable, continuous almost everywhere, on the space variables, having a growth controlled by a L^1 function on $[0, +\infty[$ or $\mathbb{R}$. A function with such properties is called in the literature, as a L^1-Carathéodory function. To avoid boring repetitions, we define them for a general unbounded interval I (see Definition 1.1.1), which will be the half-line, or the whole real line, according to each problem.

The Green's functions and their properties play a key role in some problems, for what we do more detailed considerations.

Basically these functions are solutions of a linear BVP, homogeneous or not, and they will guarantee the existence of at least one solution, and, moreover, they can provide the explicit expression of the solution for the studied BVP. In a broader sense, they can be seen as a particular case of the so called kernel functions, as they are related with the kernel of linear operators.

When dealing with linear and homogeneous ordinary differential equations on the form

$$Lu(t) = 0 \tag{0.0.2}$$

it is clear that any homogeneous solution is a linear combination of some independent functions (in the same number as the degree of the ODE). However, when the differential equation is non homogeneous

$$Lu(t) = e(t) \tag{0.0.3}$$

it is fundamental to find a particular solution for each function e and then add it to the linear combination referred.

The Green's functions method is due to George Green (1793-1841), the first mathematician to use such kind of kernels to solve BVPs.

If equation (0.0.3) coupled with homogeneous boundary conditions, has only the trivial solution for $e(t) = 0$, then the associated linear operator is invertible and its inverse operator, $L^{-1}e$, is characterized with an integral kernel, $G(t,s)$, called the Green's function. The solution of this problem is then given by

$$u(t) = L^{-1}e(t) := \int_a^b G(t,s)e(s)ds, \forall t \in [a,b]. \tag{0.0.4}$$

A remarkable characteristics of the explicit expression of the Green's functions is the fact that they are independent on the function e. After that, one needs to calculate the integral expression and then is possible to obtain some additional qualitative information about solutions: sign, oscillation properties, *a priori* bounds or their stability. All these issues transform the theory of Green's functions in a fundamental tool in the analysis of differential equations. It has been widely studied in the literature and reveals to be very important in order to use monotone iterative techniques, lower and upper solutions, fixed point theorems or variational methods (see [30] and references therein).

The equiconvergence at ∞, sometimes called as the stability at ∞, is a crucial argument to recover the compacity of the operator on unbounded domains. Indeed, with such concept , we can formulate a criterion that plays the role of the Arzèla-Ascoli theorem for bounded domains. More precisely, if, in some subset M of the space, the functions are uniformly bounded, equicontinuous on some subintervals of $[0,\infty)$ or $\mathbb{R}$, and equiconvergent at ∞, or $\pm\infty$, then M is relatively compact.

As it can easily be seen, the above notion depends on the space considered, the weights defined, and on the order of the derivatives involved. Therefore, for the reader's convenience, we specify in each problem the detailed criterion referred.

Finally, we point out that in all chapters there are examples to illustrated each theorem or, even, concrete applications to real phenomena.

Part I

STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS ON THE HALF-LINE

INTRODUCTION

Sturm-Liouville theory was initiated by Jacques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882), to study second order linear differential equations of the form,

$$\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + (\lambda w(t) - q(t))\, y = 0,$$

where p, q are positive functions, λ is a constant and w is a known function called either the density or weighting function.

The common approach to this equations deals with bounded intervals, that is, $t \in [a,b],\ a,b \in \mathbb{R},\ a < b$, and with boundary conditions of the form

$$c_1 y(a) + c_2 y'(a) = 0, \ \ c_3 y(b) + c_4 y'(b) = 0, \ \ c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

This kind of boundary conditions will, in this first part, be generalized to third and n^{th}-order BVPs, defined on unbounded intervals. Thus, in what follows, BVPs with **Sturm-Liouville boundary conditions** may also be called simply as **Sturm-Liouville problems**.

The great novelty of this part is to assume an one sided Nagumo condition. In fact, the usual bilateral Nagumo condition used in the literature requires a subquadratic growth for the nonlinearities. As far as we know, it is the first time where the unilateral Nagumo conditions are adapted to unbounded domains. In this way, the nonlinearities may have an asymmetric growth, being, for example, asymptotically unbounded for one side, remaining the subquadratic growth in the other side.

This first part is separated into two chapters, both dealing with Sturm-Liouville boundary conditions on the half-line.

In the first chapter it will be proved the existence of at least one solution for a BVP involving a third order differential equation and it is based on [86]. Other properties will be proved for such solutions like localization and asymptotic properties.

Chapter 2 is assigned to a generic n^{th}-order problem, where the main result is an existence and localization result, meaning that, it provides not only the existence, but also the localization of the unknown function and its derivatives, via lower and upper solutions method.

Lower and upper solutions method is an useful technique to deal with BVPs as, from their localization part, it can be obtained some qualitative data about solution variation and behavior (see [24, 49, 73, 82, 83]). Another important tool is the Nagumo condition, useful to obtain *a priori* estimates on some derivative of the solution, generalizing subquadratic growth assumptions on the nonlinear part of the differential equation.

As it can be seen in the references above, the usual growth condition of the Nagumo type is a bilateral one. However the same estimation holds with a similar one-sided assumption, allowing that the BVPs can include unbounded nonlinearities. In this way it generalizes the two-sided condition, as it is proved in [43, 53].

Finally is worth mentioning that, in both chapters, the nonlinearities are L^1-Carathéodory functions and, therefore, they may have discontinuities in time.

1

Third order boundary value problems

Third order differential equations arise in many areas, such as the deflection of an elastic beam having a constant or varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows (see [51] and the references therein).

In infinite intervals, third order BVPs can describe the evolution of physical phenomena, for example some draining or coating fluid-flow problems, (see [102]).

Due to the non-compactness of the interval, the discussion about sufficient conditions for the solvability of BVPs is more delicate. In the literature, existence results to such problems are, mainly, due to extension of continuous solutions on the corresponding finite intervals, under a diagonalization process and fixed point theorems, in special Banach spaces (see [4, 16, 72, 108] and the references therein).

The present chapter will study a general Sturm-Liouville type BVP, composed by a third order differential equation defined on the half line

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \; a.e. \; t \geq 0 \tag{1.0.1}$$

together with boundary conditions

$$u(0) = A, \; au'(0) + bu''(0) = B, \; u''(+\infty) = C, \tag{1.0.2}$$

with $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ a L^1- Carathéodory function (eventually discontinuous on time), where $u''(+\infty) := \lim_{t \to +\infty} u''(t)$, $a, b, A, B, C \in \mathbb{R}$ and $a > 0, b < 0$.

The setback of dealing with unbounded intervals and the possibility of studying unbounded functions can be overcome with new definitions of weighted spaces and norms.

1.1 DEFINITIONS AND AUXILIARY RESULTS

As solutions can be unbounded, the functional framework must be defined with some weight functions and the corresponding weighted norms.

Consider the space

$$X_1 = \left\{ x \in C^2(\mathbb{R}_0^+) : \lim_{t\to+\infty} \frac{x^{(i)}(t)}{\omega_i(t)} \in \mathbb{R},\ i = 0,1,2 \right\}$$

with $\omega_i(t) = 1 + t^{2-i}, i = 0,1,2$ and the norm

$$\|x\|_{X_1} = \max \left\{ \|x\|_0, \|x'\|_1, \|x''\|_2 \right\},$$

where

$$\|y\|_i = \sup_{t\geq 0} \left| \frac{y(t)}{\omega_i(t)} \right|, \text{ for } i = 0,1,2.$$

By standard arguments it can be proved that $(X_1, \|.\|_{X_1})$ is a Banach space.

Let us precise the concept of L^1-Carathéodory functions to be used forward.

Definition 1.1.1. *Let E be a normed space and I an unbounded interval ($I = \mathbb{R}_0^+$ or $I = \mathbb{R}$).*

A function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is L^1- Carathéodory if it verifies

$i)$ for each $\xi \in \mathbb{R}^n$, $t \mapsto f(t,\xi)$ is measurable on I;

$ii)$ for almost every $t \in I$, $\xi \mapsto f(t,\xi)$ is continuous in $\mathbb{R}^n$;

$iii)$ for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1(I)$ such that, for $\|\xi\|_E < \rho$,

$$|f(t,\xi)| \leq \varphi_\rho(t), \text{ a.e. } t \in I.$$

For each particular structure of the space E, and the corresponding norm, condition $iii)$ assume different forms of inequalities.

Let $\gamma_i, \Gamma_i \in C(\mathbb{R}_0^+)$, such that $\gamma_i(t) \leq \Gamma_i(t), \forall t \geq 0, i = 0, 1$ and

$$E_1 = \left\{(t, x_0, x_1, x_2) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \gamma_i(t) \leq x_i \leq \Gamma_i(t), i = 0, 1\right\}.$$

The following one-sided Nagumo condition generalizes the usual bilateral one.

Definition 1.1.2. *A function $f : E_1 \to \mathbb{R}$ is said to satisfy an one-sided Nagumo–type growth condition in E_1 if, for some positive and continuous functions ψ, h and some $\nu > 1$, such that*

$$\int_0^{+\infty} \psi(s)ds < +\infty, \ \sup_{t \geq 0} \psi(t)(1+t)^{\nu} < +\infty, \ \int_0^{+\infty} \frac{s}{h(s)} ds = +\infty, \tag{1.1.1}$$

it verifies either

$$f(t, x, y, z) \leq \psi(t) h(\|z\|_2), \forall (t, x, y, z) \in E_1, \tag{1.1.2}$$

or

$$f(t, x, y, z) \geq -\psi(t) h(\|z\|_2), \forall (t, x, y, z) \in E_1. \tag{1.1.3}$$

An important goal of this condition is to give an *a priori* bound on the second derivative of all existent solutions.

Lemma 1.1.3. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function satisfying (1.1.1) and, either (1.1.2), or (1.1.3), in E_1. Then there exists $R > 0$ (not depending on u) such that every solution u of (1.0.1), (1.0.2) satisfying*

$$\gamma(t) \leq u(t) \leq \Gamma(t), \gamma'(t) \leq u'(t) \leq \Gamma'(t), \forall t \geq 0, \tag{1.1.4}$$

verifies $\|u''\|_2 < R$.

Proof.

Let u be a solution of (1.0.1), (1.0.2) verifying (1.1.4). Consider $r > 0$ such that

$$r > \max\left\{\left|\frac{B - a\Gamma'(0)}{b}\right|, \left|\frac{B - a\gamma'(0)}{b}\right|, |C|\right\}. \tag{1.1.5}$$

By the previous inequality it's impossible that $|u''(t)| > r, \forall t \geq 0$, because

$$|u''(0)| = \left|\frac{B - au'(0)}{b}\right| \leq \max\left\{\left|\frac{B - a\Gamma'(0)}{b}\right|, \left|\frac{B - a\gamma'(0)}{b}\right|\right\} < r.$$

If $|u''(t)| \leq r, \forall t \geq 0$, taking $R > \frac{r}{2}$ the proof is complete as

$$\|u''\|_2 = \sup_{t\geq 0}\left|\frac{u''(t)}{2}\right| \leq \frac{r}{2} < R.$$

In the following, it will be proved that even when there exists $t > 0$ such that $|u''(t)| > r$, the norm $\|u''\|_2$ remains bounded, in all possible cases, either f verifies (1.1.2) or (1.1.3).

Suppose there exists $t > 0$ such that $|u''(t)| > r$, that is $u''(t) > r$ or $u''(t) < -r$. In the first case, by (1.1.1), one can take $R > r$ such that

$$\int_r^R \frac{s}{h(s)}ds > M \max\left\{M_1 + \sup_{t\geq 0}\frac{\Gamma'(t)}{1+t}\frac{\nu}{\nu-1}, M_1 - \inf_{t\geq 0}\frac{\gamma'(t)}{1+t}\frac{\nu}{\nu-1}\right\}$$

with $M := \sup_{t\geq 0}\psi(t)(1+t)^\nu$ and $M_1 := \sup_{t\geq 0}\frac{\Gamma'(t)}{(1+t)^\nu} - \inf_{t\geq 0}\frac{\gamma'(t)}{(1+t)^\nu}$.

If condition (1.1.2) holds, then, by (1.1.5), there are $t_*, t_+ \in \mathbb{R}^+$ such that $t_* < t_+, u''(t_*) = r$ and $u''(t) > r, \forall t \in (t_*, t_+]$. Therefore

$$\begin{aligned}
\int_{u''(t_*)}^{u''(t_+)} \frac{s}{h(s)}ds &= \int_{t_*}^{t_+} \frac{u''(s)}{h(u''(s))}u'''(s)ds \leq \int_{t_*}^{t_+} \psi(s)u''(s)ds \\
&\leq M\int_{t_*}^{t_+}\frac{u''(s)}{(1+s)^\nu}ds \\
&= M\int_{t_*}^{t_+}\left[\left(\frac{u'(s)}{(1+s)^\nu}\right)' + \frac{\nu u'(s)}{(1+s)^{1+\nu}}\right]ds \\
&\leq M\left(M_1 + \sup_{t\geq 0}\frac{\Gamma'(t)}{1+t}\int_0^{+\infty}\frac{\nu}{(1+s)^\nu}ds\right) < \int_r^R \frac{s}{h(s)}ds.
\end{aligned}$$

Consequently $u''(t_+) < R$ and as t_* and t_+ are arbitrary in $\mathbb{R}^+$, then $u''(t) < R, \forall t > 0$. Similarly, it can be proved the case where there are $t_-, t_* \in \mathbb{R}^+$ such that $t_- < t_*$ and $u''(t_*) = -r, u''(t) < -r, \forall t \in (t_-, t_*)$.

Therefore $\|u''\|_2 < \frac{R}{2} < R, \forall t \geq 0$.

Now consider that f verifies (1.1.3). By (1.1.5), consider that there are $t_-, t_* \in \mathbb{R}^+$ such that $t_- < t_*$ and $u''(t_*) = r, u''(t) > r, \forall t \in (t_-, t_*)$. Therefore, following similar steps as before

$$\begin{aligned}\int_{u''(t_*)}^{u''(t_-)} \frac{s}{h(s)} ds &= \int_{t_*}^{t_-} \frac{u''(s)}{h(u''(s))} u'''(s) ds \leq \int_{t_-}^{t_*} \psi(s) u''(s) ds \\ &\leq \int_{t_-}^{t_*} \psi(s) u''(s) ds \leq M \int_{t_-}^{t_*} \frac{u''(s)}{(1+s)^{\nu}} ds \quad (1.1.6) \\ &= M \left(M_1 + \sup_{t \geq 0} \frac{\Gamma'(t)}{1+t} \frac{\nu}{\nu - 1} \right) < \int_r^R \frac{s}{h(s)} ds.\end{aligned}$$

So $u''(t_-) < R$ and by the arbitrariness of t_- and t_* in $\mathbb{R}^+$, then $u''(t) < R, \forall t > 0$. The case where there are $t_*, t_+ \in \mathbb{R}^+$, with $t_* < t_+$, such that $u''(t_*) = -r, u''(t) < -r, \forall t \in (t_*, t_+]$ is proved in the same way. □

The exact solution for the associated linear problem can be obtained by Green's functions method.

Lemma 1.1.4. *If $e \in L^1(\mathbb{R}_0^+)$, then the BVP*

$$\begin{cases} u'''(t) + e(t) = 0, \ t \geq 0, \\ u(0) = A, \ au'(0) + bu''(0) = B, \ u''(+\infty) = C \end{cases} \quad (1.1.7)$$

has a unique solution in X_1. Moreover, this solution can be expressed as

$$u(t) = g(t) + \int_0^{+\infty} G(t,s) e(s) ds \quad (1.1.8)$$

where

$$g(t) = \frac{Ct^2}{2} + \frac{B - bC}{a} t + A, \ G(t,s) = \begin{cases} -\frac{b}{a} t + st - \frac{s^2}{2}, 0 \leq s \leq t \\ \frac{1}{2} t^2 - \frac{b}{a} t, \quad 0 \leq t \leq s < +\infty. \end{cases}$$

Moreover, $u'(t) = g'(t) + \int_0^{+\infty} G_1(t,s) e(s) ds$ with

$$G_1(t,s) = \begin{cases} -\frac{b}{a} + s, \quad 0 \leq s \leq t \\ -\frac{b}{a} + t, \quad 0 \leq t \leq s < +\infty. \end{cases} \quad (1.1.9)$$

The lack of compactness is overcome by the following lemma which gives a general criterium for relative compactness, (see [4]).

Lemma 1.1.5. *A set $M \subset X_1$ is relatively compact if the following conditions hold:*

i) all functions from M are uniformly bounded;

ii) all functions from M are equicontinuous on any compact interval of $\mathbb{R}_0^+$;

iii) all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that

$$\left| \frac{u^{(i)}(t)}{\omega_i(t)} - \frac{u^{(i)}(+\infty)}{\omega_i(+\infty)} \right| < \epsilon,$$

for all $t > t_\epsilon$, $u \in M$ and $i = 0, 1, 2$.

The well known Schauder's fixed point theorem will be the existence tool:

Theorem 1.1.6 ([112])**.** *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P as at least one fixed point in Y.*

An important tool to bound the solution and its derivatives is the Lower and Upper Solution Method. Let us define the usual lower and upper functions:

Definition 1.1.7. *Given $a > 0, b < 0$, and $A, B, C \in \mathbb{R}$, a function $\alpha \in C^3(\mathbb{R}_0^+) \cap X_1$ is a lower solution of problem (1.0.1), (1.0.2) if*

$$\begin{cases} \alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t)), t \geq 0, \\ \alpha(0) \leq A, \ a\alpha'(0) + b\alpha''(0) \leq B, \ \alpha''(+\infty) < C. \end{cases}$$

A function $\beta \in C^3(\mathbb{R}_0^+) \cap X_1$ is an upper solution if it satisfies the reversed inequalities.

1.2 EXISTENCE AND LOCALIZATION RESULT

The main result of this chapter will be given by next theorem.

Theorem 1.2.1. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function. Suppose there are $\alpha, \beta \in C^3(\mathbb{R}_0^+) \cap X_1$ lower and upper solutions of the problem (1.0.1), (1.0.2), respectively, such that*

$$\alpha'(t) \leq \beta'(t), \forall t \geq 0. \tag{1.2.1}$$

If f verifies the one-sided Nagumo condition (1.1.2), or (1.1.3), in the set

$$E_* = \left\{(t,x,y,z) \in \mathbb{R}_0^+ \times \mathbb{R}^3, \alpha(t) \leq x \leq \beta(t), \alpha'(t) \leq y \leq \beta'(t)\right\},$$

and

$$f(t,\alpha(t),y,z) \geq f(t,x,y,z) \geq f(t,\beta(t),y,z), \tag{1.2.2}$$

for (t,y,z) fixed and $\alpha(t) \leq x \leq \beta(t)$, then the problem (1.0.1), (1.0.2) has at least one solution $u \in C^3(\mathbb{R}_0^+) \cap X_1$ and there exists $R > 0$ such that

$$\alpha(t) \leq u(t) \leq \beta(t),\ \alpha'(t) \leq u'(t) \leq \beta'(t),\ \|u''\|_2 < R,\ \forall t \geq 0.$$

Remark 1.2.2. *By 1.2.1 and Definition 1.1.7 next inequality is valid*

$$\alpha(t) \leq \beta(t), \forall t \geq 0$$

and, therefore, E_* is well defined and inequalities 1.2.2 make sense.

Proof.

Let $\alpha, \beta \in C^3(\mathbb{R}_0^+) \cap X_1$ be, respectively, lower and upper solutions of (1.0.1), (1.0.2) verifying (1.2.1).

Consider the truncated and perturbed equation

$$u'''(t) = f\left(t, \delta_0(t), \delta_1(t), u''(t)\right) + \frac{1}{1+t^2}\frac{u'(t)-\delta_1(t)}{1+|u'(t)-\delta_1(t)|},\ t \geq 0 \tag{1.2.3}$$

where functions $\delta_j : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}, j = 0, 1$, are given by

$$\delta_j(t) := \delta_j(t, u(t)) = \begin{cases} \beta^{(j)}(t) & , u^{(j)}(t) > \beta^{(j)}(t) \\ u^{(j)}(t) & , \alpha^{(j)}(t) \leq u^{(j)}(t) \leq \beta^{(j)}(t) \\ \alpha^{(j)}(t) & , u^{(j)}(t) < \alpha^{(j)}(t). \end{cases} \quad (1.2.4)$$

Notice that the relation $\alpha(t) \leq \beta(t)$ is obtained by integration from (1.2.1), by the boundary conditions (1.0.2) and by Definition 1.1.7.

The proof will include three steps:

Step 1: *If u is a solution of problem (1.2.3), (1.0.2), then*

$$\alpha(t) \leq u(t) \leq \beta(t), \; \alpha'(t) \leq u'(t) \leq \beta'(t), \; \forall t \geq 0.$$

Suppose, by contradiction, that there exists $t \in \mathbb{R}_0^+$ with $\alpha'(t) > u'(t)$ and define

$$\inf_{t \geq 0}(u'(t) - \alpha'(t)) = u'(t_*) - \alpha'(t_*) < 0.$$

- If $t_* \in \mathbb{R}^+$ then $u''(t_*) = \alpha''(t_*)$ and $u'''(t_*) - \alpha'''(t_*) \geq 0$. Therefore, by (1.2.2) and Definition 1.1.7, the following contradiction holds

$$\begin{aligned} 0 &\leq u'''(t_*) - \alpha'''(t_*) \\ &= f(t_*, \delta_0(t_*), \delta_1(t_*), u''(t_*)) + \frac{1}{1+t_*^2} \frac{u'(t_*) - \alpha'(t_*)}{1 + |u'(t_*) - \alpha'(t_*)|} - \alpha'''(t_*) \\ &\leq f(t_*, \alpha(t_*), \alpha'(t_*), \alpha''(t_*)) + \frac{1}{1+t_*^2} \frac{u'(t_*) - \alpha'(t_*)}{1 + |u'(t_*) - \alpha'(t_*)|} - \alpha'''(t_*) \\ &\leq \frac{1}{1+t_*^2} \frac{u'(t_*) - \alpha'(t_*)}{1 + |u'(t_*) - \alpha'(t_*)|} < 0. \end{aligned}$$

- If $t_* = 0$,

$$\min_{t \geq 0}(u'(t) - \alpha'(t)) := u'(0) - \alpha'(0) < 0$$

and

$$u''(0) - \alpha''(0) \geq 0.$$

By Definition 1.1.7 and since $a > 0, b < 0$, it yields the contradiction

$$\begin{aligned} 0 &\geq bu''(0) - b\alpha''(0) \geq B - au'(0) - B + a\alpha'(0) \\ &= a(\alpha'(0) - u'(0)) > 0. \end{aligned}$$

- If $t_* = +\infty$

$$\inf_{t\geq 0}(u'(t) - \alpha'(t)) := u'(+\infty) - \alpha'(+\infty) < 0,$$

$$u''(+\infty) - \alpha''(+\infty) \leq 0,$$

and the following contradiction holds

$$0 \geq u''(+\infty) - \alpha''(+\infty) > C - C = 0.$$

So $\alpha'(t) \leq u'(t), \forall t \geq 0$. In a similar way it can be proved that $\beta'(t) \geq u'(t), \forall t \geq 0$.

Integrating $\alpha'(t) \leq u'(t) \leq \beta'(t)$ on $[0, t]$ for $t \geq 0$, by (1.0.2) and Definition 1.1.7 it can be proved that $\alpha(t) \leq u(t) \leq \beta(t), \forall t \geq 0$.

Step 2: *If u is a solution of the modified problem (1.2.3), (1.0.2) then there exists $R > 0$, not depending on u , such that*

$$\left\|u''\right\|_2 < R. \tag{1.2.5}$$

By the previous step, all solutions of equation (1.2.3) are solutions of (1.0.1), and as f verifies the one-sided Nagumo condition (1.1.2), or (1.1.3), this claim is a direct application of Lemma 1.1.3.

Step 3: *Problem (1.2.3), (1.0.2) has at least one solution.*

Take $\rho > \max\{\|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \|\beta'\|_1, R\}$, with R given by (1.2.5).

Define the operator $T : X_1 \to X_1$, given by

$$Tu(t) = g(t) + \int_0^{+\infty} G(t,s)F(u(s))ds,$$

with

$$g(t) := \frac{C}{2}t^2 + \frac{B - bC}{a}t + A$$

and

$$F(u(s)) := f(s,\delta_0(s),\delta_1(s),u''(s)) + \frac{1}{1+s^2}\frac{u'(s)-\delta_1(s)}{1+|u'(s)-\delta_1(s)|}.$$

As f is a L^1-Carathéodory function, for any $u \in X_1$ with $\|u\|_{X_1} < \rho$, then $F \in L^1$ because

$$\begin{aligned}\int_0^{+\infty} |F(u(s))|\, ds &\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2}\frac{|u'(s)-\delta_1(s)|}{1+|u'(s)-\delta_1(s)|} ds \\ &\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2} ds < +\infty. \qquad (1.2.6)\end{aligned}$$

By Lemma 1.1.4, the fixed points of T are solutions of problem (1.2.3), (1.0.2). So it is enough to prove that T has a fixed point.

Claim 1: *$T : X_1 \to X_1$ is well defined.*

By Lebesgue dominated theorem and Lemma 1.1.4,

$$\lim_{t\to+\infty} \frac{(Tu)(t)}{1+t^2} \leq \frac{C}{2} + \frac{1}{2}\int_0^{+\infty} F(u(s))ds < +\infty.$$

Analogously, by (1.1.9),

$$\begin{aligned}\lim_{t\to+\infty} \frac{(Tu)'(t)}{1+t} &= \lim_{t\to+\infty} \frac{g'(t)}{1+t} + \int_0^{+\infty} \lim_{t\to+\infty} \frac{G_1(t,s)}{1+t} F(u(s))ds \\ &\leq C + \int_0^{+\infty} F(u(s))ds < +\infty,\end{aligned}$$

and

$$\lim_{t\to+\infty} \frac{(Tu)''(t)}{2} \leq \frac{C}{2} + \frac{1}{2}\lim_{t\to+\infty}\int_t^{+\infty} F(u(s))ds = \frac{C}{2} < +\infty.$$

Therefore $Tu \in X_1$.

Claim 2: *T is continuous.*

Consider a convergent sequence $u_n \to u$ in X_1. Then there exists $r_1 > 0$ such that $\|u_n\|_{X_1} < r_1$ and

$$\begin{aligned}\|Tu_n - Tu\|_{X_1} &\leq \int_0^{+\infty} \max\left\{\begin{array}{c} \sup_{t\geq 0}\left|\frac{G(t,s)}{1+t^2}\right|, \\ \sup_{t\geq 0}\left|\frac{G_1(t,s)}{1+t}\right|, \frac{1}{2}\end{array}\right\} |F(u_n(s)) - F(u(s))|\, ds \\ &\leq \int_0^{+\infty} |F(u_n(s)) - F(u(s))|\, ds \longrightarrow 0\,, \qquad (1.2.7)\end{aligned}$$

as $n \to +\infty$.

Claim 3: *T is compact.*

Let
$$M(s) := \max\left\{\sup_{t\geq 0}\frac{|G(t,s)|}{1+t^2}, \sup_{t\geq 0}\frac{|G_1(t,s)|}{1+t}\right\}.$$

Consider a bounded set $B \subset X_1$ defined by $B := \{u \in X_1 : \|u\|_{X_1} < r_1\}$, for some $r_1 > 0$ such that
$$r_1 > \max\left\{\rho, \frac{|C|}{2} + \int_0^{+\infty} M(s)\left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds\right\},$$
with ρ given by (1.2.6).

Claim 3.1: *TB is uniformly bounded.*

For any $u \in B$, as $\|\alpha\|_0 \leq \|\delta_0\|_0 \leq \|\beta\|_0$, $\|\alpha'\|_1 \leq \|\delta_1\|_1 \leq \|\beta'\|_1$, by (1.1.2) one has
$$\begin{aligned}\|Tu\|_0 &= \sup_{t\geq 0}\frac{|Tu(t)|}{1+t^2} \leq \sup_{t\geq 0}\frac{|g(t)|}{1+t^2} + \int_0^{+\infty}\sup_{t\geq 0}\frac{|G(t,s)|}{1+t^2}|F(u(s))|\, ds \\ &\leq \frac{|C|}{2} + \int_0^{+\infty} M(s)\left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds < r_1,\end{aligned}$$

$$\begin{aligned}\|Tu\|_1 &= \sup_{t\geq 0}\frac{|(Tu)'(t)|}{1+t} \leq \sup_{t\geq 0}\frac{|g'(t)|}{1+t} + \int_0^{+\infty}\sup_{t\geq 0}\frac{|G_1(t,s)|}{1+t}|F(u(s))|\, ds \\ &\leq |C| + \int_0^{+\infty} M(s)\left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds < r_1,\end{aligned}$$

and
$$\|Tu\|_2 = \sup_{t\geq 0}\frac{|(Tu)''(t)|}{2} \leq \frac{|C|}{2} < r_1.$$

Thus $\|Tu\|_{X_1} < r_1$, TB is uniformly bounded, and, moreover, $TB \subset B$.

Claim 3.2: *TB is equicontinuous.*

For $T > 0$ and $t_1, t_2 \in [0, T]$,

$$\left|\frac{Tu(t_1)}{1+t_1^2}-\frac{Tu(t_2)}{1+t_2^2}\right| \leq \left|\frac{g(t_1)}{1+t_1^2}-\frac{g(t_2)}{1+t_2^2}\right| + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^2}-\frac{G(t_2,s)}{1+t_2^2}\right| |F(u(s))|\, ds \longrightarrow 0,\ as\ t_1 \to t_2.$$

Analogously

$$\left|\frac{(Tu)'(t_1)}{1+t_1}-\frac{(Tu)'(t_2)}{1+t_2}\right| = \left|\frac{g'(t_1)}{1+t_1}-\frac{g'(t_2)}{1+t_2}\right| + \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1}-\frac{G_1(t_2,s)}{1+t_2}\right| |F(u(s))|\, ds \longrightarrow 0,\ as\ t_1 \to t_2$$

and

$$\left|\frac{(Tu)''(t_1)}{2}-\frac{(Tu)''(t_2)}{2}\right| = \left|\int_{t_1}^{t_2} F(s)ds\right|$$

$$\leq \int_{t_1}^{t_2}\left(\varphi_\rho(s)+\frac{1}{1+s^2}\right) ds \longrightarrow 0,\ as\ t_1 \to t_2.$$

Claim 3.3: *TB is equiconvergent at infinity.*

Indeed,

$$\left|\frac{Tu(t)}{1+t^2}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t^2}\right| \leq \left|\frac{g(t)}{1+t^2}-\frac{C}{2}\right| + \int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2}-\lim_{t\to+\infty}\frac{G(t,s)}{1+t^2}\right| |F(u(s))|\, ds \longrightarrow 0, \quad as \quad t \to +\infty,$$

$$\left|\frac{(Tu)'(t)}{1+t}-\lim_{t\to+\infty}\frac{(Tu)'(t)}{1+t}\right| \leq \left|\frac{g'(t)}{1+t}-C\right| + \int_0^{+\infty}\left|\frac{G_1(t,s)}{1+t}-\lim_{t\to+\infty}\frac{G_1(t,s)}{1+t}\right| |F(u(s))| ds \longrightarrow 0, \quad as \quad t \to +\infty,$$

and

$$\left| \frac{(Tu)''(t)}{2} - \lim_{t \to +\infty} (Tu)''(t) \right| = \left| \int_t^{+\infty} F(u(s))ds \right|$$
$$\leq \int_t^{+\infty} \left(\varphi_\rho(s) + \frac{1}{1+s^2} \right) ds \longrightarrow 0, \ as \ \ t \to +\infty.$$

So, by Lemma 1.1.5, TB is relatively compact.

As T is completely continuous then by Schauder Fixed Point Theorem 1.1.6, T has at least one fixed point $u \in X_1$. □

1.3 EXAMPLE

Consider the next third order BVP

$$\begin{cases} u'''(t) = \frac{1}{(t+1)^2} \left(-\arctan\left(u(t)\right) - 10|u''(t)|e^{u''(t)} \right), t \geq 0, \\ u(0) = A, \ au'(0) + bu''(0) = B, \ u''(+\infty) = C, \end{cases} \tag{1.3.1}$$

with $A \in (-1, 0]$, $a > 0$, $b < 0$ such that $-2(a+b) \leq B \leq 0$ and $C \in (-2, 0)$.

Define

$$E_{ex1} = \left\{ (t, x, y, z) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : -(t+1)^2 \leq x \leq 0, -2t - 2 \leq y \leq 0 \right\}.$$

Function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(t, x, y, z) := \frac{1}{(t+1)^2} \left(-\arctan x - 10|z|e^z \right),$$

verifies on E_{ex1} the inequality $|f(t, x, y, z)| \leq \frac{K_\rho}{(t+1)^2} := \varphi_\rho(t)$, for some $K_\rho > 0$ and ρ such that $\max\{2, \|z\|_2\} < \rho$. Therefore f is L^1-Carathéodory.

Functions $\alpha(t) = -(t+1)^2$ and $\beta(t) \equiv 0$ are, respectively, lower and upper solutions of problem (1.3.1) with $\alpha(t) \leq \beta(t)$ and $\alpha'(t) \leq \beta'(t)$, $\forall t \geq 0$, verifying (1.2.2).

As

$$f(t, x, y, z) \leq \frac{1}{(t+1)^2} \frac{\pi}{2},$$

the one-sided Nagumo-type growth condition (1.1.2) holds in E_{ex1} with

$$\psi(t) := \frac{1}{(t+1)^2},\ \nu \in (1,2),\ \text{and } h(|z|) := \frac{\pi}{2}.$$

Therefore, by Theorem 1.2.1, there is at least a solution u of (1.3.1) with

$$-(t+1)^2 \leq u(t) \leq 0,\ -2t-2 \leq u'(t) \leq 0,\ \|u''\|_2 < R,\ \forall t \geq 0.$$

Moreover, from the localization part of the theorem, one can precise some qualitative properties of this solution: it is nonpositive, nonincreasing and, as $C \neq 0$, this solution is unbounded.

Notice that f does not satisfy the usual two-sided Nagumo-type condition. In fact, if there exist $\psi_1, h_1 \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ satisfying

$$|f(t,x,y,z)| \leq \psi_1(t)\ h_1(|z|), \forall (t,x,y,z) \in E_{ex1},$$

with $\int_0^{+\infty} \frac{s}{h_1(s)} ds = +\infty$, then, in particular,

$$-f(t,x,y,z) \leq \psi_1(t)\ h_1(|z|),\ \forall (t,x,y,z) \in E_{ex1}.$$

So, for $x = 0,\ y, z \in \mathbb{R}$, one has

$$-f(t,0,y,z) = \frac{10}{(t+1)^2}|z|e^z \leq \psi_1(t)\ h_1(|z|).$$

Considering $\psi_1(t) := \frac{1}{(t+1)^2}$, the following contradiction holds:

$$+\infty > \int_0^{+\infty} \frac{s}{10se^s} ds \geq \int_0^{+\infty} \frac{s}{h_1(s)} ds = +\infty.$$

2

General n^{th}-order problems

Like on the previous chapter, n^{th}-order BVPs on infinite intervals occur in different areas. For example fourth order differential equations can model the bending of an elastic beam and, in this sense, they are called beam equations. Other higher order problems are related with the study of radially symmetric solutions of nonlinear elliptic equations, fluid dynamics, boundary layer theory, semiconductor circuits and soil mechanics, either on bounded domains (see [11, 31, 43, 83]), either on the real line ([3, 36, 62, 63, 74]).

The study of BVPs on bounded domains is vast but on infinite intervals is scarce. Different methods like fixed point theorems, shooting methods, upper and lower technique, are used to prove the existence of solutions. However, these solutions are usually bounded.

Lower and upper solutions method, coupled with the Nagumo-type condition, guarantee the existence of at least one solution lying on the strip defined by lower and upper solutions (see [74]) but, to the best of our knowledge, there are no results when the nonlinearity satisfies only the one-sided Nagumo-type condition, on unbounded intervals.

This chapter concerns the study of a general Sturm-Liouville type BVP composed by a n^{th}-order differential fully equation defined on the half line ($n \geq 2$)

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \quad a.e. \ t \geq 0 \tag{2.0.1}$$

and

$$\begin{cases} u^{(i)}(0) = A_i, \\ u^{(n-2)}(0) + a u^{(n-1)}(0) = B, \\ u^{(n-1)}(+\infty) = C, \end{cases} \tag{2.0.2}$$

with $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ a L^1- Carathéodory function, $a < 0$, A_i, B, $C \in \mathbb{R}$ for $i = 0, 1, \ldots, n-3$, and $u^{(n-1)}(+\infty) := \lim_{t \to +\infty} u^{(n-1)}(t)$.

The functional setting will be adapted to the n^{th}-order case, namely the weight space, the corresponding norms and the notion of L^1-Carathéodory.

As an application of this result, we include a particular case of a fourth order problem with a beam equation, referred in [34].

2.1 PRELIMINARY RESULTS

A new admissible space will be needed:

For polynomial functions $\omega_i(t) = 1 + t^{n-1-i}, i = 0, 1, ..., n-1$ let us define the space

$$X_2 = \left\{ x \in C^{n-1}(\mathbb{R}_0^+) : \lim_{t \to +\infty} \frac{x^{(i)}(t)}{\omega_i(t)} \in \mathbb{R}, \ i = 0, 1, ..., n-1 \right\}$$

with the norm $\|x\|_{X_2} = \max \left\{ \|x\|_0, \|x'\|_1, ..., \|x^{(n-1)}\|_{n-1} \right\}$, where

$$\|y\|_i = \sup_{t \geq 0} \left| \frac{y(t)}{\omega_i(t)} \right|, \text{ for } i = 0, 1, ..., n-1.$$

It is clear that $(X_2, \|.\|_{X_2})$ is a Banach space.

Let $\gamma_i, \Gamma_i \in C(\mathbb{R}_0^+), \gamma_i(t) \leq \Gamma_i(t), \forall t \geq 0, \ i = 0, 1, ..., n-2$ and define

$$E_2 = \left\{ (t, x_0, ..., x_{n-1}) \in \mathbb{R}_0^+ \times \mathbb{R}^n : \gamma_i(t) \leq x_i \leq \Gamma_i(t), i = 0, 1, ..., n-2 \right\}.$$

Now the one-sided growth condition can be formulated in the following way:

Definition 2.1.1. *A function $f : E_2 \to \mathbb{R}$ is said to satisfy an one-sided Nagumo-type growth condition in E_2 if, for some positive and continuous functions ψ, h and some $\nu > 1$, such that*

$$\int_0^{+\infty} \psi(s) ds < +\infty, \ \sup_{t \geq 0} \psi(t)(1+t)^\nu < +\infty, \ \int_0^{+\infty} \frac{s}{h(s)} ds = +\infty, \tag{2.1.1}$$

it verifies either

$$f(t, x_0, ..., x_{n-1}) \leq \psi(t) h(\|x_{n-1}\|_{n-1}), \forall (t, x_0, ..., x_{n-1}) \in E_2, \quad (2.1.2)$$

or

$$f(t, x_0, ..., x_{n-1}) \geq -\psi(t) h(\|x_{n-1}\|_{n-1}), \forall (t, x_0, ..., x_{n-1}) \in E_2. \quad (2.1.3)$$

Now the *a priori* estimation is obtained on $u^{(n-1)}$, given by the following lemma, which proof follows the same technique as in Lemma 1.1.3 and, for this reason, is omitted.

Lemma 2.1.2. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ be a L^1- Carathéodory function satisfying (2.1.1) and (2.1.2), or (2.1.3), in E_2. Then there exists $R > 0$ (not depending on u) such that every u solution of (2.0.1), (2.0.2) satisfying*

$$\gamma_i(t) \leq u^{(i)}(t) \leq \Gamma_i(t), \forall t \geq 0, \; i = 0, 1, ..., n-2, \quad (2.1.4)$$

verifies $\left\| u^{(n-1)} \right\|_{n-1} < R$.

The exact solution for the associated linear problem can be obtained by a Green function.

Lemma 2.1.3. *If $e \in L^1(\mathbb{R}_0^+)$, then the BVP*

$$\begin{cases} u^{(n)}(t) + e(t) = 0, \; a.e. \; t \geq 0, \\ u^{(i)}(0) = A_i, \; i = 0, 1, ..., n-3, \\ u^{(n-2)}(0) + a u^{(n-1)}(0) = B, \\ u^{(n-1)}(+\infty) = C \end{cases} \quad (2.1.5)$$

has a unique solution in X_2. Moreover, this solution can be expressed as

$$u(t) = p(t) + \int_0^{+\infty} G(t,s) e(s) ds \quad (2.1.6)$$

where

$$p(t) = \sum_{k=0}^{n-3} \frac{A_k}{k!} t^k + \frac{B - aC}{(n-2)!} t^{n-2} + \frac{C}{(n-1)!} t^{n-1}$$

and

$$G(t,s)=\begin{cases}\sum\limits_{k=0}^{n-2}\dfrac{(-1)^k}{(k+1)!(n-2-k)!}s^{k+1}t^{n-2-k}-\dfrac{at^{n-2}}{(n-2)!}, 0\le s\le t\ <+\infty\\ \frac{1}{(n-1)!}t^{n-1}-\frac{a}{(n-2)!}t^{n-2}, \qquad 0\le t\le s<+\infty.\end{cases}$$

General n^{th}-order definitions of lower and upper functions are presented next.

Definition 2.1.4. *Given $a<0$ and $A_i, B, C\in\mathbb{R}, i=0,1,...,n-3$, a function $\alpha\in C^n(\mathbb{R}_0^+)\cap X_2$ is a lower solution of problem (2.0.1), (2.0.2) if*

$$\begin{cases}\alpha^{(n)}(t)\ge f(t,\alpha(t),\alpha'(t),...,\alpha^{(n-1)}(t)),\ t\ge 0,\\ \alpha^{(i)}(0)\le A_i,\\ \alpha^{(n-2)}(0)+a\alpha^{(n-1)}(0)\le B,\\ \alpha^{(n-1)}(+\infty)<C.\end{cases}$$

A function $\beta\in C^n(\mathbb{R}_0^+)\cap X_2$ is an upper solution if it satisfies the reversed inequalities.

2.2 EXISTENCE AND LOCALIZATION RESULT

The existence theorem to the n^{th}-order case follows similar arguments of Theorem 1.2.1, and the proof is omitted.

Theorem 2.2.1. *Let $f:\mathbb{R}_0^+\times\mathbb{R}^n\to\mathbb{R}$ be a L^1-Carathéodory function. Suppose there are $\alpha,\beta\in C^n(\mathbb{R}_0^+)\cap X_2$ lower and upper solutions of the problem (2.0.1), (2.0.2), respectively, such that*

$$\alpha^{(n-2)}(t)\le\beta^{(n-2)}(t),\forall t\ge 0. \tag{2.2.1}$$

If f verifies the one-sided Nagumo condition (2.1.2), or (2.1.3), in the set

$$E_*=\left\{(t,x_0,...,x_{n-1})\in\mathbb{R}_0^+\times\mathbb{R}^n:\alpha^{(i)}(t)\le x_i\le\beta^{(i)}(t), i=0,...,n-2\right\},$$

and

$$f(t, \alpha(t), ..., \alpha^{(i)}(t), ..., u_{n-2}, u_{n-1}) \geq f(t, u_0, ..., u_i, ..., u_{n-2}, u_{n-1}) \quad (2.2.2)$$
$$\geq f(t, \beta(t), ..., \beta^{(i)}(t), ..., u_{n-2}, u_{n-1}),$$

for (t, u_{n-2}, u_{n-1}) *fixed when* $\alpha^{(i)}(t) \leq u_i \leq \beta^{(i)}(t), i = 0, ...n-3$, *then problem (2.0.1), (2.0.2) has at least one solution* $u \in C^n(\mathbb{R}_0^+) \cap X_2$ *and there exists* $R > 0$ *such that*

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), i = 0, 1, ...n-2 \text{ and } \left\| u^{(n-1)} \right\|_{n-1} < R, \ \forall t \geq 0.$$

Remark 2.2.2. *Notice that by integration on* $[0, t]$ *of (2.2.1) and Definition 2.1.4, lower and upper solutions and their derivatives (until order* $n-3$*) are well ordered, that is,*

$$\alpha^{(i)}(t) \leq \beta^{(i)}(t), \ i = 0, 1, ..., n-3, \ \forall t \geq 0$$

and E_* *is well defined.*

2.3 EXAMPLE

Consider the next fourth order BVP

$$\begin{cases} u^{(iv)}(t) = \frac{-u(t)|u'''(t) - 6|e^{u'''(t)} - e^{-t}(6t + 2 - u''(t))}{1+t^2}, & t \geq 0, \\ u(0) = A, \ u'(0) = 0, \ u''(0) + au'''(0) = 0, \ u'''(+\infty) = C, \end{cases} \quad (2.3.1)$$

with $A \geq 0$, $-\frac{1}{3} \leq a < 0$ and $0 < C < 6$.

This BVP is a particular case of (2.0.1), (2.0.2) with $A_0 = A, A_1 = 0, B = 0$ and

$$f(t, x, y, z, w) = \frac{-x|w - 6|e^w - e^{-t}(6t + 2 - z)}{1 + t^2}. \quad (2.3.2)$$

Moreover, functions $\alpha(t) \equiv A$ and $\beta(t) = t^3 + t^2 + A$ are, respectively, lower and upper solutions for (2.3.1), and Nagumo condition with (2.1.2) is verified with

$$\psi(t) = \frac{1}{1 + t^2}, \ 1 < \nu < 2, \ h(|w|) \equiv 1,$$

on

$$E_{ex2} = \left\{ (t,x,y,z,w) \in \mathbb{R}_0^+ \times \mathbb{R}^4 : \begin{array}{c} A \leq x \leq t^3 + t^2 + A, \\ 0 \leq y \leq 3t^2 + 2t, \\ 0 \leq z \leq 6t + 2 \end{array} \right\}.$$

Also f verifies (2.2.2) and all assumptions of Theorem 2.2.1 are fulfilled, therefore, there is at least a non trivial solution u of (2.3.1) such that

$$\begin{gathered} A \leq u(t) \leq t^3 + t^2 + A, \\ 0 \leq u'(t) \leq 3t^2 + 2t, \\ 0 \leq u''(t) \leq 6t + 2, \\ \|u'''\|_3 \leq R, \ \forall t \geq 0. \end{gathered}$$

Remark that, this solution is unbounded and, from the location part, it is nondecreasing and convex.

It is important to stress that the nonlinearity (2.3.2) does not satisfy the usual two-sided Nagumo-type condition. Therefore the existent results in the literature can not be applied to problem (2.3.1).

In fact, if there exist $\psi_2, h_2 \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ satisfying

$$|f(t,x,y,z,w)| \leq \psi_2(t) h_2(|w|), \forall (t,x,y,z,w) \in E_{ex2},$$

with $\displaystyle\int_0^{+\infty} \frac{s}{h_2(s)} ds = +\infty$ then, in particular,

$$-f(t,x,y,z,w) \leq \psi_2(t) h_2(|w|),$$

and, for $t \geq 0, \ x = 1, \ 0 \leq y \leq 3t^2 + 2t, \ z = 6t + 2$, and $w \in \mathbb{R}$,

$$-f(t,1,y,6t+2,w) = \frac{|w-6|e^w}{1+t^2} \leq \psi_2(t) h_2(|w|),$$

For $\psi_2(t) = \frac{1}{1+t^2}$ one has $|w-6|e^w \leq h_2(|w|)$ and the following contradiction holds

$$+\infty > \int_0^{+\infty} \frac{s}{(s-6)e^s} ds \geq \int_0^{+\infty} \frac{s}{h_2(s)} ds = +\infty.$$

Part II

Homoclinic solutions and Lidstone problems on the whole real line

INTRODUCTION

Qualitative analysis of differential equations has had an increasingly important role, specially the analytic study of their asymptotic behavior and stability.

A **homoclinic** orbit is a trajectory of a flow of a dynamical system which joins a saddle equilibrium point to itself. If a path in the phase space of a dynamical system joins two different equilibrium points, receives the name of a **heteroclinic** orbit.

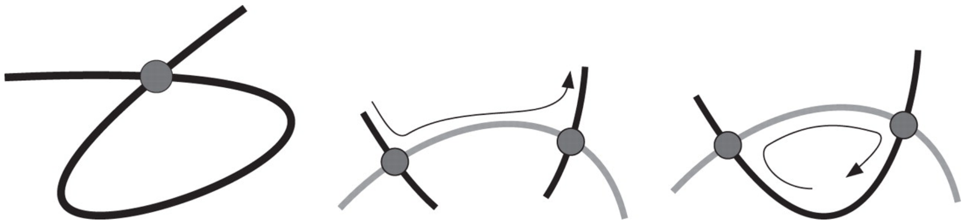

Figure 1: Homoclinic trajectory, heteroclinic connection and heteroclinic cycle

The interest in these trajectories goes far beyond mathematics itself, as homoclinic and heteroclinic solutions appear in a variety of mathematical models born in Mechanics, Chemistry or Biology.

The history of these homoclinic and heteroclinic solutions is already long. In addition to the phase portrait analysis, whose applicability is restricted to autonomous differential equations of second order, the study of these solutions started by a geometric approach. Poincaré, Melnikov and Smale were some of the first names to cover this topic in the 19^{th} century. At the end of the last century a more functional and analytical approach gave new tools like variational methods and the theory of critical points. It is worth highlighting Ambrosetti, Ekeland and Rabinowitz (see [32] and references therein).

This part is separated into three chapters, and each one provides the existence of homoclinic solutions for higher order nonlinear BVPs, not necessarily autonomous.

The first chapter will be addressed to problems with second order equations. Three different applications will be presented to illustrate the main results of the chapter: a problem with discontinuity in time;

an application to a Duffing equation; and another over a forced cantilever beam equation with damping.

The second chapter will ensure the existence of homoclinic solutions to fourth order BVPs. A generic example will be given and an application to a Bernoulli-Euler-v.Karman BVP will complete the chapter.

Finally, last chapter will center the attention on Lidstone's BVPs, putting a link between the solutions of Lidstone BVPs in the whole real line and homoclinic solutions. The results of this last chapter of this part will be applied to an infinite beam resting on granular foundations with moving loads.

3

Homoclinic solutions for second order problems

The existence of homoclinic solutions for autonomous and nonautonomous differential equations and Hamiltonian systems is an important subject in qualitative theory. It can be considered as a special case of the so-called *convergent solutions*, i.e., solutions defined in the half-line (or the real line), and having a finite limit to $+\infty$ (respectively $\pm\infty$), see [14].

In this chapter it is considered the second order discontinuous equation in the real line,

$$u''(t) - ku(t) = f\left(t, u(t), u'(t)\right), a.e.\ t \in \mathbb{R}, \tag{3.0.1}$$

with $k > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$ a L^1- Carathéodory function. The final purpose is looking for homoclinic orbits to 0, that is, nontrivial solutions of (3.0.1) such that

$$u(\pm\infty) := \lim_{t \to \pm\infty} u(t) = 0, u'(\pm\infty) := \lim_{t \to \pm\infty} u'(t) = 0. \tag{3.0.2}$$

Several works prove the existence of homoclinic and heteroclinic solutions for small perturbations (see [35, 116]), or deal with some superquadratic or subquadratic conditions at infinity (see [98, 103]) or asymptotically quadratic ([39]). Another point of view is to obtain an homoclinic orbit as a limit of $2kT$ -periodic solutions of a certain sequence of periodic boundary value problems (see [9, 52, 61]). The main arguments used in this method apply variational methods, upper and lower solutions and fixed point theory ([15, 20, 101, 108]).

Equation (3.0.1) arises in several real phenomena, for instance, as the study of traveling wave fronts for parabolic reaction-diffusion equations with a local reaction term, and generalizes several classical equations such as Duffing-type equations ([54, 94]) or Liénard-like systems ([114]).

In this chapter we combine the method of lower and upper solutions, not necessarily ordered, as suggested in [53, 82]. Moreover, our result improves the literature, as the existence and localization of homoclinic solutions is proved without extra assumptions on the growth, sign or asymptotic behavior of the nonlinear part.

The chapter is based on the work [87], and is organized as it follows: first section contains some definitions and auxiliary results to be used forward and the tools used to deal with the lack of compactness. The existence and localization results for homoclinic solutions are presented, some of them are obtained in presence of non-ordered lower and upper solutions, since they are defined as an adequate pair of functions. Last sections include an example of a discontinuous problem and applications to a Duffing-type equation that models the forced vibrations of a cantilever beam in a nonuniform field of two permanent magnets.

3.1 PRELIMINARIES

Define the space

$$X_{H2} = \left\{ x \in C^1(\mathbb{R}) : \lim_{|t|\to+\infty} x(t) \in \mathbb{R} \right\}$$

with the norm $\|x\|_{X_{H2}}$=max $\left\{ \|x\|_\infty, \|x'\|_\infty \right\}$, where $\|y\|_\infty := \sup_{t\in\mathbb{R}} |y(t)|$.

In this way $(X_{H2}, \|.\|_{X_{H2}})$ is a Banach space (see [110, 113]).

An important property of functions on space X_{H2} is shown in next lemma.

Lemma 3.1.1. *Let* $x \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $n \geq 1$. *If* $x(+\infty) = l \in \mathbb{R}$ *then* $x^{(n)}(+\infty) = 0$, *for* $n \geq 1$.

Proof.

In the case where $x(+\infty) = l$, for any $\delta_0 > 0$ there is $T_0 > 0$ such that for $t > T_0$ one has $|x(t) - l| < \delta_0$.

For $n = 1$, take $h > 0$, $\delta_0 = \frac{h\ \delta_1}{2}$ and $t > T_1$, for some $T_1 > 0$. Therefore, for $t > \max\{T_0, T_1\}$, one has

$$\begin{aligned}|x'(t)| &= \lim_{h\to 0}\frac{|x(t+h)-x(t)|}{h} = \lim_{h\to 0}\frac{|x(t+h)-l+l-x(t)|}{h}\\ &\leq \lim_{h\to 0}\frac{|x(t+h)-l|+|x(t)-l|}{h} \leq \lim_{h\to 0}\frac{\frac{h\ \delta_1}{2}+\frac{h\ \delta_1}{2}}{h} = \delta_1,\end{aligned}$$

for any $\delta_1 > 0$, that is $x'(+\infty) = 0$.

For $n > 1$ the proof follows by mathematical induction.

The case $x(-\infty) = l$ can be proved by the same technique. □

The following result will play an important role in the proof of the main result, giving a solution of some linear second order problem via Green's functions:

Lemma 3.1.2 ([3]). *If $h \in L^1(\mathbb{R})$, then problem*

$$\begin{cases} u''(t) - ku(t) = h(t), \ a.e. \ t \in \mathbb{R} \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \tag{3.1.1}$$

has a unique solution in X_{H2}. Moreover, this solution can be expressed as

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds \tag{3.1.2}$$

where

$$G(t,s) = -\frac{1}{2\sqrt{k}}e^{-\sqrt{k}|s-t|}. \tag{3.1.3}$$

Proof.

The homogeneous solution of the linear equation is given by

$$u(t) = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}, \text{ for } c_1, c_2 \in \mathbb{R}$$

As the null function is the only solution of the homogeneous problem associated to (3.1.1), its solution is given by

$$u(t) = -\frac{1}{2\sqrt{k}}\int_{-\infty}^{+\infty} e^{-\sqrt{k}|s-t|}h(s)ds.$$

For $G(t,s) := -\frac{1}{2}e^{-\sqrt{k}|s-t|}$ one has

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds.$$

□

Some trivial properties can easily be proved for Green's functions.

Remark 3.1.3. *The above Green's functions verify the following properties:*

- $G(t,s)$ *and* $\dfrac{\partial G(t,s)}{\partial t}$ *are continuous,*
- $\lim_{|t|\to+\infty} G(t,s) = 0,$
- $\lim_{|t|\to+\infty} \dfrac{\partial G(t,s)}{\partial t} = 0.$

To deal with the lack of compactness of set X_{H2}, next compactness criterion plays a key role, following arguments suggested in [37, 93, 110].

Theorem 3.1.4. *A set $M \subset X_{H2}$ is compact if the following conditions hold:*

i) both $\{t \to x(t) : x \in M\}$ and $\{t \to x'(t) : x \in M\}$ are uniformly bounded;

ii) both $\{t \to x(t) : x \in M\}$ and $\{t \to x'(t) : x \in M\}$ are equicontinuous in any compact interval of $\mathbb{R}$;

iii) both $\{t \to x(t) : x \in M\}$ and $\{t \to x'(t) : x \in M\}$ are equiconvergent at $\pm\infty$, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|f(t) - f(\pm\infty)| < \epsilon$ and $|f'(t) - f'(\pm\infty)| < \epsilon$, for all $|t| > T(\epsilon)$ and $f \in M$.

Proof.

In order to prove that the subset M is relatively compact in X_{H2}, as we are in a Banach space, we only need to show that M is totally compact, or, bounded in X_{H2}, that is, for $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by (i)-(iii), there exist constants $A > 0, \delta > 0$, and an integer $N > 0$, such that

- $|x(t_1) - x(t_2)| \leq \frac{\epsilon}{3}, |x'(t_1) - x'(t_2)| \leq \frac{\epsilon}{3}$ with $t_1, t_2 < -N$ or $t_1, t_2 > N$ and $x \in M, \|x\|_{X_{H2}} \leq A$;

- $|x(t_1) - x(t_2)| \leq \frac{\epsilon}{3}, |x'(t_1) - x'(t_2)| \leq \frac{\epsilon}{3}$ with $t_1, t_2 \in [-N, N]$ and
  $|t_1 - t_2| < \delta, x \in X_{H2}$.

Define $X_{[-N,N]} = \left\{ x|_{[-N,N]} : x \in X_{H2} \right\}$. For $x \in X_{[-N,N]}$ define

$$\|x\|_N = \max \left\{ \sup_{t \in [-N,N]} |x(t)|, \sup_{t \in [-N,N]} |x'(t)| \right\}$$

It can be proved that $X_{[-N,N]}$ is a Banach space with the norm $\|.\|_N$.

Let $M_{[-N,N]} = \{t \to x(t), t \in [-N, N] : x \in M\}$. Then $M_{[-N,N]}$ is a subset of $X_{[-N,N]}$. By Arzèla-Ascoli theorem, $M_{[-N,N]}$ is relatively compact in $X_{[-N,N]}$. Thus, there exist $x_1, x_2, ..., x_k \in M$ such that $\|x - x_i\|_N \leq \frac{\epsilon}{3}$, for any $x \in M$ and $i = 1, 2, ..., k$.

Therefore, for $x \in M$, we find that for $j = 0, 1$,

$$\begin{aligned}
\left\| x^{(j)} - x_i^{(j)} \right\|_X = & \max \left\{ \sup_{t \in \mathbb{R}} \left| x^{(j)}(t) - x_i^{(j)}(t) \right| \right\} \\
= & \max \left\{ \begin{array}{c} \sup\limits_{t \leq -N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right|, \\ \sup\limits_{|t| \leq N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right|, \\ \sup\limits_{t \geq N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right| \end{array} \right\} \\
\leq & \max \left\{ \sup_{t \leq -N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right|, \frac{\epsilon}{3}, \sup_{t \geq N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right| \right\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\sup_{t \leq -N} |x(t) - x_i(t)| \leq & \sup_{t \leq -N} |x(t) - x(-N)| + |x(-N) - x_i(-N)| \\
& + \sup_{t \leq -N} |x_i(-N) - x_i(t)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Similarly we can prove that all $\sup\limits_{|t| > N} \left| x^{(j)}(t) - x_i^{(j)}(t) \right| \leq \epsilon$.

So, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_1}, U_{x_2}, ..., U_{x_k}\}$, that is, M is totally bounded in X_{H2}. Hence M is relatively compact in X_{H2}. $\square$

To provide the localization part of the main result it is used lower and upper solutions technique, based on the following definition:

Definition 3.1.5. *A function $\alpha \in X_{H2}$ is said to be lower solution of problem (3.0.1), (3.0.2) if*

$$\alpha''(t) - k\,\alpha(t) \geq f\left(t, \alpha(t), \alpha'(t)\right), a.e.\ t \in \mathbb{R}, \text{ and } \alpha(\pm\infty) \leq 0.$$

A function $\beta \in X_{H2}$ is an upper solution if the reversed inequalities hold.

Usually, in the literature, these functions have some order relation: well ordered or reversed ordered. However, next definition can be applied to $\alpha(t)$ and $\beta(t)$ with no definite order.

Definition 3.1.6. *Functions $\alpha, \beta \in X_{H2}$ are a pair of lower and upper solutions of problem (3.0.1), (3.0.2), respectively, if*

$$\begin{cases} \alpha''(t) - k\,\overline{\alpha}(t) \geq f\left(t, \overline{\alpha}(t), \alpha'(t)\right), t \in \mathbb{R}, \\ \beta''(t) - k\,\beta(t) \leq f\left(t, \beta(t), \beta'(t)\right), t \in \mathbb{R} \\ \qquad \overline{\alpha}(\pm\infty) \leq 0,\ \beta(\pm\infty) \geq 0, \end{cases}$$

where $\overline{\alpha}(t) = \alpha(t) - \sup_{t\in\mathbb{R}} |\alpha(t) - \beta(t)|$.

3.2 EXISTENCE AND LOCALIZATION OF HOMOCLINICS

First result requires that lower and upper solutions are well ordered to guarantee the existence of homoclinic solutions of problem (3.0.1), (3.0.2).

Theorem 3.2.1. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function not identical to zero and $\alpha, \beta \in X_{H2}$ be lower and upper solutions of problem (3.0.1), (3.0.2), respectively, with*

$$\alpha(t) \leq \beta(t), \forall t \in \mathbb{R}. \tag{3.2.1}$$

If $f(t,x,y)$ is monotone in y (nonincreasing or nondecreasing) for $(t,x) \in \mathbb{R}^2$ fixed, then problem (3.0.1), (3.0.2) has a homoclinic solution $u \in X_{H2}$ such that $\alpha(t) \leq u(t) \leq \beta(t), \forall t \in \mathbb{R}$.

Proof.

Consider the modified equation

$$u''(t) - ku(t) = f\left(t, \delta(t, u(t)), u'(t)\right), \text{ a.e. } t \in \mathbb{R}, \tag{3.2.2}$$

where function $\delta : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\delta(t, u(t)) = \begin{cases} \beta(t) & , u(t) > \beta(t) \\ u(t) & , \alpha(t) \leq u(t) \leq \beta(t) \\ \alpha(t) & , u(t) < \alpha(t). \end{cases}$$

Step 1: *The modified problem (3.2.2), (3.0.2) has a solution.*

Define the operator $T : X_{H2} \to X_{H2}$ by

$$Tu(t) = \int_{-\infty}^{+\infty} G(t,s) F_u(s) ds.$$

where

$$F_u(t) = f\left(t, \delta(t, u(t)), u'(t)\right)$$

and $G(t,s)$ is the Green Function given by Lemma 3.1.2. So it is enough to prove that T has a fixed point, which is done in the following claims:

Claim 1.1: $T : X_{H2} \to X_{H2}$ *is well defined.*

Let $u \in X_{H2}$. As f is a L^1-Carathéodory function then Tu is continuous. For $r_0 > 0$ such that

$$r_0 > \max\left\{\|\alpha\|_\infty, \|\beta\|_\infty\right\}, \tag{3.2.3}$$

there exists φ_{r_0} with $|f(t,x,y)| \le \varphi_{r_0}(t)$, for $\sup_{t\in\mathbb{R}}\{|x(t)|,|y(t)|\} < r_0$ and *a.e.* $t \in \mathbb{R}$. As Tu and $(Tu)'$ are continuous, passing to the limit, by the Lebesgue dominated theorem and Remark 3.1.3,

$$\begin{aligned}\lim_{|t|\to\infty}(Tu)(t) &= \int_{-\infty}^{+\infty}\lim_{|t|\to\infty} G(t,s)F_u(s)ds = 0,\\ \lim_{|t|\to\infty}(Tu)'(t) &= \int_{-\infty}^{+\infty}\lim_{|t|\to\infty}\frac{\partial G(t,s)}{\partial t}F_u(s)ds = 0,\end{aligned}$$

and, therefore, $Tu \in X_{H2}$.

Claim 1.2: *T is compact.*

Let

$$M(s) := \max\left\{\sup_{t\in\mathbb{R}}|G(t,s)|, \sup_{t\in\mathbb{R}}\left|\frac{\partial G(t,s)}{\partial t}\right|\right\}.$$

Consider a bounded set $B \subset X_{H2}$ defined by

$$B := \{u \in X_{H2} : \|u\|_{X_{H2}} < r_1\},$$

for some $r_1 > 0$ such that $r_1 > \max\left\{r_0, \int_{-\infty}^{+\infty} M(s)\varphi_{r_0}(s)ds\right\}$, with r_0 given by (3.2.3). Then, for $t \in \mathbb{R}$,

$$|Tu(t)| \le \int_{-\infty}^{+\infty} M(s)|F_u(s)|ds \le \int_{-\infty}^{+\infty} M(s)\varphi_r(s)ds < r_1,$$

and, analogously $|(Tu)'(t)| < r_1$. Therefore TB is bounded and $TB \subset B$.

For $a > 0$ and $t_1, t_2 \in [-a,a]$, because of the continuity of the Green's functions and its derivative, one has

$$\lim_{t_1\to t_2}|Tu(t_1) - Tu(t_2)| \le \int_{-\infty}^{+\infty}\lim_{t_1\to t_2}|G(t_1,s) - G(t_2,s)||F_u(s)|ds = 0,$$

$$\lim_{t_1\to t_2}\left|(Tu)'(t_1) - (Tu)'(t_2)\right| \le \int_{-\infty}^{+\infty}\lim_{t_1\to t_2}\left|\frac{\partial G}{\partial t}(t_1,s) - \frac{\partial G}{\partial t}(t_2,s)\right||F_u(s)|ds = 0.$$

So, TB is equicontinuous.

To prove that TB is equiconvergent at $\pm\infty$ note that

$$\begin{aligned}
\left|Tu(t) - \lim_{t\to\pm\infty}(Tu(t))\right| &\leq \int_{-\infty}^{+\infty} |G(t,s)||F_u(s)|ds \\
&\leq \int_{-\infty}^{+\infty} |G(t,s)|\varphi_r(s)ds \longrightarrow 0, t\to\pm\infty \\
\left|(Tu)'(t) - \lim_{t\to\pm\infty}(Tu)'(t)\right| &\leq \int_{-\infty}^{+\infty} \left|\frac{\partial G}{\partial t}(t,s)\right| |F_u(s)|ds \\
&\leq \int_{-\infty}^{+\infty} \left|\frac{\partial G}{\partial t}(t,s)\right| \varphi_r(s)ds \longrightarrow 0, t\to\pm\infty.
\end{aligned}$$

Therefore, by Theorem 3.1.4, T is compact and, by Theorem 1.1.6, T has at least one fixed point $u \in X_{H2}$.

Step 2: *Every solution of the modified problem (3.2.2), (3.0.2) is a solution of the initial problem (3.0.1), (3.0.2).*

Let u be a solution of problem (3.2.2), (3.0.2). In order to obtain this step it is sufficient to prove that

$$\alpha(t) \leq u(t) \leq \beta(t), \forall t \in \mathbb{R}.$$

Suppose, by contradiction, that there exists $t \in \mathbb{R}$ such that $\alpha(t) > u(t)$ and define

$$\inf_{t\in\mathbb{R}}(u(t) - \alpha(t)) < 0.$$

This infimum can not be attained at $\pm\infty$. Otherwise, by (3.0.2) and Definition 3.1.5 this contradiction holds:

$$0 > u(\pm\infty) - \alpha(\pm\infty) \geq 0.$$

So, there is $t_* \in \mathbb{R}$ such that

$$\min_{t\in\mathbb{R}}(u(t) - \alpha(t)) = u(t_*) - \alpha(t_*) < 0.$$

Then there exists an interval $[t_-, t_+]$ such that $t_* \in [t_-, t_+]$ and $u(t) - \alpha(t) < 0, u''(t) - \alpha''(t) \geq 0$ *a.e.* $t \in [t_-, t_+]$. Also $u'(t) - \alpha'(t) \leq 0$, for $t \in [t_-, t_*]$ and $u'(t) - \alpha'(t) \geq 0$, for $t \in [t_*, t_+]$.

If $f(t,x,y)$ is nonincreasing in y, for $t \in [t_*, t_+]$ this contradiction is achieved

$$\begin{aligned}
0 &\leq \int_{t_*}^{t} u''(s) - \alpha''(s)ds = \int_{t_*}^{t} \left[f(s, \delta\left(s, u(s)\right), u'(s)) + ku(s) - \alpha''(s) \right] ds \\
&\leq \int_{t_*}^{t} \left[f(s, \alpha(s), \alpha'(s)) + ku(s) - \alpha''(s) \right] ds \\
&\leq k \int_{t_*}^{t} u(s) - \alpha(s)ds < 0.
\end{aligned}$$

By the previous arguments, a similar contradiction holds if f is nondecreasing, but with an integration on $[t_-, t_*] \subset [t_-, t_+]$.

So $\alpha(t) \leq u(t), \forall t \in \mathbb{R}$. In a similar way it can be proved that $\beta(t) \geq u(t), \forall t \in \mathbb{R}$. □

If the nonlinearity f verifies an anti-symmetric-type property, there is also homoclinic solutions for the symmetric equation

$$-u''(t) + ku(t) = f\left(t, u(t), u'(t)\right), t \in \mathbb{R}. \tag{3.2.4}$$

Theorem 3.2.2. *Let $\alpha, \beta \in X_{H2}$ be lower and upper solutions of problem (3.0.1), (3.0.2), respectively, verifying (3.2.1). If $f : \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function, with $f(t,x,y)$ monotone in y, for $(t,x) \in \mathbb{R}^2$ fixed, and satisfying*

$$f\left(t, -x, -y\right) = -f\left(t, x, y\right), \ \forall (t,x,y) \in \mathbb{R}^3, \tag{3.2.5}$$

then there is a pair of homoclinic solutions $(u, -u) \in X_{H2}^2$ such that u is a solution of problem (3.0.1), (3.0.2) and $-u$ solution of (3.2.4), (3.0.2), verifying

$$\begin{aligned}
\alpha(t) &\leq u(t) \leq \beta(t), \\
-\beta(t) &\leq -u(t) \leq -\alpha(t), \ \forall t \in \mathbb{R}.
\end{aligned}$$

Proof.

Let $\alpha \in X_{H2}$ be a lower and upper solutions of problem (3.0.1), (3.0.2). Then, by (3.2.5),

$$\begin{aligned} -\alpha''(t) + k\,\alpha(t) &= -\left[\alpha''(t) - k\,\alpha(t)\right] \\ &\leq -f\left(t, \alpha(t), \alpha'(t)\right) = f\left(t, -\alpha(t), -\alpha'(t)\right), \text{ for } t \in \mathbb{R}. \end{aligned}$$

That is $-\alpha(t)$ is an upper solution of problem (3.2.4), (3.0.2).

Analogously it can be proved that $-\beta(t)$ is a lower solution of problem (3.2.4), (3.0.2).

So, by Theorem 3.2.1, there is a solution $-u$ of problem (3.2.4), (3.0.2), such that

$$-\beta(t) \leq -u(t) \leq -\alpha(t), \ \forall t \in \mathbb{R}.$$

□

The well-ordered relation (3.2.1) can be removed if lower and upper functions are defined as a pair of functions, applying a translation technique suggested in [44].

In this case, the main theorem can be formulated in the following way:

Theorem 3.2.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a L^1-Carathéodory function and $\alpha, \beta \in X_{H2}$ a pair of lower and upper solutions of problem (3.0.1), (3.0.2), respectively, according to Definition 3.1.6.*

If $f(t,x,y)$ is monotone in y (nonincreasing or nondecreasing) for $(t,x) \in \mathbb{R}^2$ fixed, then problem (3.0.1), (3.0.2) has a homoclinic solution $u \in X_{H2}$ such that $\overline{\alpha}(t) \leq u(t) \leq \beta(t)$.

The proof is similar to Theorem 3.2.1 replacing the truncature function δ by $\overline{\delta} : \mathbb{R}^2 \to \mathbb{R}$ given as

$$\overline{\delta}(t, u(t)) = \begin{cases} \beta(t) & , u(t) > \beta(t) \\ u(t) & , \overline{\alpha}(t) \leq u(t) \leq \beta(t) \\ \overline{\alpha}(t) & , u(t) < \overline{\alpha}(t). \end{cases}$$

Notice that α and β do not need to be well ordered or even ordered at all.

3.3 EXAMPLE OF A DISCONTINUOUS BVP

Consider the second order, nonlinear and discontinuous BVP

$$\begin{cases} u''(t) - u(t) = \frac{sgn(t)u^3(t)+0.1-100u'(t)}{1+t^2}, t \in \mathbb{R} \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \tag{3.3.1}$$

where $sgn(t) = \begin{cases} 1 & ,t \geq 0 \\ -1 & ,t < 0. \end{cases}$.

The nonlinear and discontinuous function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(t,x,y) := \frac{sgn(t)\; x^3 + 0.1 - 100y}{1+t^2}$$

is monotone in y for $(t,x) \in \mathbb{R}^2$ fixed and for $|x|, |y| < \rho$, and a L^1- Carathéodory function with $\varphi_\rho(t) = \frac{\rho^3+0.1+100\rho}{1+t^2}$.

Functions $\alpha(t) = \arctan(t)$ and $\beta(t) \equiv 0$ are, respectively, a pair of lower and upper solutions of problem (3.3.1) according to Definition 3.1.6, with $\overline{\alpha}(t) = \arctan(t) - \pi/2$.

Therefore, by Theorem 3.2.3, there is at least a non-positive solution u of (3.3.1) with $\arctan(t) - \pi/2 \leq u(t) \leq 0, \forall t \in \mathbb{R}$.

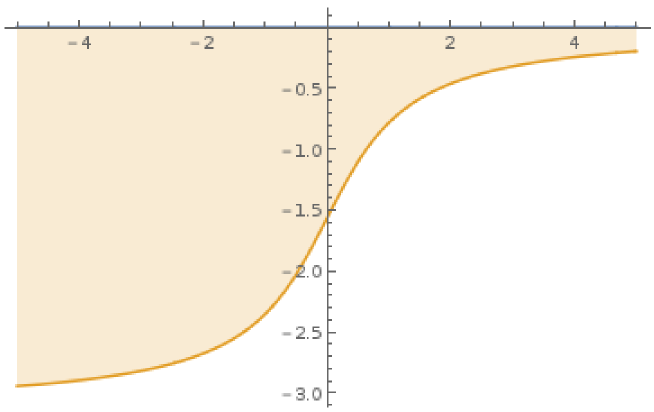

Figure 2: Admissible region for solution u

Notice that the null function is not a solution for the problem and f is discontinuous on t.

3.4 DUFFING EQUATION

In [8] the authors consider equation

$$-u''(t) + u(t) = a(t)\,|u(t)|^{p-1}\,u(t), t \in \mathbb{R} \tag{3.4.1}$$

with $p > 1$, which models the forced vibrations of a cantilever beam in the nonuniform field of two permanent magnets.

The structure and behavior of function $a : \mathbb{R} \to \mathbb{R}$ is a key point for the existence of homoclinic solutions. Applying the main result it can be proved that there exists at least one nontrivial solution in cases not covered, as far as we know, by results in the existent literature.

For example, if $a(t) = -\frac{1}{1+t^2}$, $p = 3$, $k = 0.1$, let us seek a nontrivial and homoclinic solution for

$$\begin{cases} u''(t) - 0.1\,u(t) = \frac{|u(t)|^2 u(t)}{1+t^2}, t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \tag{3.4.2}$$

The nonlinear function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(t,x) = \frac{|x|^2\,x}{1+t^2}$$

is a L^1 Carathéodory function with $|x| < \rho$ and $\varphi_\rho(t) = \frac{\rho^3}{1+t^2}$. Functions $\alpha(t) = \frac{1}{3+t^2} - 0.3$ and $\beta(t) \equiv 0.3$ are lower and upper solutions, respectively, of problem (3.4.2).

Therefore, by Theorem 3.2.2, there are at least two homoclinic solutions: u of (3.4.2) and $-u$ of problem

$$\begin{cases} -u''(t) + 0.1\,u(t) = \frac{|u(t)|^2 u(t)}{1+t^2}, t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \tag{3.4.3}$$

with $\frac{1}{3+t^2} - 0.3 \le u(t) \le 0.3$, and $-0.3 \le -u(t) \le -\frac{1}{3+t^2} + 0.3$, for $t \in \mathbb{R}$.

Note that the null function is not a solution, and therefore, u and $-u$ are nontrivial solutions.

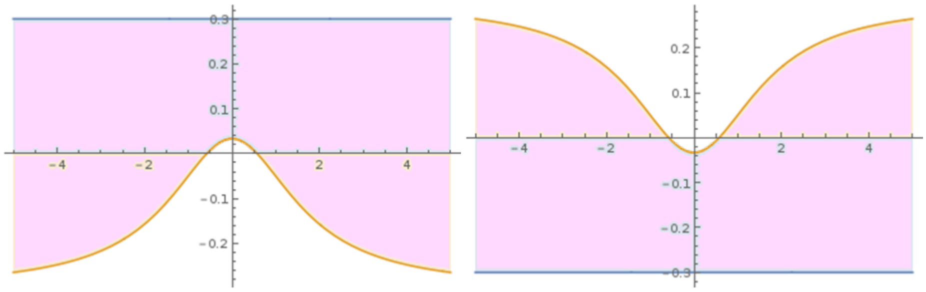

Figure 3: Admissible regions for both solutions u and $-u$, respectively

3.5 FORCED CANTILEVER BEAM EQUATION WITH DAMPING

The second order differential equation

$$x''(t) + bx'(t) - x + x^3 = F\cos(\omega t). \tag{3.5.1}$$

can model the forced vibrations of a cantilever beam in a nonuniform field of two magnets.

As it is illustrated in the Figure 4, a slender steel beam is clamped

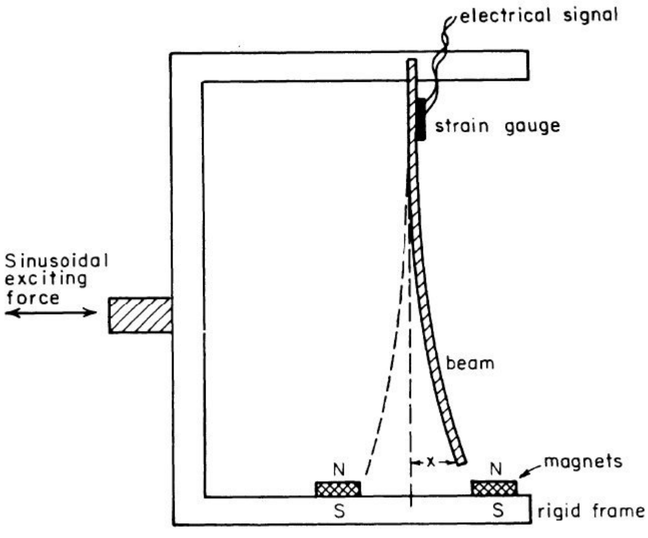

Figure 4: Interaction between a cantilever beam, two magnets and an excitation force

in a rigid framework which supports the magnets. Their attractive forces overcome the elastic ones, which would otherwise keep the beam straight. In the absence of some external force, the beam settles with its tip close to one or the other of the magnets. The variable x represents a measure of the beam's position, say its tip displacement.

As example, lets consider the following equation

$$u''(t) + b(t)u'(t) + c(t)g(t,u) = 0, \tag{3.5.2}$$

with $b(t) = -\frac{0.01}{1+t^2}, c(t) = 1, g(t,u) = -u - \frac{100u^4}{1+t^2}$.

This class of ODE arises in diffusion phenomena in biomathematics. For more details see [13, 67].

Note that in this case the BVP

$$\begin{cases} u''(t) - u(t) = \frac{0.01u'(t)+100u^4(t)}{1+t^2}, t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases} \tag{3.5.3}$$

is not covered by any kind of existence results, to the best of our knowledge.

The nonlinear function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(t,x,y) = \frac{0.01y + 100x^4}{1+t^2}$$

is monotone in y for $(t,x) \in \mathbb{R}^2$ fixed, and for $|x|, |y| < \rho$, is a L^1-Carathéodory function with $\varphi_\rho(t) = \frac{0.01\rho+\rho^4}{1+t^2}$.

Functions $\alpha(t) = \frac{1}{1+t^2}$ and $\beta(t) \equiv 0.5$ are, respectively, lower and upper solutions of problem (3.5.3), according to Definition 3.1.6, with $\overline{\alpha}(t) = \frac{1}{1+t^2} - 0.5$.

Therefore, by Theorem 3.2.3, there is at least an homoclinic solution u of (3.5.3) such that

$$\frac{1}{1+t^2} - 0.5 \leq u(t) \leq 0.5, \forall t \in \mathbb{R}.$$

4

Homoclinic solutions to fourth order problems

This chapter provides sufficient conditions for the existence of homoclinic solutions of fourth order nonlinear ODEs. Different applications are presented to illustrate new results, as the nonlinear Bernoulli-Euler-v. Karman problem, Extended Fisher-Kolmogorov problem or the Swift-Hohenberg problem. The method will use Green's functions to formulate a new modified integral equation which is equivalent to the original nonlinear one. Moreover, in an adequate function space, the corresponding nonlinear integral operator is compact, and it can be applied an existence result by Schauder's fixed point theorem.

It is study the existence of homoclinic solutions to the fourth order, nonlinear and not necessarily periodic, differential equation

$$u^{(iv)}(t)+k\,u(t)=f(t,u(t),u'(t),u''(t),u'''(t)),\; t\in\mathbb{R}, \tag{4.0.1}$$

with $k>0$ and $f:\mathbb{R}^5\to\mathbb{R}$ a continuous function verifying an adequate asymptotic condition.

Note that no further condition will be necessary on the nonlinearity $f(t,x,y,z,w)$, to obtain the existence of homoclinic orbits to 0, that is, nontrivial solutions of (4.0.1) such that

$$u(\pm\infty):=\lim_{t\to\pm\infty}u(t)=0,\;\; u'(\pm\infty):=\lim_{t\to\pm\infty}u'(t)=0. \tag{4.0.2}$$

In the last decades, the study of autonomous and non-autonomous fourth order differential equations attracted many researchers. To be more precise, equations of the type

$$u^{(iv)}(t)+ku''(t)+g(u(t))=0,\;\; t\in\mathbb{R}, \tag{4.0.3}$$

with $k \in \mathbb{R}$, and g a locally Lipschitz function, arises in several theoretical cases and real phenomena such as:

- if $k < 0$, it is known as the **Extended Fisher-Kolmogorov equation** and if $k > 0$, it is referred to as **Swift-Hohenberg equation** (see [92]);
- if $g(u) = u - u^2$, it is applied in the dynamic phase-space analogy of a nonlinearly supported elastic strut (see [60]);
- if $g(u) = u^3 - u$, it models the pattern formation in many physical, chemical or biological systems (see [19]);
- if $g(u) = u^5 - u^3 + u$, it is used to study the localization and spreading of deformation of a strut confined by an elastic foundation (see [10, 91]);
- if $g(u) = (u+1)^+ - 1$, where $(u+1)^+ = \max\{u+1, 0\}$, equation (4.0.3) arises in the search of traveling waves solutions, ([95]), in the study of deflection in railway tracks, ([1]), and undersea pipelines, ([18]).

The existence of homoclinic solutions were proved by several methods and techniques. Some examples, without pretending to be exhaustive, are shown in [97], where it is considered the above nonlinearities by variational arguments and the Palais-Smale condition.

For equation

$$u^{(iv)}(t) + ku''(t) + a(t)u(t) - b(t)u^2(t) - c(t)u^3(t)=0,$$

in [101], it is proved the existence of one nontrivial homoclinic solution with $a(t)$ and $c(t)$ positive bounded and continuous functions, and $b(t)$ a bounded continuous function, applying Mountain Pass Theorem, and, in [70], the existence of nontrivial homoclinic solutions in the nonperiodic case. In [65], the authors show the existence of two homoclinic solutions for some nonperiodic fourth order equations with a perturbation.

This chapter put the emphasis in a perturbation with an unknown function where the nonlinearity is given by a generic continuous function, with dependence on u and all derivatives till order three.

As far as we know it is the first time where it is considered such perturbation associated to generic nonlinearity, which has to verify only an asymptotic condition (see assumption (4.2.1)).

The arguments are based in the explicit form of the Green's functions associated to the linear perturbation of (4.0.1), in a compactness criterion and fixed point theory.

The chapter is organized as it follows: first it is defined an adequate space, the explicit expressions of the associated Green's functions and other main tools, such as the criterion used to deal with the lack of compactness and the fixed-point theorem. Existence results for homoclinic solutions are presented next, together with the relation of asymptotic properties of the nonlinearity on some qualitative data of homoclinic solutions. Finally an example and some applications will be shown to illustrate the applicability of the main theorem.

4.1 DEFINITIONS AND AUXILIARY RESULTS

Let us define the space

$$X_{H4} = \left\{ x \in C^3(\mathbb{R}) : \lim_{|t| \to +\infty} x(t) = 0 \right\}$$

with the norm $\|x\|_{X_{H4}} = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\}$, where $\|\omega\|_\infty = \sup_{t \in \mathbb{R}} |\omega(t)|$.

In this way $(X_{H4}, \|.\|_{X_{H4}})$ is a Banach space.

The following result will play an important role in the proof of the main result, giving a solution of some linear fourth order problem via Green's functions:

Lemma 4.1.1. *If $h \in L^1(\mathbb{R})$, then, for some $k > 0$, the problem*

$$\begin{cases} u^{(iv)}(t) + ku(t) = h(t), \ t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases} \tag{4.1.1}$$

has a unique solution in X_{H4}. Moreover, this solution can be expressed as

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds \tag{4.1.2}$$

where

$$G(t,s) = \frac{\sqrt[4]{k}}{2k} e^{\frac{-\sqrt[4]{k}|s-t|}{\sqrt{2}}} \sin\left(\frac{\sqrt[4]{k}|s-t|}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{4.1.3}$$

Proof.

The homogeneous solution of the linear equation is given by

$$u(t) = e^{At}\left(c_1 \cos(At) + c_2 \sin(At)\right) + e^{-At}\left(c_3 \cos(At) + c_4 \sin(At)\right)$$

with $A = \sqrt[4]{\frac{k}{4}}$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

As the null function is the only solution of the homogeneous problem, Green's functions can be defined and the general solution of (4.1.1) is given by

$$u(t) = \frac{\sqrt[4]{k}}{2k} \int_{-\infty}^{+\infty} e^{-\sqrt[4]{\frac{k}{4}}|s-t|} \sin\left(\sqrt[4]{\frac{k}{4}}|s-t| + \frac{\pi}{4}\right) h(s)ds.$$

For $G(t,s) := \frac{\sqrt[4]{k}}{2k} e^{-A|s-t|} \sin\left(A|s-t| + \frac{\pi}{4}\right)$ one can write

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds.$$

□

The following properties of the Green function can easily be proved.

Remark 4.1.2. *For $i = 0,1,2,3$, defining*

$$G_i^-(t,s) := \frac{\sqrt[4]{k}^{\,i+1}}{2k} e^{\frac{-\sqrt[4]{k}(s-t)}{\sqrt{2}}} \sin\left(\frac{\sqrt[4]{k}(s-t)}{\sqrt{2}} + \frac{\pi(3i+1)}{4}\right)$$

$$G_i^+(t,s) := \frac{\sqrt[4]{k}^{\,i+1}}{2k} e^{\frac{-\sqrt[4]{k}(t-s)}{\sqrt{2}}} \sin\left(\frac{\sqrt[4]{k}(t-s)}{\sqrt{2}} + \frac{\pi(3i+1)}{4}\right),$$

then, for $j = 0,1,2,3$,

$$u^{(j)}(t) = \int_{-\infty}^{t} G_j^-(t,s)h(s)ds + (-1)^j \int_t^{+\infty} G_j^+(t,s)h(s)ds; \quad (4.1.4)$$

$$\lim_{|t|\to\infty} \frac{\partial^j G(t,s)}{\partial t^j} = 0; \quad (4.1.5)$$

$$\left| \frac{\partial^j G(t,s)}{\partial t^j} \right| \leq \frac{\left(\sqrt[4]{k}\right)^{j+1}}{2k}. \quad (4.1.6)$$

The following theorem is a key argument to deal with the lack of compactness of the set X_{H4}:

Theorem 4.1.3 ([37])**.** *Let $M \subset (C_l, \mathbb{R})$ with*

$$C_l := \left\{ x \in C[0,+\infty) : \exists \lim_{t\to+\infty} x(t) \right\}.$$

Then M is compact if the following conditions hold:

i) M is bounded in C_l;

ii) functions $f \in M$ are equicontinuous on any compact interval of $[0,+\infty)$;

iii) functions from M are equiconvergent, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|f(t) - f(+\infty)| < \epsilon$, for all $t > T(\epsilon)$ and $f \in M$.

The proof of this result can easily be applied of compact intervals of the form $[-T, T]$, for some $T > 0$, as it is suggested in [93], to obtain a similar result to the set X_{H4}.

Theorem 4.1.4. *A set $M \subset X_{H4}$ is relatively compact if the following conditions hold:*

i) M is bounded in X_{H4};

ii) the functions belonging to M are equicontinuous on any compact interval of $\mathbb{R}$;

iii) the functions from M are equiconvergent at $\pm\infty$, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|f^{(i)}(t) - f^{(i)}(\pm\infty)| < \epsilon$, for all $|t| > T(\epsilon), i = 0,1,2,3$ and $f \in M$.

4.2 EXISTENCE RESULTS

This section contains an existence result for homoclinic solutions of problem (4.0.1), (4.0.2) without monotone, periodic or extra assumptions on the nonlinear part.

Theorem 4.2.1. *Let* $f : \mathbb{R}^5 \to \mathbb{R}$ *be a continuous function not identical to zero. If for each* $r > 0$ *with* $\max\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty, \|w\|_\infty\} < r$ *there exists a positive function* $\phi_r \in L^1(\mathbb{R})$ *such that*

$$|f(t, x, y, z, w)| < \phi_r(t), \tag{4.2.1}$$

then problem (4.0.1), (4.0.2) has a homoclinic solution $u \in X_{H4}$*.*

Proof.

Define

$$F_u(t) := f(t, u(t), u'(t), u''(t), u'''(t))$$

and consider the operator $T : X_{H4} \to X_{H4}$ given by

$$Tu(t) = \int_{-\infty}^{+\infty} G(t, s) F_u(s) ds,$$

with $G(t, s)$ defined by (4.1.3).

As f is a continuous function verifying (4.2.1) and $u \in X_{H4}$, it is obvious that $F_u \in L^1(\mathbb{R})$, and, by Lemma 4.1.1, the fixed points of T are solutions of problem (4.0.1), (4.0.2). So, it is enough to prove that T has a fixed point.

Clearly $Tu \in C^3(\mathbb{R})$ and by (4.1.5) and Lebesgue's Dominated Convergence Theorem,

$$\lim_{|t| \to \infty} (Tu)(t) = \int_{-\infty}^{+\infty} \lim_{|t| \to \infty} G(t, s) F_u(s) ds = 0$$

and, for $i = 1, 2, 3$,

$$\lim_{|t| \to \infty} (Tu)^{(i)}(t) = \int_{-\infty}^{+\infty} \lim_{|t| \to \infty} \frac{\partial^{(i)} G(t, s)}{\partial t^i} F_u(s) ds = 0,$$

Therefore $Tu \in X_{H4}$, and $T : X_{H4} \to X_{H4}$ is well defined.

Now for any bounded subset $B \subset X_{H4}$ and any $u \in B$ with $\|u\|_{X_{H4}} \leq r_1$, by (4.1.6) and (4.2.1) one has

$$|Tu(t)| \leq \int_{-\infty}^{+\infty} |G(t,s)||F_u(s)|ds \leq \frac{\sqrt[4]{k}}{2k} \int_{-\infty}^{+\infty} \phi_{r_1} < +\infty, \forall t \in \mathbb{R},$$

and, therefore, $\{Tu(t) : Tu \in B\}$ is relatively compact in $\mathbb{R}$.

For $a > 0$ and $t_1, t_2 \in [-a, a]$, one has, as $t_1 \to t_2$,

$$|Tu(t_1) - Tu(t_2)| = \int_{-\infty}^{+\infty} |G(t_1,s) - G(t_2,s)||F_u(s)|ds \longrightarrow 0,$$

and

$$|(Tu)^{(i)}(t_1) - (Tu)^{(i)}(t_2)| =$$

$$= \int_{-\infty}^{+\infty} \left| \frac{\partial^{(i)} G}{\partial t^i}(t_1,s) - \frac{\partial^{(i)} G}{\partial t^i}(t_2,s) \right| |F_u(s)|ds \longrightarrow 0, \text{ for } i = 0,1,2,3.$$

So the set $\{u : [a, -a] \to \mathbb{R}\} \subset B$ is equicontinuous.

By the continuity of f for any $\epsilon > 0$ there exists $t_+ > 0$ and $\delta > 0$ such that when $|u(t) - v(t)| \leq \epsilon$, for $t > t_+$, then

$$|F_u(t_+) - F_v(t_+)| \leq \delta.$$

So, for $i = 1,2,3$, and by (4.1.6)

$$\left|(Tu)^{(i)}(t) - (Tv)^{(i)}(t)\right| = \int_{-\infty}^{+\infty} \left| \frac{\partial^{(i)} G}{\partial t^i}(t,s) \right| |F_u(s) - F_v(s)|ds \longrightarrow 0,$$

as $t \to +\infty$.

Analogously, when $|u(t) - v(t)| \leq \epsilon$, for $t < -t_+$, then

$$|F_u(-t_+) - F_v(-t_+)| \leq \delta.$$

So, T is equiconvergent at $\pm\infty$, and by Theorem 4.1.4, TB is relatively compact.

Consider now a subset $D \subset X_{H4}$ defined as

$$D := \left\{u \in X_{H4} : \|u\|_{X_{H4}} < r_2\right\},$$

with

$$r_2 > \max\left\{r, \int_{-\infty}^{+\infty} M\, \phi_r(s)ds\right\},$$

where $r > 0$ is given by (4.2.1) and

$$M := \max\left\{1, \frac{1}{2\sqrt[4]{k^3}}, \frac{1}{2\sqrt{k}}, \frac{1}{2\sqrt[4]{k}}\right\},$$

with $G_3^-(t,s)$ and $G_3^+(t,s)$ given by Remark (4.1.2).

For $t \in \mathbb{R}$, by (4.1.6) and (4.2.1),

$$\begin{aligned}
\|Tu\| &= \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{+\infty} G(t,s)F_u(s)ds \right| \\
&\leq \int_{-\infty}^{+\infty} \frac{1}{2\sqrt[4]{k^3}} |f(s,u(s),u'(s),u''(s),u'''(s))| ds \\
&\leq \int_{-\infty}^{+\infty} \frac{1}{2\sqrt[4]{k^3}} \phi_r(s)ds < r_2,
\end{aligned}$$

$$\begin{aligned}
\|(Tu)^{(i)}\| &= \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{+\infty} \frac{\partial^{(i)} G}{\partial t^i}(t,s)F_u(s)ds \right| \\
&\leq \int_{-\infty}^{+\infty} \frac{\left(\sqrt[4]{k}\right)^{i+1}}{2k} \phi_r(s)ds < r_2, \text{ for } i = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
\|(Tu)'''\| &= \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} G_3^-(t,s)F_u(s)ds - \int_{t}^{+\infty} G_3^+(t,s)F_u(s)ds \right| \\
&\leq \int_{-\infty}^{+\infty} \sup_{t\in\mathbb{R}} \left(|G_3^-(t,s)| + |G_3^-(t,s)| \right) \phi_r(s)ds \\
&\leq \int_{-\infty}^{+\infty} \phi_r(s)ds < r_2. \qquad (4.2.2)
\end{aligned}$$

Therefore, $TD \subset D$ and, by Theorem 1.1.6, T has at least a fixed point $u \in X_{H4}$. □

With more information on the asymptotic behavior of the nonlinearity it is possible to derive more data on solutions of (4.0.1):

Lemma 4.2.2. *Let $k > 0$, u be a solution of (4.0.1), (4.0.2) and f a continuous function verifying*

$$\lim_{\substack{|t|\to+\infty \\ (x,y)\to(0,0)}} f(t,x,y,z,w) = 0, \tag{4.2.3}$$

Then $u^{(i)}(\pm\infty) = 0,\ i = 0,1,2,3,4.$

Proof.

Let us rewrite equation (4.0.1) as

$$\frac{d}{dt}\left(e^t(u'''(t) - u''(t) + u'(t) - u(t))\right) = \delta_1(t)e^t \tag{4.2.4}$$

with $\delta_1(t) = f(t,u(t),u'(t),u''(t),u'''(t)) - (k+1)u(t)$.

By (4.2.3), for any $\epsilon > 0$ there is $\sigma > 0$ such that $|\delta_1(t)| < \epsilon$, for every $t > \sigma$, $|u(t)| < \sigma$, and $|u'(t)| < \sigma$.

Fix $\epsilon > 0$ and integrate (4.2.4) over $]\sigma, t[$, for any $t > \sigma$, to obtain

$$e^t\left(u'''(t) - u''(t) + u'(t) - u(t)\right) = C + \int_\sigma^t \delta_1(s)e^s ds,$$

for some $C \in \mathbb{R}$, and, subsequently,

$$\begin{aligned} \left|u'''(t) - u''(t) + u'(t) - u(t)\right| &\leq |C|e^{-t} + \epsilon e^{-t}\int_\sigma^t e^s ds \\ &\leq |C|e^{-t} + \epsilon\left(1 - e^{\sigma - t}\right), \end{aligned}$$

for $t > \sigma$.

By letting $t \to +\infty$ and by the arbitrariness of ϵ, it can be defined

$$\delta_2(t) := u'''(t) - u''(t) + u'(t) - u(t), \tag{4.2.5}$$

for some continuous function δ_2 vanishing as $t \to +\infty$. Rewriting again equation (4.2.4)

$$\frac{d}{dt}\left(e^t\left(u''(t) - 2u'(t) + 3u(t)\right)\right) := \delta_3(t)e^t \tag{4.2.6}$$

with $\delta_3(t) = \delta_2(t) + 4u(t)$. Arguing as for (4.2.4), it may be defined

$$\delta_4(t) := u''(t) - 2u'(t) + 3u(t), \tag{4.2.7}$$

for some continuous function $\delta_4(t)$ vanishing as $t \to +\infty$.

Since both $u(t), u'(t) \to 0$ this implies that $u''(t) \to 0$. Similarly, from (4.2.5) it can be demonstrated that $u'''(t) \to 0$, whereas from (4.0.1), $u^{(iv)}(t) \to 0$. □

4.3 EXAMPLE

Consider the fourth order BVP

$$\begin{cases} u^{(iv)}(t) + u(t) = \frac{u(t)(u''(t)-(u(t))^2)+(u'(t))^2(u'''(t))^3+1}{1+t^2}, t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \quad (4.3.1)$$

Function $f(t,x,y,z,w) = \frac{x(z-x^2)+y^2w^3+1}{1+t^2}$ is continuous and verifies (4.2.1) for $\max\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty, \|w\|_\infty\} < r_1, (r_1 > 0)$, with

$$\phi_{r_1}(t) := \frac{r_1^2 + r_1^3 + r_1^5 + 1}{1+t^2}.$$

Therefore, by Theorem 4.2.1 there exists a non-negative homoclinic solution of problem (4.3.1) with the phase portrait and its graphic given by next figures.

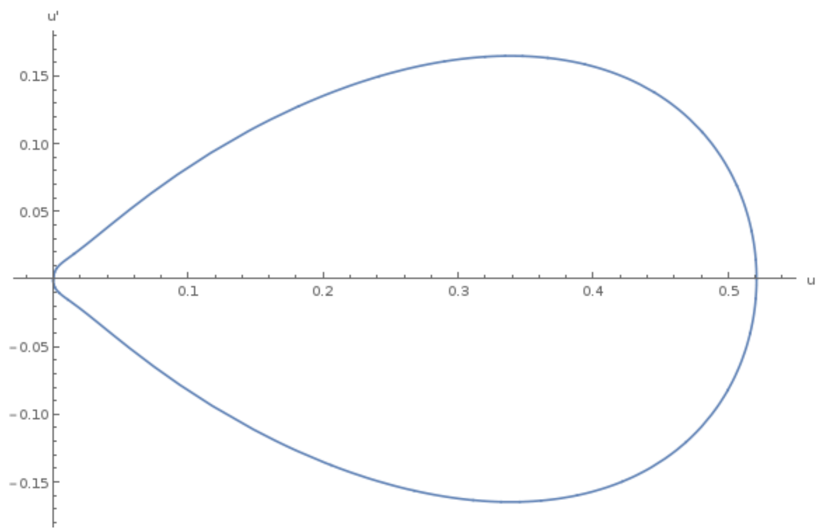

Figure 5: Phase portrait of the homoclinic solution u of (4.3.1)

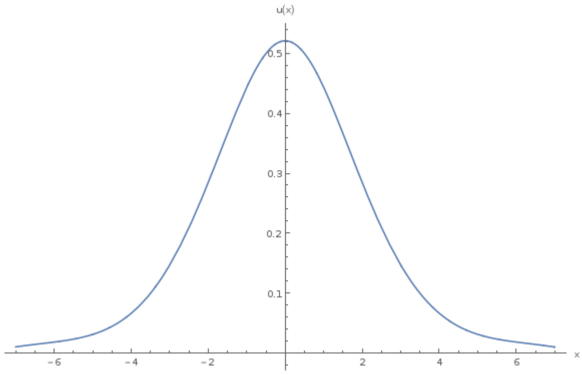

Figure 6: Graph of the homoclinic solution u of (4.3.1)

4.4 BERNOULLI-EULER-V. KARMAN PROBLEM

In [63] it is considered the nonlinear Bernoulli-Euler-v. Karman BVP

$$\begin{cases} EIu^{(iv)}(t) + ku(t) = \frac{3}{2}EA(u'(t))^2u''(t) + \omega(t), \ t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases} \tag{4.4.1}$$

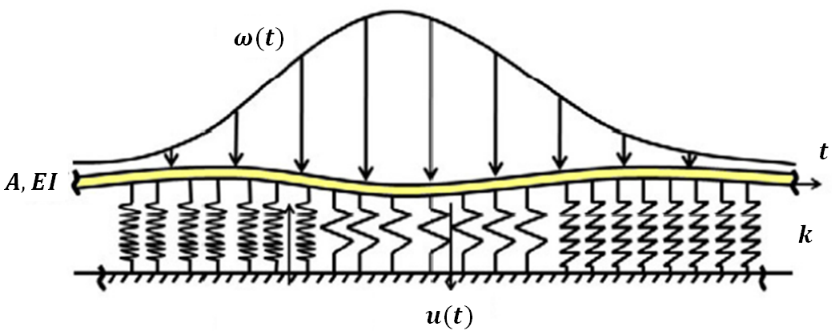

Figure 7: Infinite nonlinear beam resting on nonuniform elastic foundations

which is related to the analysis of moderately large deflections of infinite nonlinear beams resting on elastic foundations under localized external loads. More precisely, E is the Young's modulus, I the mass moment of inertia, $k\ u(t)$ the spring force upward, in which k is a spring constant (for simplicity the weight of the beam is neglected), A the cross-sectional area of the beam and $\omega(t)$ the applied loading downward (see figure 7).

An example of this family of problems is given by

$$\begin{cases} u^{(iv)}(t) + 3u(t) = \frac{3.4+u^3(t)-u''(t)(u'(t))^2}{1+t^4}, \ t \in \mathbb{R}, \\ u(\pm\infty) = u'(\pm\infty) = 0. \end{cases} \tag{4.4.2}$$

Here the loading force $\omega(t) = \frac{3.4}{1+t^4}$ and the nonlinear function $g : \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$g(t,x,y,z) := \frac{x^3 - zy^2}{1+t^4}.$$

The function $f(t,x,y,z) := g(t,x,y,z) + \omega(t)$ is continuous and verifies (4.2.1) for $\max\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty\} < r_2$, $(r_2 > 0)$, with

$$\phi_{r_2}(t) := \frac{3.4 + 2r_2^3}{1+t^4}.$$

By Theorem 4.2.1 there is a nontrivial homoclinic solution u^*. Moreover, as f verifies (4.2.3), by Corollary 4.2.2, this homoclinic solution u^* of (4.4.2) verifies $(u^*)^{(i)}(\pm\infty) = 0$, for $i = 0,1,2,3,4$.

4.5 EXTENDED FISHER-KOLMOGOROV AND SWIFT-HOHENBERG PROBLEMS

In [65], the authors consider a fourth order differential equation which can be written as

$$u^{(iv)}(t) + u(t) = 2u(t) - au''(t) - u^3(t), \ t \in \mathbb{R}. \tag{4.5.1}$$

In the literature, when $a < 0$, this equation corresponds to the well-known Extended Fisher-Kolmogorov (EFK) equation, proposed in [38], to study phase transitions. If $a > 0$, (4.5.1) is related to Swift-Hohenberg (SH) equation, which is a general model for pattern-forming process, to describe random thermal fluctuations in the Boussinesq equation (see [100]) and in the propagation of lasers (see [69]).

In this sense, equation

$$u^{(iv)}(t) + u(t) = \frac{(1+u(t))\left(1+u''(t) - u^2(t)\right)}{1+t^4}, \ t \in \mathbb{R}, \tag{4.5.2}$$

can be seen as a generalized (EFK), or (SH), where the coefficient of $u''(t)$ depends on the unknown function and it has not a definite signal. In both cases of the coefficient sign the nonlinear function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(t,x,z) := \frac{(1+x)(1+z-x^2)}{1+t^4}$$

is continuous and for $\max\{\|x\|_\infty, \|z\|_\infty\} < r_3$, $(r_3 > 0)$, f verifies (4.2.1) with

$$\phi_{r_3}(t) := \frac{(1+r_3)(1+r_3+r_3^2)}{1+t^4}.$$

Therefore, by Theorem 4.2.1, there is a homoclinic solution u^* of problem (4.5.2), (4.0.2). As it is illustrated in the next two figures, this homoclinic is a sign-changing function.

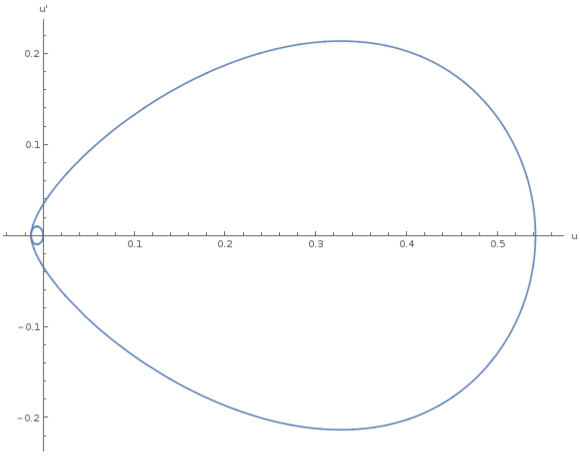

Figure 8: Phase portrait of the homoclinic solution of (4.5.2), (4.0.2)

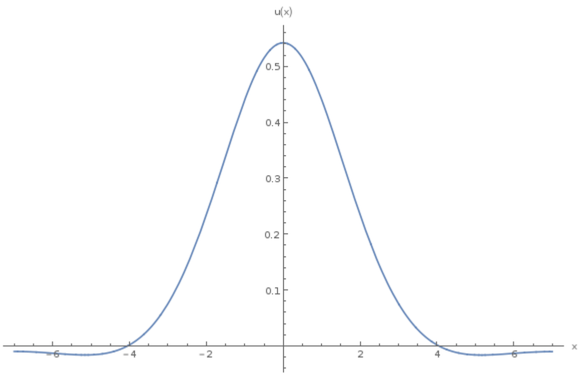

Figure 9: Graph of the homoclinic solution of (4.5.2), (4.0.2)

5

Lidstone boundary value problems

George James Lidstone (1870-1952) was an English mathematician who worked, among other things, in the study of polynomial interpolation. In 1929 he introduced a generalization of Taylor's series, where the innovation part was an approximation of a given function in the neighborhood of two points instead of one.

Essentially this interpolating polynomial is a solution of a BVP given by an elementary even order differential equation and boundary conditions defined on a bounded interval

$$\begin{cases} u^{(2m)}(t) = 0, t \in [a, b] \\ u^{(j)}(a) = A_j, u^{(j)}(b) = B_j, j = 0, 1, \dots, m-1. \end{cases}$$

In the field of approximation theory the Lidstone interpolating polynomial of degree $(2m-1)$ matches $u(t)$ and its $(m-1)$ even derivatives at both ends of the compact interval.

The homogeneous differential equation can be generalized and, coupled with boundary conditions, generates the next BVP

$$\begin{cases} u^{(2m)}(t) = f(t, u(t), u'(t), \dots, u^{(2m-1)}(t)), t \in [0, 1] \\ u^{(j)}(0) = A_j, u^{(j)}(1) = B_j, j = 0, 1, \dots, m-1. \end{cases}$$

This kind of BVP are known as **Lidstone boundary value problems**.

The particular case $m = 2$ frequently occurs in engineering and other branches of physical sciences. For instance, the deflection of a uniformly loaded rectangular plate, supported over the entire surface by an elastic foundation and rigidly supported along the edges, leads to this type of problem, or to model the deformations of an elastic beam where the type of boundary conditions considered depends on how the beam is supported at the two endpoints (see [55] and the references therein).

In this specific case, Lidstone boundary conditions,

$$u(a) = u''(a) = u(b) = u''(b) = 0,$$

means that both endpoints of the beam are simply supported.

Recently, it was introduced the so-called **complementary Lidstone boundary value problems** (see [6, 7, 106]) with differential equations of odd order together with odd boundary derivatives conditions only, that is of the type

$$\begin{cases} u^{(2m-1)}(t) = f(t, u(t), u'(t), ..., u^{(2m-2)}(t)), t \in [a, b] \\ u(a) = A_0,\ u^{(2j-1)}(a) = A_j, u^{(2j-1)}(b) = B_j, j = 1, ..., m. \end{cases}$$

These types of problems with full nonlinearities, that is, with dependence on even and odd derivatives, are very scarce (see [43, 45, 81]). However, as far as we know, Lidstone or complementary Lidstone problems were never applied to the whole real line.

This chapter is concerned with the study of a fully nonlinear differential equation on the real line

$$u^{(iv)}(t) + k^4 u(t) = f(t, u(t), u'(t), u''(t), u'''(t)), t \in \mathbb{R}, \qquad (5.0.1)$$

where $k \in \mathbb{R}, f : \mathbb{R}^5 \to \mathbb{R}$ is a continuous function and two Lidstone-type boundary conditions: the classical ones, with even derivatives,

$$u(\pm\infty) = u''(\pm\infty) = 0; \qquad (5.0.2)$$

with $u^{(i)}(\pm\infty) := \lim_{t\to\pm\infty} u^{(i)}(t), i = 0, 2$ and the so-called complementary Lidstone boundary conditions

$$u(\pm\infty) = u'(\pm\infty) = 0. \qquad (5.0.3)$$

Notice that solutions of problem (5.0.1), (5.0.3) are homoclinic solutions and in this way the results of this chapter complement and generalize the ones achieved in the previous one.

The main arguments are based on the explicit form of Green's functions associated to problem (5.0.1), (5.0.2), in a compactness criterion and fixed point theory.

The problem (5.0.1), (5.0.2) can model several real phenomena in beam theory ([1, 16]), suspension bridges ([13, 22]) and elasticity theory, among others. Equation (5.0.1) is often referred as a beam equation, because it describes the deflection of an elastic beam under a certain force. The boundary conditions (5.0.2) mean that the beam is simply supported at infinity.

The chapter is organized as it follows: first section contains the definition of an adequate space, a technical lemma that allows to relate solutions of (5.0.1), (5.0.2) with homoclinic solutions, the explicit expressions of Green's functions and other auxiliary tools, such as the criterion used to deal with the lack of compactness and the fixed-point theorem. Next section contains the main theorem and, finally, last section shows an application to the study of the bending of an infinite beam on elastic foundations.

5.1 AUXILIARY DEFINITIONS AND GREEN'S FUNCTIONS

The space of admissible functions to be used forward will be

$$X_L = \left\{ x \in C^3(\mathbb{R}) : \lim_{|t|\to+\infty} x(t) = 0 \right\},$$

equipped with the norm $\|x\|_{X_L} = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\}$, where $\|\omega\|_\infty = \sup_{t\in\mathbb{R}} |\omega(t)|$.

In this way $(X_L, \|.\|_{X_L})$ is a Banach space.

The following result will play an important role in the proof of the main result, giving an explicit solution of some linear fourth order problem via Green's functions:

Lemma 5.1.1. *If $h \in L^1(\mathbb{R})$, then for $k \in \mathbb{R}$, the linear problem*

$$\begin{cases} u^{iv}(t) + k^4 u(t) = h(t), t \in \mathbb{R}, \\ u(\pm\infty) = u''(\pm\infty) = 0 \end{cases} \tag{5.1.1}$$

has a unique solution in X_L which can be expressed as

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds$$

where

$$G(t,s) = \frac{e^{-k_*|s-t|}}{\sqrt{2}^5 k_*^3} \sin\left(k_*|s-t| + \frac{\pi}{4}\right), \tag{5.1.2}$$

with $k_* = \frac{k\sqrt{2}}{2}$.

Proof.

The homogeneous solution of the linear equation is given by

$$u(t) = e^{k_* t}\left(c_1 \cos(k_* t) + c_2 \sin(k_* t)\right) + e^{-k_* t}\left(c_3 \cos(k_* t) + c_4 \sin(k_* t)\right)$$

with $c_1, c_2, c_3, c_4 \in \mathbb{R}$ and the general solution of the homogeneous problem associated to (5.1.1), is given by

$$u(t) = \frac{1}{2k^3} \int_{-\infty}^{+\infty} e^{-k_*|s-t|} \sin\left(k_*|s-t| + \frac{\pi}{4}\right) h(s)ds.$$

For $G(t,s) := G(t,s) = \frac{e^{-k_*|s-t|}}{\sqrt{2}^5 k_*^3} \sin\left(k_*|s-t| + \frac{\pi}{4}\right)$, one can write

$$u(t) = \int_{-\infty}^{+\infty} G(t,s)h(s)ds.$$

□

Some properties of these Green's functions are in the following remark:

Remark 5.1.2. *For $i = 0,1,2,3$, defining*

$$G_i^-(t,s) := \frac{e^{k_*(s-t)}}{\sqrt{2}^{5-i} k_*^{3-i}} \sin\left(k_*(t-s) + \frac{\pi(3i+1)}{4}\right),$$

$$G_i^+(t,s) := \frac{e^{k_*(t-s)}}{\sqrt{2}^{5-i} k_*^{3-i}} \sin\left(k_*(s-t) + \frac{\pi(3i+1)}{4}\right),$$

one has

$$u^{(i)}(t) = \int_{-\infty}^{t} G_i^-(t,s)h(s)ds + (-1)^i \int_{t}^{+\infty} G_i^+(t,s)h(s)ds. \tag{5.1.3}$$

The following properties of the Green function can easily be proved

$$\lim_{|t|\to+\infty} G(t,s) = \lim_{t\to+\infty} G_i^-(t,s) = \lim_{t\to-\infty} G_i^+(t,s) = 0, \tag{5.1.4}$$

$$|G_i(t,s)| \leq \frac{1}{\sqrt{2}^{5-i} k_*^{3-i}}, i = 0,1,2,3. \tag{5.1.5}$$

Next theorem is a key argument to deal with the lack of compactness :

Theorem 5.1.3. *For a set $D \subset X_L$ to be relatively compact, it is necessary and sufficient that:*

i) $\{x(t) : x \in D\}$ *is relatively compact in* $\mathbb{R}$ *for any* $t \in \mathbb{R}$*;*

ii) *for each* $a > 0$ *the family* $D_a := \{x : [-a,a] \to \mathbb{R}\} \subset D$ *is equicontinuous;*

iii) *D is stable at $\pm\infty$, i.e., for arbitrary functions x and y in D, and any $\epsilon > 0$, there exist $T > 0$ and $\delta > 0$ such that if $|x^{(i)}(T) - y^{(i)}(T)| \leq \delta$ then $|x^{(i)}(t) - y^{(i)}(t)| \leq \epsilon$ for $t > T$, and if $|x^{(i)}(-T) - y^{(i)}(-T)| \leq \delta$ then $|x^{(i)}(t) - y^{(i)}(t)| \leq \epsilon$ for $t < -T$, for each $i = 0,1,2,3$.*

Proof.

The proof is a direct application of [93], Theorem 1. □

5.2 EXISTENCE RESULT

The main result of this chapter is given by the following theorem :

Theorem 5.2.1. *Let $f : \mathbb{R}^5 \to \mathbb{R}$ be a continuous function. If for each $r > 0$ with $\max\{\|x\|_\infty, \|y\|_\infty, \|z\|_\infty, \|w\|_\infty\} < r$ there exists a positive function $\phi_r : \mathbb{R} \to [0,+\infty)$ such that*

$$|f(t,x,y,z,w)| < \phi_r(t) \text{ and } \int_{-\infty}^{+\infty} \phi_r(t)dt < +\infty, \tag{5.2.1}$$

then problem (5.0.1), (5.0.2) has a solution $u \in X_L$, which is also a homoclinic solution.

Proof.

Define

$$F_u(t) := f(t, u(t), u'(t), u''(t), u'''(t))$$

and consider the operator $T : X_L \to X_L$ given by

$$Tu(t) = \int_{-\infty}^{+\infty} G(t,s)F_u(s)ds,$$

with $G(t,s)$ defined by (5.1.2).

As f is a continuous function, $u \in X_L$, and verifies (5.2.1), it is obvious that $F_u \in L^1(\mathbb{R})$, and, by Lemma 5.1.1, fixed points of T are solutions of problem (5.0.1), (5.0.2). So, it is enough to prove that T has a fixed point.

Clearly $Tu \in C^3(\mathbb{R})$ and, by Lebesgue's dominated convergence theorem and (5.1.4),

$$\begin{aligned}
\lim_{|t|\to+\infty} (Tu)(t) &= \int_{-\infty}^{+\infty} \lim_{|t|\to+\infty} G(t,s)F_u(s)ds = 0, \\
\lim_{|t|\to+\infty} (Tu)''(t) &= \int_{-\infty}^{+\infty} \lim_{|t|\to+\infty} G_2(t,s)F_u(s)ds = 0,
\end{aligned}$$

and

$$\lim_{|t|\to+\infty} (Tu)'(t) = \int_{-\infty}^{t} \lim_{t\to+\infty} G_1^- F_u(s)ds - \int_{t}^{+\infty} \lim_{t\to-\infty} G_1^+ F_u(s)ds = 0,$$
$$\lim_{|t|\to+\infty} (Tu)'''(t) = \int_{-\infty}^{t} \lim_{t\to+\infty} G_3^- F_u(s)ds - \int_{t}^{+\infty} \lim_{t\to-\infty} G_3^+ F_u(s)ds = 0.$$

Therefore $Tu \in X_L$, and $T : X_L \to X_L$ is well defined.

Let $B \subset X_L$ be a bounded subset . That is, there is $r_1 > 0$ such that, for any $u \in B$, one has $\|u\|_{X_L} < r_1$. By (5.1.5) and (5.2.1), for $i = 0, 1, 2, 3$,

$$\begin{aligned}
\left|(Tu(t))^{(i)}\right| &\leq \int_{-\infty}^{+\infty} |G_i(t,s)||F_u(s)|ds \\
&\leq \frac{1}{\sqrt{2}^{5-i} k_*^{3-i}} \int_{-\infty}^{+\infty} \phi_{r_1}(s)ds < +\infty, \forall t \in \mathbb{R},
\end{aligned}$$

and, therefore, $\{Tu(t) : Tu \in B\}$ is relatively compact in $\mathbb{R}$.

For some $a > 0$ and $t_1, t_2 \in [-a, a]$, as $t_1 \to t_2$,

$$\begin{aligned}
|Tu(t_1) - Tu(t_2)| &= \int_{-\infty}^{+\infty} |G(t_1,s) - G(t_2,s)||F_u(s)|ds \longrightarrow 0, \\
\left|(Tu)''(t_1) - (Tu)''(t_2)\right| &= \int_{-\infty}^{+\infty} |G_2(t_1,s) - G_2(t_2,s)||F_u(s)|ds \longrightarrow 0,
\end{aligned}$$

and for $i = 1, 3$,

$$\begin{aligned}
& \int_{-\infty}^{t} \left|G_i^-(t_1,s) - G_i^-(t_2,s)\right| |F_u(s)|ds \\
+ & \int_{t}^{+\infty} \left|G_i^+(t_1,s) - G_i^+(t_2,s)\right| |F_u(s)|ds \longrightarrow 0.
\end{aligned}$$

So the set $\{u : [-a, a] \to \mathbb{R}\} \subset B$ is equicontinuous.

As the stability at $\pm\infty$, by the continuity of f, for any $\epsilon > 0$, there exists $t_+ > 0$ and $\delta > 0$ such that when $|u(t) - v(t)| \leq \epsilon$, for $t > t_+$, then

$$|F_u(t_+) - F_v(t_+)| \leq \delta.$$

So, for $i = 0, 1, 2, 3$,

$$\begin{aligned}
\left|(Tu)^{(i)}(t) - (Tv)^{(i)}(t)\right| \leq & \int_{-\infty}^{t} \left|G_i^-(t,s)\right| |F_u(s) - F_v(s)|ds \\
& + \int_{t}^{+\infty} \left|G_i^+(t,s)\right| |F_u(s) - F_v(s)|ds \longrightarrow 0,
\end{aligned}$$

as $t \to +\infty$.

Analogously, when $|u(t) - v(t)| \leq \epsilon$, for $t < -t_+$, then

$$|F_u(-t_+) - F_v(-t_+)| \leq \delta.$$

So, T is stable at $\pm\infty$ and by Theorem 5.1.3, TB is relatively compact.

Consider now a subset $D \subset X_L$ defined as

$$D := \{u \in X_L : \|u\|_{X_L} \leq r_2\},$$

with

$$r_2 > \max\left\{r, \int_{-\infty}^{+\infty} M\ \phi_r(s)ds\right\},$$

where $r > 0$ is given by (5.2.1) and

$$M := \max\left\{1, \frac{1}{\sqrt{2}^5 k_*^3}, \frac{1}{2k_*^2}, \frac{1}{\sqrt{2}^3 k_*}\right\},$$

For $t \in \mathbb{R}$, by (5.1.5) and (5.2.1),

$$\begin{aligned}
\|Tu\|_\infty &= \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{+\infty} G(t,s)F_u(s)ds\right| \\
&\leq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}^5 k_*^3}\left|f(s,u(s),u'(s),u''(s),u'''(s))\right| ds \\
&\leq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}^5 k_*^3}\phi_r(s)ds < r_2,
\end{aligned}$$

$$\|(Tu)''\|_\infty = \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{+\infty} G_2(t,s)F_u(s)ds\right| \leq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}^3 k_*}\phi_r(s)ds < r_2,$$

and

$$\begin{aligned}
\|(Tu)'\|_\infty &= \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} G_1^-(t,s)F_u(s)ds - \int_{t}^{+\infty} G_1^+(t,s)F_u(s)ds\right| \\
&\leq \int_{-\infty}^{+\infty} \sup_{t\in\mathbb{R}}\left(\left|G_1^-(t,s)\right| + \left|G_1^-(t,s)\right|\right)\phi_r(s)ds \\
&\leq \frac{1}{2k_*^2}\int_{-\infty}^{+\infty} \phi_r(s)ds < r_2, \\
\|(Tu)'''\|_\infty &= \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t} G_3^-(t,s)F_u(s)ds - \int_{t}^{+\infty} G_3^+(t,s)F_u(s)ds\right| \\
&\leq \int_{-\infty}^{+\infty} \sup_{t\in\mathbb{R}}\left(\left|G_3^-(t,s)\right| + \left|G_3^-(t,s)\right|\right)\phi_r(s)ds \\
&\leq \int_{-\infty}^{+\infty} \phi_r(s)ds < r_2.
\end{aligned}$$

Therefore, $TD \subset D$ and, by Theorem 1.1.6, T has at least a fixed point $u \in X_L$.

This fixed point is a solution (5.0.1), (5.0.2) and, moreover, a homoclinic solution of (5.0.1), (5.0.2), by Lemma 3.1.1. □

Remark 5.2.2. *By Lemma 3.1.1, the solution of problem (5.0.1), (5.0.2) given by the previous theorem, is also a solution of the complementary Lidstone problem (5.0.1), (5.0.3).*

5.3 an infinite beam resting on granular foundations

Soil improvement via stone columns (filling a cylindrical cavity with granular material) is achieved by accelerating the consolidation of the soft soil due to the shortened drainage path, by an increase in the load-carrying capacity and/or by a decrease in the settlement due to the inclusion of stronger granular material.

Apart from improving the ground below the foundations of residential as well as industrial buildings, stone columns are also installed in soft soils or loose sand for rail roads and roadways due to the stringent settlement restrictions.

Many studies are available on the analysis of rails, treated as infinite beams on elastic foundations, subjected to concentrated moving loads as well as dynamic loads, using different techniques. For details see [76, 77, 89] and the references therein.

A longitudinal section of a rail idealized as an infinite beam resting on a ballast layer of a granular fill-stone column-reinforced soft soil system, is sketched in Fig.10.

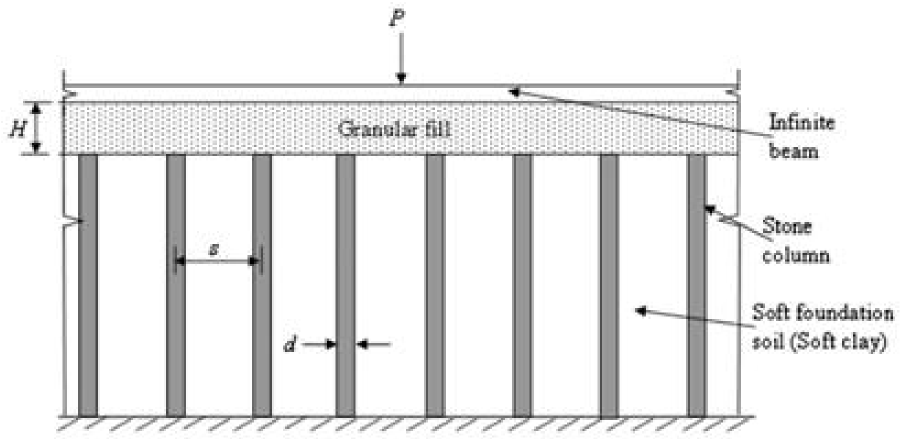

Figure 10: Railway resting on reinforced granular fill-poor soil system

The beam is founded on a granular fill layer of thickness H overlying saturated soft soil. The shear modulus of the granular fill layer is G. The diameter and the spacing of the stone columns are d and s, respectively.

In [76], the differential equation of an infinite beam with a uniform cross section and a moving load can be written as

$$EI\frac{d^4w}{d\xi^4}+\rho v^2\frac{d^2w}{d\xi^2}-c\frac{dw}{d\xi}+q=P(\xi),$$

where EI is the flexural rigidity of the infinite beam, ξ is the distance from the point of action of load at time t has been considered as $\xi = x - vt$, where v is the constant velocity at which the load is moving on the infinite beam, $w(\xi)$ is the transverse displacement of the beam at ξ, ρ is the mass per unit length of the beam, c is the coefficient of viscous damping per unit length of the beam, $P(\xi)$ is the applied load intensity and q is the reaction of the granular fill on the beam, a function that involves the shear modulus G and the thickness of the granular fill layer H.

Suppose that, (see [76]),

$$q:=\left(1-\frac{1}{1+\xi^2}\right)a\;w-GH\;\frac{d^2w}{d\xi^2},$$

for some positive parameters a, b and d. Then an example of this type of problems is given by the Lidstone boundary value problem in the whole real line, composed by the differential equation

$$\frac{d^4w}{d\xi^4}+\frac{a}{EI}w=\frac{1}{1+\xi^2}\frac{1}{EI}\left[\left(GH-\rho v^2\right)\frac{d^2w}{d\xi^2}+cv\frac{dw}{d\xi}+aw+P(\xi)\right], \tag{5.3.1}$$

together with the boundary conditions (5.0.2).

This problem (5.3.1), (5.0.2) is a particular case of the initial problem (5.0.1), (5.0.2) with $k^4 = \frac{a}{EI}$ and

$$f(\xi,x_1,x_2,x_3,x_4):=\frac{1}{1+\xi^2}\frac{1}{EI}\left[\left(GH-\rho v^2\right)x_3+cvx_2+a\;x_1+P(\xi)\right]$$

is a continuous function.

If the applied load $P(\xi)$ is bounded, that is, there is $K > 0$ such that $\|P\| \le K$, and not identically to 0, then f verifies (5.2.1) with

$$\phi_r(\xi):=\frac{1}{1+\xi^2}\frac{1}{EI}\left[|GH-\rho v^2|r+(cv+a)\,r+K\right]$$

By Theorem 5.2.1 there is a nontrivial solution w of problem (5.3.1), (5.0.2), which is, by Lemma 3.1.1, a homoclinic solution.

Part III

FUNCTIONAL BOUNDARY VALUE PROBLEMS

INTRODUCTION

Many phenomena of real life have a retrospective effect, i.e., their status in the future may depend not only from the present but also from what happened in the past. One of the mathematical processes appropriate to study this effect distributed over time, it is given by Functional Differential Equations (FDEs). It should be noted that, the concept of FDEs generalizes the common differential equations into functions with a continuous argument.

Let us precise a little more the meaning of "functional". In Algebra, we deal with algebraic equations involving one or more unknown real numbers. Functional equations are much like algebraic equations, except that the unknown quantities are functions rather than real numbers.

From an historic point a view, as far as we know, the first time where functional equations were studied, was in the work of the fourteenth century mathematician Nicole Oresme (1323-1382) who provided an indirect definition of linear functions by means of a functional equation: in modern terminology, we have three distinct real numbers x, y, and z, and, associated to each one, a variable (the "intensity" of the quality at each point) which we can write as $f(x)$, $f(y)$, and $f(z)$, respectively (for more details see [96]). The function f, considered as a linear function, is defined by the relation

$$\frac{y-x}{z-y} = \frac{f(y)-f(x)}{f(z)-f(y)}, \text{ for all distinct values of } x, y, z.$$

FDEs only appear, to the best of our knowledge, in the second half of last century (see, for example, [41, 57, 66]).

However, the word "functional" was restrict to delay, advanced or neutral differential equations. This concept was adapted to a global unknown functional variable in, for instance, [23, 27]. If the functional part appear in the differential equation, then it covers differential, integral or integro-differential equations, delay, neutral or advanced equations, among others. If the functional variation exists in the boundary conditions, so these boundary values problems include the classical two-point or multipoint conditions, but also nonlocal, integral boundary data, and cases where the global behavior of the unknown variable and its derivatives are involved. As an illustration of this type of functional problem with functional boundary conditions, we refer the problem in [85], with a functional variation in u, u' and u'' in the differential equation:

$$-\left(\phi\left(u'''(x)\right)\right)' = f(x, u''(x), u'''(x), u, u', u''),$$

for *a.e.* $x \in]a, b[$, where ϕ is an increasing homeomorphism, $I := [a, b]$, and $f : I \times \mathbb{R}^2 \times (C(I))^3 \to \mathbb{R}^2$ is a L^1-Carathéodory function, and the boundary conditions

$$\begin{aligned}
0 &= L_1\left(u\left(a\right), u, u', u''\right) \\
0 &= L_2\left(u'\left(a\right), u, u', u''\right) \\
0 &= L_3\left(u''\left(a\right), u''\left(b\right), u'''\left(a\right), u'''\left(b\right), u, u', u''\right) \\
0 &= L_4\left(u''\left(a\right), u''\left(b\right)\right)
\end{aligned}$$

where L_i, $i = 1, 2, 3, 4$, are suitable functions with L_1 and L_2 not necessarily continuous, satisfying some monotonicity assumptions.

In all the above references, functional boundary value problems are considered on bounded intervals. On unbounded domains the techniques are more delicate due to the lack of compactness of the correspondent operators. By this reason, for example, the usual Arzèla-Ascoli Theorem can not be applied.

The three chapters of this third part will present methods and techniques in order to consider some of these type of functional problems to unbounded domains, namely, the half-line or the whole real line.

In Chapter 6 it will be proved an existence and localization result for a second order BVP with functional boundary conditions. An application to an Emden-Fowler equation will be shown to illustrate the main result of the chapter.

Chapter 7 deals with third order BVPs with functional boundary conditions. These type of problems can be observed, for example, in a Falkner-Skan equation and may describe the behavior of a viscous flow over a flat plate. The localization of a solution and, moreover, some of its qualitative properties, will be presented in this chapter.

Last chapter, Chapter 8, concerns the study of ϕ -Laplacian equations. An existence and localization result will be proved and, in order to demonstrate the applicability of the main result, two examples will be shown.

6

Second order problems

Previous chapters have shown that some real phenomena are modeled by differential equations of various orders with different types of boundary conditions such as Sturm-Liouville, Homoclinic or Lidstone - type. There are, however, other problems with functional conditions, that is, situations where the boundary data do not depend on particular points but on the global variation of the unknown function. These may, for example, be provided with integral, differential, maximum or minimum arguments.

In order to cover a wide range of applications, in this chapter it will be studied the general second order differential equation

$$u''(t) = f(t, u(t), u'(t)),\ t \geq 0 \tag{6.0.1}$$

coupled with functional conditions as follows

$$\begin{cases} L(u, u(0), u'(0)) = 0 \\ u'(+\infty) = B, \end{cases} \tag{6.0.2}$$

with $L : C(\mathbb{R}_0^+) \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function verifying some monotone assumption and $B \in \mathbb{R},\ \ u'(+\infty) := \lim_{t \to +\infty} u'(t)$.

Notice that this functional dependence allows, not only conditions on the boundary, but also multipoint conditions, that is, requirements on one or more interior points.

BVP (6.0.1), (6.0.2) covers a huge variety of problems such as separated, multipoint, nonlocal, integrodifferential, periodic, anti-periodic and with maximum or minimum arguments. For example, in the case of integral conditions, it covers problems that arise naturally in the description of physical phenomena, for instance thermal conduction, semiconductor and hydrodynamic problems (see [21, 49, 64, 72, 84, 99, 108, 111, 113] and references therein).

In most cases positive solutions are searched in compact intervals. However results on the solvability of BVPs on unbounded intervals (half-line or real line) are scarce.

The main technique relies on the lower and upper solutions. Rather than the existence of bounded or unbounded solutions, their localization provides some qualitative data, like, for example, signal variation and behavior (see [25, 82]). Some results are concerned with the existence of bounded or positive solutions, as in [71, 109], and the references therein. For problem (6.0.1), (6.0.2) it is proved the existence of two types of solution, depending on B : if $B \neq 0$ the solution is unbounded; if $B = 0$ the solution is bounded.

This chapter follows the paper [33]. In this way it is organized as it follows: first some auxiliary results are defined such as the adequate space functions, some weighted norms, a criterion to overcome the lack of compactness, and the definition of lower and upper solutions. Next section contains the main result, an existence and localization theorem, which proof combines lower and upper solution technique with the fixed-point theory. Finally, last two sections contain one example and an application to some problem composed by an Emden-Fowler-type equation with a infinite multipoint conditions, which are not covered by the existent literature.

6.1 DEFINITIONS AND AUXILIARY RESULTS

Consider the space of admissible functions

$$X_F = \left\{ x \in C^1(\mathbb{R}_0^+) : \lim_{t\to+\infty} \frac{x(t)}{1+t} \in \mathbb{R}, \lim_{t\to+\infty} x'(t) \in \mathbb{R} \right\}$$

equipped with the norm $\|x\|_{X_F} = \max\{\|x\|_0, \|x'\|_1\}$, where

$$\|\omega\|_0 := \sup_{t\geq 0} \frac{|\omega(t)|}{1+t} \text{ and } \|\omega'\|_1 := \sup_{t\geq 0} |\omega'(t)|.$$

In this way $(X_F, \|.\|_{X_F})$ is a Banach space.

Solutions of the linear problem associated to (6.0.1) and usual boundary conditions are defined with Green's function, which can be obtained by standard calculus.

Lemma 6.1.1. *Let* $th, h \in L^1(\mathbb{R}_0^+)$ *and* $A, B \in \mathbb{R}$*. Then the linear BVP*

$$\begin{cases} u''(t) = h(t), t \geq 0, \\ u(0) = A, \\ u'(+\infty) = B, \end{cases} \tag{6.1.1}$$

has a unique solution in X_F*, given by*

$$u(t) = A + Bt + \int_0^{+\infty} G(t,s)h(s)ds \tag{6.1.2}$$

where

$$G(t,s) = \begin{cases} -s, & 0 \leq s \leq t \\ -t, & t \leq s < +\infty. \end{cases} \tag{6.1.3}$$

Proof.

If u is a solution of problem (6.1.1), then the general solution for the differential equation is:

$$u(t) = c_1 + c_2\, t + \int_0^t (t-s)h(s)ds,$$

where $c_1, c_2 \in \mathbb{R}$. Since u should satisfy the boundary conditions, one has

$$c_1 = A,\ c_2 = B - \int_0^{+\infty} h(s)ds.$$

The solution becomes

$$u(t) = A + Bt - t\int_0^{+\infty} h(s)ds + \int_0^t (t-s)h(s)ds.$$

And by computation

$$u(t) = A + Bt + \int_0^{+\infty} G(t,s)h(s)ds,$$

with G given by (6.1.3).

Conversely, if u is a solution of (6.1.2), it is easy to show that it satisfies the differential equation in (6.1.1). Also $u(0) = A$ and $u'(+\infty) = B$. □

The lack of compactness of X_F is overcome by the following lemma which gives a general criterion for relative compactness, referred in [3].

Lemma 6.1.2. *A set $M \subset X_F$ is relatively compact if the following conditions hold:*

i) all functions from M are uniformly bounded;

ii) all functions from M are equicontinuous on any compact interval of $\mathbb{R}_0^+$;

iii) all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that

$$\left|\frac{x(t)}{1+t} - \lim_{t\to+\infty}\frac{x(t)}{1+t}\right| < \epsilon, \left|x'(t) - \lim_{t\to+\infty} x'(t)\right| < \epsilon,$$

for all $t > t_\epsilon$ and $x \in M$.

The functions considered as lower and upper solutions for the initial problem are defined as it follows:

Definition 6.1.3. *Given $B \in \mathbb{R}$, a function $\alpha \in X_F$ is a lower solution of problem* (6.0.1), (6.0.2) *if*

$$\begin{cases} \alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), t \geq 0, \\ L(\alpha, \alpha(0), \alpha'(0)) \geq 0, \\ \alpha'(+\infty) < B. \end{cases}$$

A function $\beta \in X_F$ is an upper solution if it satisfies the reverse inequalities.

6.2 EXISTENCE AND LOCALIZATION RESULTS

In this section it is proved the existence of at least one solution for the problem (6.0.1), (6.0.2), and, moreover, some localization data, following the arguments applied in [33].

Theorem 6.2.1. *Let* $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function, and for each* $\rho > 0$*, there exists a positive function* φ_ρ *with* $\varphi_\rho, t\varphi_\rho \in L^1(\mathbb{R}_0^+)$ *such that for* $(x(t), y(t)) \in \mathbb{R}^2$ *with* $\sup_{t\geq 0}\left\{\frac{|x(t)|}{1+t}, |y(t)|\right\} < \rho$*,*

$$|f(t,x,y)| \leq \phi_\rho(t), t \geq 0. \tag{6.2.1}$$

Moreover, if $L(x_1, x_2, x_3)$ *is nondecreasing on* x_1 *and* x_3 *and there are* α, β*, lower and upper solutions of* (6.0.1), (6.0.2)*, respectively, such that*

$$\alpha(t) \leq \beta(t), \forall t \geq 0, \tag{6.2.2}$$

then problem (6.0.1), (6.0.2) *has at least one solution* $u \in X_F$ *with* $\alpha(t) \leq u(t) \leq \beta(t)$*, for* $t \geq 0$.

Proof.

Let α, β be, respectively, lower and upper solutions of (6.0.1), (6.0.2) verifying (6.2.2). Consider the modified problem

$$\begin{cases} u''(t)=f(t,\delta(t,u(t)),u'(t)) + \frac{1}{1+t^3}\frac{u(t)-\delta(t,u(t))}{1+|u(t)-\delta(t,u(t))|}, t \geq 0, \\ u(0) = \delta(0, u(0) + L(u, u(0), u'(0))), \\ u'(+\infty) = B. \end{cases} \tag{6.2.3}$$

where $\delta : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is given by $\delta(t,x) = \begin{cases} \beta(t) & ,x > \beta(t) \\ x & ,\alpha(t) \leq x \leq \beta(t) \\ \alpha(t) & ,x < \alpha(t). \end{cases}$

For clearness, the proof will follow several steps:

Step 1: *If u is a solution of (6.2.3) then* $\alpha(t) \leq u(t) \leq \beta(t), \forall t \geq 0.$

Let u be a solution of the modified problem (6.2.3) and suppose, by contradiction, that there exists $t \geq 0$ such that $\alpha(t) > u(t)$. Therefore

$$\inf_{t\geq 0}(u(t) - \alpha(t)) < 0.$$

If there is $t_* > 0$ such that

$$\min_{t\geq 0}(u(t)-\alpha(t)) := u(t_*)-\alpha(t_*) < 0,$$

one has $u'(t_*) = \alpha'(t_*)$ and $u''(t_*) - \alpha''(t_*) \geq 0$. By Definition 6.1.3 the following contradiction holds

$$\begin{aligned} 0 \leq\; & u''(t_*)-\alpha''(t_*) \\ =& f(t_*,\delta(t_*,u(t_*)),u'(t_*)) + \frac{1}{1+t_*^3}\frac{u(t_*)-\delta(t_*,u(t_*))}{1+|u(t_*)-\delta(t_*,u(t_*))|} - \alpha''(t_*) \\ =\; & f(t_*,\alpha(t_*),\alpha'(t_*)) + \frac{1}{1+t_*^3}\frac{u(t_*)-\alpha(t_*)}{1+|u(t_*)-\alpha(t_*)|} - \alpha''(t_*) \\ \leq\; & \frac{u(t_*)-\alpha(t_*)}{1+|u(t_*)-\alpha(t_*)|} < 0. \end{aligned}$$

So $u(t) \geq \alpha(t),\ \forall t > 0$.

If the infimum is attained at $t = 0$ then

$$\min_{t\geq 0}(u(t)-\alpha(t)) := u(0)-\alpha(0) < 0.$$

As u is solution of (6.2.3), by the definition of δ the following contradiction is achieved

$$\begin{aligned} 0 >\; & u(0)-\alpha(0) = \delta(0,u(0)+L(u,u(0),u'(0))) - \alpha(0) \geq \alpha(0)-\alpha(0) \\ =\; & 0. \end{aligned}$$

If

$$\inf_{t\geq 0}(u(t)-\alpha(t)) := u(+\infty)-\alpha(+\infty) < 0,$$

then $u'(+\infty) - \alpha'(+\infty) \leq 0$. As u is solution of (6.2.3), by Definition 6.1.3, this contradiction holds

$$0 \geq u'(+\infty)-\alpha'(+\infty) = B - \alpha'(+\infty) > 0.$$

Therefore $u(t) \leq \alpha(t),\ \forall t \geq 0$.

In a similar way it can be proved that $u(t) \geq \beta(t),\ \forall t \geq 0$.

Step 2: *Problem* (6.2.3) *has at least one solution.*

Let $u \in X_F$ and define the operator $T : X_F \to X_F$

$$Tu(t) = \Delta + Bt + \int_0^{+\infty} G(t,s)F_u(s)ds,$$

with

$$F_u(s) := f(s, \delta(s,u(s)), u'(s)) + \frac{1}{1+s^3}\frac{u(s) - \delta(s,u(s))}{1 + |u(s) - \delta(s,u(s))|},$$

$\Delta := \delta(0, u(0) + L(u, u(0), u'(0)))$ and G is the Green function given by (6.1.3).

Therefore, problem (6.2.3) becomes

$$\begin{cases} u''(t) = F_u(t), t \geq 0 \\ u(0) = \Delta, \\ u'(+\infty) = B, \end{cases} \tag{6.2.4}$$

and if $tF_u(t), F_u(t) \in L^1(\mathbb{R}_0^+)$, by Lemma 6.1.1 it is enough to prove that T has a fixed point.

Step 2.1: *T is well defined.*

As f is a continuous function, $Tu \in C^1(\mathbb{R}_0^+)$ and, by (6.2.1), for any $u \in X_F$ with $\rho > \max\{\|\alpha\|_{X_F}, \|\beta\|_{X_F}\}$

$$\int_0^{+\infty} |F_u(s)|\, ds \leq \int_0^{+\infty} \left(\phi_\rho(s) + \frac{1}{1+s^3}\right) ds < +\infty.$$

That is $F_u(t)$ and $tF_u(t) \in L^1(\mathbb{R}_0^+)$. By Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{t\to+\infty} \frac{(Tu)(t)}{1+t} &= \lim_{t\to+\infty} \frac{\Delta + Bt}{1+t} + \int_0^{+\infty} \lim_{t\to+\infty} \frac{G(t,s)}{1+t} F_u(s)ds \\ &\leq B + \int_0^{+\infty} \left(\phi_\rho(s) + \frac{1}{1+s^3}\right) ds < +\infty, \end{aligned}$$

and analogously for

$$\lim_{t\to+\infty} (Tu)'(t) = B - \lim_{t\to+\infty} \int_t^{+\infty} F_u(s)ds = B < +\infty.$$

Therefore $Tu \in X_F$.

Step 2.2: *T is continuous.*

Consider a convergent sequence $u_n \to u$ in X_F, there exists $\rho_1 > 0$ such that $\max\{\|\alpha\|_{X_F}, \|\beta\|_{X_F}\} < \rho_1$.

With $M := \sup_{t\geq 0} \dfrac{|G(t,s)|}{1+t}$, one has

$$\begin{aligned} \|Tu_n - Tu|_{X_F} &= \max\left\{\|Tu_n - Tu|_0, \|(Tu_n)' - (Tu)'\|_1\right\} \\ &\leq \int_0^{+\infty} M|F_{u_n}(s) - F_u(s)|ds \\ &\quad + \int_t^{+\infty} |F_{u_n}(s) - F_u(s)|ds \longrightarrow 0, \end{aligned}$$

as $n \to +\infty$.

Step 2.3: *T is compact.*

Let $B \subset X_F$ be any bounded subset. Therefore there is $r > 0$ such that $\|u\|_{X_F} < r$, $\forall u \in B$.

For each $u \in B$, and for $\max\{r, \|\alpha\|_{X_F}, \|\beta\|_{X_F}\} < r_1$

$$\begin{aligned} \|Tu\|_0 &= \sup_{t\geq 0} \frac{|Tu(t)|}{1+t} \leq \sup_{t\geq 0} \frac{|\Delta + Bt|}{1+t} \\ &\quad + \int_0^{+\infty} \sup_{t\geq 0} \frac{|G(t,s)|}{1+t} |F_u(s)|ds \\ &\leq \sup_{t\geq 0} \frac{|\Delta + Bt|}{1+t} + \int_0^{+\infty} M\left(\phi_{r_1}(s) + \frac{1}{1+s^3}\right) ds < +\infty, \\ \|(Tu)'\|_1 &= \sup_{t\geq 0} |(Tu)'(t)| \leq |B| + \int_t^{+\infty} |F_u(s)|ds \\ &\leq |B| + \int_t^{+\infty} \left(\phi_{r_1}(s) + \frac{1}{1+s^3}\right) ds < +\infty. \end{aligned}$$

So $\|Tu\|_{X_F} = \max\{\|Tu\|_0, \|(Tu)'\|_1\} < +\infty$, that is, TB is uniformly bounded in X_F.

TB is equicontinuous, because, for $L > 0$ and $t_1, t_2 \in (0, L]$, one has, as $t_1 \to t_2$,

$$\begin{aligned}\left|\frac{Tu(t_1)}{1+t_1}-\frac{Tu(t_2)}{1+t_2}\right| &\leq \left|\frac{\Delta+Bt_1}{1+t_1}-\frac{\Delta+Bt_2}{1+t_2}\right| \\ &\quad + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1}-\frac{G(t_2,s)}{1+t_2}\right| |F(u(s))|ds \\ &\leq \left|\frac{\Delta+Bt_1}{1+t_1}-\frac{\Delta+Bt_2}{1+t_2}\right| \\ &\quad + \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1}-\frac{G(t_2,s)}{1+t_2}\right| (\phi_{r_1}(s)+\frac{1}{1+s^3})ds \longrightarrow 0, \\ \left|(Tu)'(t_1)-(Tu)'(t_2)\right| &= \left|\int_{t_1}^{+\infty}F_u(s)ds-\int_{t_2}^{+\infty}F_u(s)ds\right| \\ &\leq \int_{t_1}^{t_2}|F_u(s)|ds \leq \int_{t_1}^{t_2}\left(\phi_{r_1}(s)+\frac{1}{1+s^3}\right)ds \longrightarrow 0.\end{aligned}$$

So TB is equicontinuous.

Moreover TB is equiconvergent at infinity, because, as $t \to +\infty$,

$$\begin{aligned}&\left|\frac{Tu(t)}{1+t}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t}\right| \leq \left|\frac{\Delta+Bt}{1+t}-B\right| + \int_0^{+\infty}\left|\frac{G(t,s)}{1+t}+1\right| |F_u(s)|ds \\ &\leq \left|\frac{\Delta+Bt}{1+t}-B\right| + \int_0^{+\infty}\left|\frac{G(t,s)}{1+t}+1\right| \left(\phi_{\rho_1}+\frac{1}{1+s^3}\right)ds \to 0, \quad (6.2.5)\end{aligned}$$

and

$$\begin{aligned}\left|(Tu)'(t)-\lim_{t\to+\infty}(Tu)'(t)\right| &= \int_t^{+\infty}|F_u(s)|ds \\ &\leq \int_t^{+\infty}\left(\phi_{\rho_1}+\frac{1}{1+s^3}\right)ds \longrightarrow 0, \text{ as } t\to+\infty.\end{aligned}$$

So, by Lemma 6.1.2, TB is relatively compact.

Then by Schauder's Fixed Point Theorem 1.1.6, T has at least one fixed point $u_1 \in X_F$.

Step 3: *u_1 is a solution of problem* (6.0.1), (6.0.2).

By Step 1, as u_1 is a solution of (6.2.3) then $\alpha(t) \leq u_1(t) \leq \beta(t)$, for all $t \geq 0$. So, the differential equation (6.0.1) is obtained. It remains to prove that $\alpha(0) \leq u_1(0)+L(u_1,u_1(0),u_1'(0)) \leq \beta(0)$.

Suppose, by contradiction, that $\alpha(0) > u_1(0)+L(u_1,u_1(0),u_1'(0))$. Then

$$u_1(0) = \delta(0, u_1(0)+L(u_1,u_1(0),u_1'(0))) = \alpha(0)$$

and by the monotony of L and Definition 6.1.3, the following contradiction holds

$$\begin{aligned} 0 &> u_1(0) + L(u_1, u_1(0), u_1'(0)) - \alpha(0) \\ &= L(u_1, \alpha(0), u_1'(0)) \geq L(\alpha, \alpha(0), \alpha'(0)) \geq 0. \end{aligned}$$

So $\alpha(0) \leq u_1(0) + L(u_1, u_1(0), u_1'(0))$ and in a similar way it can be proved that $u_1(0) + L(u_1, u_1(0), u_1'(0)) \leq \beta(0)$.

Therefore, u_1 is a solution of (6.0.1), (6.0.2). □

A similar result can be obtained if f is a L^1-Carathéodory function and equation (6.0.1) is replaced by

$$u''(t) = f(t, u(t), u'(t)), \ a.e. \ t \geq 0. \tag{6.2.6}$$

However in this case it must be assumed an extra assumption on f:

Theorem 6.2.2. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ be a L^1- Carathéodory function such that $f(t, x, y)$ is monotone on y. If there are α, β, lower and upper solutions of (6.2.6), (6.0.2), respectively, verifying (6.2.2) and $L(x_1, x_2, x_3)$ is nondecreasing on x_1 and x_3, then problem (6.2.6), (6.0.2) has at least one solution $u \in X_F$ with $\alpha(t) \leq u(t) \leq \beta(t)$, $\forall t \geq 0$.*

Proof.

The proof is similar to Theorem 6.2.1 except the first step.

Let u be a solution of the modified problem composed by

$$u''(t) = f(t, \delta(t, u(t)), u'(t)) + \frac{1}{1+t^3} \frac{u(t) - \delta(t, u(t))}{1 + |u(t) - \delta(t, u(t))|}, \ a.e. \ t \geq 0,$$

and the boundary conditions

$$\begin{aligned} u(0) &= \delta(0, u(0) + L(u, u(0), u'(0))), \\ u'(+\infty) &= B. \end{aligned}$$

If, by contradiction, there is $t_* > 0$ such that

$$\min_{t \geq 0} (u(t) - \alpha(t)) := u(t_*) - \alpha(t_*) < 0,$$

then $u'(t_*) = \alpha'(t_*), u''(t_*) - \alpha''(t_*) \geq 0$, and there exists an interval $I_- :=]t_-, t_*[$ where $u(t) < \alpha(t),\ u'(t) \leq \alpha'(t),\ \forall t \in I_-$.

By Definition 6.1.3 and if $f(t,x,y)$ is nondecreasing on y, this contradiction holds for $t \in I_-$:

$$\begin{aligned} 0 &\leq u''(t) - \alpha''(t) = \\ &= f(t, \delta(t, u(t)), u'(t)) + \frac{1}{1+t^3} \frac{u(t) - \delta(t, u(t))}{1 + |u(t) - \delta(t, u(t))|} - \alpha''(t) \\ &\leq f(t, \alpha(t), \alpha'(t)) + \frac{1}{1+t^3} \frac{u(t) - \alpha(t)}{1 + |u(t) - \alpha(t)|} - \alpha''(t) \\ &\leq \frac{u(t) - \alpha(t)}{1 + |u(t) - \alpha(t)|} < 0. \end{aligned}$$

The same remains valid if f is nonincreasing, considering an interval $I_+ :=]t_*, t_+[$ where $u(t) < \alpha(t), u'(t) \geq \alpha'(t), \forall t \in I_+$.

So in both cases $u(t) \geq \alpha(t), \forall t \geq 0$.

The remaining steps are identically to the proof of Theorem 6.2.1, and it will be omitted. □

6.3 EXAMPLE

Consider the second order problem in the half-line with functional boundary conditions

$$\begin{cases} u''(t) = \frac{\sin(u(t)+1) + (u'(t))^3 + u(t)e^{-t}}{1+t^3},\ t \geq 0, \\ 4u^2(0) + \min_{t \geq 0} u(t) + u'(0) - 2 = 0, \\ u'(+\infty) = 0,5. \end{cases} \tag{6.3.1}$$

Remark that the above problem is a particular case of (6.0.1), (6.0.2) with

$$\begin{aligned} f(t,x,y) &= \frac{\sin(x+1) + y^3 + xe^{-t}}{1+t^3}, \\ B &= 0,5, \\ L(a,b,c) &= 4b^2 + \min_{t \geq 0} a(t) + c - 2. \end{aligned}$$

As f is continuous in $\mathbb{R}_0^+$ then for $u \in X_F$, assumption (6.2.1) holds with $\varphi_\rho = \frac{k}{1+t^3}$, for some $k > 0$ and $\rho > 1$.

The function $L(a, b, c)$ is not decreasing in a and c, and $\alpha(t) \equiv -1$ and $\beta(t) = t$ are lower and upper solutions for (6.3.1), respectively, then, by Theorem 6.2.1, there is at least an unbounded solution u of (6.3.1) such that

$$-1 \leq u(t) \leq t, \forall t \geq 0.$$

6.4 EMDEN-FOWLER EQUATION

Emden-Fowler-types equations (see [107]) can model the heat diffusion perpendicular to parallel planes by

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\alpha}{x}\frac{\partial u(x,t)}{\partial x} + af(x,t)g(u) + h(x,t) = \frac{\partial u(x,t)}{\partial t},\ 0 < x < t,$$

where $f(x,t)g(u) + h(x,t)$ means the nonlinear heat source and $u(x,t)$ gives the temperature at time t.

In the steady-state case, and with $h(x,t) \equiv 0$, last equation becomes

$$u''(x) + \frac{\alpha}{x}u'(x) + af(x)g(u) = 0,\ x \geq 0. \tag{6.4.1}$$

If $f(x) \equiv 1$ and $g(u) = u^n$, (6.4.1) is called the Lane-Emden equation of the first kind, whereas in the second kind one has $g(u) = e^u$. Both cases are used in the study of thermal explosions. For more details see [59].

In the literature, Emden-Fowler-types equations are associated to Dirichlet or Neumann boundary conditions (see [56, 105]). To the best of author's knowledge, is the first time where some Emden-Fowler is considered together with functional boundary conditions on the half-line.

Consider that one looks for nonnegative solutions for the problem composed by the discontinuous differential equation

$$u''(x) = \frac{u'(x)}{1+x^3} + \frac{u^4(x)}{e^x},\ a.e.\ x > 0, \tag{6.4.2}$$

coupled with the infinite multi-point conditions

$$\begin{cases} \sum_{n=1}^{+\infty} a_n u(\eta_n) - u(0) + u'(0) = 0, \\ u'(+\infty) = \delta, \ (0 < \delta < 1), \end{cases} \tag{6.4.3}$$

where a_n and η_n are nonnegative sequences such that

$$a_1\eta_1 \geq a_2\eta_2 \geq ... \geq a_n\eta_n \geq ..., \sum_{n=1}^{+\infty} a_n u(\eta_n)$$

and $\sum_{n=1}^{+\infty} a_n\eta_n$ are convergent with $\sum_{n=1}^{+\infty} a_n (\eta_n + k) \leq 1 - k$, $(0 < k < 1)$.

This is a particular case of (6.2.6), (6.0.2), where

$$\begin{aligned} f(x,y,z) &= \frac{z}{1+x^3} + \frac{y^4}{e^x}, \\ B &= \delta, \\ L(v,y,z) &= \sum_{n=1}^{+\infty} a_n v(\eta_n) - y + z. \end{aligned}$$

$$|f(x,y,z)| \leq \frac{k_1}{1+x^3} + \frac{k_2}{e^x} := \varphi_r(x), \ k_1, k_2 > 0, \ r > 1.$$

As $\varphi_r(x), x\varphi_r(x) \in L^1(\mathbb{R}_0^+)$ thus f is L^1-Carathéodory, and, moreover, f is monotone on z (is nondecreasing).

As $L(v,y,z)$ is not decreasing in v and z, and functions $\alpha(x) \equiv 0$ and $\beta(x) = x + k$, are lower and upper solutions for problem (6.4.2), (6.4.3), respectively, then, by Theorem 6.2.2, there is at least an unbounded and nonnegative solution u of (6.4.2), (6.4.3) such that

$$0 \leq u(x) \leq x + k, \ \forall x \geq 0.$$

7

Third order functional problems

In this chapter it is consider a third order BVP, composed by a fully differential equation

$$u'''(t)=f(t,u(t),u'(t),u''(t)),\ t\geq 0, \tag{7.0.1}$$

where $f:\mathbb{R}_0^+\times\mathbb{R}^3\to\mathbb{R}$ is a L^1- Carathéodory function, and the functional boundary conditions on the half-line

$$\begin{aligned} L_0(u,u(0)) &= 0,\\ L_1(u,u'(0)) &= 0,\\ L_2(u,u''(+\infty)) &= 0, \end{aligned} \tag{7.0.2}$$

with $L_i : C(\mathbb{R}_0^+)\times\mathbb{R}\to\mathbb{R}, i=0,1,2$ continuous functions verifying some monotone assumptions and

$$u''(+\infty):=\lim_{t\to+\infty}u''(t).$$

There is an extensive literature on BVP defined in bounded domains, as this type of problems is an adequate tool to describe countless phenomena of real life, such as models on chemical engineering, heat conduction, thermo-elasticity, plasma physics, fluids flow,... (see, for instance, [22, 42, 48, 58, 64, 68, 75, 90]). However, on the real line or half-line the results are scarcer (see, for example, [3, 111] and the references therein).

In some backgrounds the models require different kinds of nonlocal or integral boundary conditions. In this way, it is useful to consider generalized boundary data, which include usual and non classic boundary conditions. In fact, if BVP contains a functional dependence on the unknown functions, or in its derivatives, either in the differential equation, or in the boundary data, these functional BVP allow a much more variety of problems such as separated, multipoint, non-local, integro-differential, with maximum or minimum arguments,..., as it can be seen, for instance, in [26, 29, 46, 49, 50, 84].

To the author's best knowledge, it is the first time where this type of functional boundary conditions are applied to third order BVP on the half-line. From the different arguments used it can be highlighted weighted norms, fixed point theory and lower and upper solutions method. This last technique provides a location result, which is particularly useful to get some qualitative properties on the solution, such as positivity, monotony, convexity,...

The chapter is organized as it follows: in the first section some auxiliary results are defined such as the adequate space of admissible functions, the weighted norms, an existence result for a linear BVP via Green's functions, an *a priori* bound for the second derivative from a Nagumo-type condition, a criterion to overcome the lack of compactness, and the definition of lower and upper solutions. Next section contains the main result of the chapter - an existence and localization theorem, which proof combines lower and upper solution technique with the fixed point theory. Finally an application to a Falkner-Skan equation is shown to illustrate the main result, which is not covered by previous works in the literature, as far as we know.

7.1 DEFINITIONS AND *a priori* BOUNDS

Consider the space of admissible functions

$$X_{F3}=\left\{ x \in C^2(\mathbb{R}_0^+) : \lim_{t\to+\infty} \frac{x(t)}{1+t^2} \in \mathbb{R}, \lim_{t\to+\infty} \frac{x'(t)}{1+t} \in \mathbb{R}, \lim_{t\to+\infty} x''(t) \in \mathbb{R} \right\},$$

with the norm $\|x\|_{X_{F3}} = \max\left\{\|x\|_0, \|x'\|_1, \|x''\|_2\right\}$, where

$$\|\omega\|_0 := \sup_{t\geq 0} \frac{|\omega(t)|}{1+t^2}, \|\omega\|_1 := \sup_{t\geq 0} \frac{|\omega(t)|}{1+t} \text{ and } \|\omega\|_2 := \sup_{t\geq 0} |\omega(t)|.$$

Defining in this way, $(X_{F3}, \|.\|_{X_{F3}})$ is a Banach space.

The solutions of the linear problem associated to (7.0.1), with the two-point boundary conditions in the half line, can be defined with Green's functions:

Lemma 7.1.1. *Let $t^2 h, th, h \in L^1(\mathbb{R}_0^+)$. Then the linear BVP*

$$\begin{cases} u'''(t) = h(t), \ a.e. t \geq 0, \\ u(0) = A, \\ u'(0) = B, \\ u''(+\infty) = C, \end{cases} \tag{7.1.1}$$

with $A, B, C \in \mathbb{R}$, has a unique solution given by

$$u(t) = A + Bt + \frac{Ct^2}{2} + \int_0^{+\infty} G(t,s)h(s)ds \tag{7.1.2}$$

where

$$G(t,s) = \begin{cases} \frac{s^2}{2} - ts, & 0 \leq s \leq t \\ -\frac{t^2}{2}, & 0 \leq t \leq s < +\infty. \end{cases} \tag{7.1.3}$$

Proof.

If u is a solution of problem (7.1.1), then the general solution for the differential equation is:

$$u(t) = c_1 + c_2 t + c_3 t^2 + \int_0^t \left(\frac{s^2}{2} - ts + \frac{t^2}{2} \right) h(s)ds,$$

where c_1, c_2, c_3 are real constants. Since $u(t)$ should satisfy the boundary conditions,

$$c_1 = A, c_2 = B, c_3 = \frac{C}{2} - \frac{1}{2} \int_0^{+\infty} h(s)ds,$$

and, therefore,

$$u(t) = A + Bt + \frac{Ct^2}{2} - \frac{t^2}{2} \int_0^{+\infty} h(s)ds + \int_0^t \left(\frac{s^2}{2} - ts + \frac{t^2}{2} \right) h(s)ds,$$

which can be written as (7.1.2), with $G(t,s)$ given by (7.1.3). $\square$

Some trivial properties of (7.1.3) will play an important role forward:

Lemma 7.1.2. *Function $G(t,s)$ defined by (7.1.3) verifies*

i) $\lim_{t\to+\infty} \frac{G(t,s)}{1+t^2} \in \mathbb{R},\ \forall s \geq 0;$

ii) $G_1(t,s) := \frac{\partial G(t,s)}{\partial t} := \begin{cases} -s, & 0 \leq s \leq t \\ -t, & 0 \leq t \leq s < +\infty. \end{cases};$

iii) $\lim_{t\to+\infty} \frac{G_1(t,s)}{1+t} \in \mathbb{R},\ \forall s \geq 0.$

Let $\gamma, \Gamma \in X_{F3}$ be such that $\gamma(t) \leq \Gamma(t), \gamma'(t) \leq \Gamma'(t), \forall t \geq 0$ and $\gamma''(+\infty) \leq \Gamma''(+\infty)$. Consider the set

$$E_{F3} = \left\{ (t,x,y,z) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \begin{array}{c} \gamma(t) \leq x \leq \Gamma(t), \\ \gamma'(t) \leq y \leq \Gamma'(t), \\ \gamma''(+\infty) \leq z(+\infty) \leq \Gamma''(+\infty) \end{array} \right\}.$$

The following Nagumo condition allows some *a priori* bounds on the second derivative of the solution:

Definition 7.1.3. *A function $f : E_{F3} \to \mathbb{R}$ is said to satisfy a Nagumo-type growth condition in E_{F3} if, for some positive continuous functions ψ, h and some $\nu > 1$, such that*

$$\sup \psi(t)(1+t)^{\nu} < +\infty,\ \int_0^{+\infty} \frac{s}{h(s)} ds = +\infty, \tag{7.1.4}$$

it verifies

$$|f(t,x,y,z)| \leq \psi(t)h(|z|), \forall (t,x,y,z) \in E_{F3}. \tag{7.1.5}$$

Lemma 7.1.4. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function satisfying (7.1.4) and (7.1.5) in E_{F3}. Then for every solution u of (7.0.1) satisfying, for $t \geq 0$,*

$$\begin{array}{c} \gamma(t) \leq u(t) \leq \Gamma(t), \\ \gamma'(t) \leq u'(t) \leq \Gamma'(t), \\ \gamma''(+\infty) \leq u''(+\infty) \leq \Gamma''(+\infty), \end{array} \tag{7.1.6}$$

there exists $R > 0$ (not depending on u) such that $\|u''\|_2 < R$.

Proof.

Let u be a solution of (7.0.1) verifying (7.1.6). Consider $r > 0$ such that

$$r > \max\left\{|\gamma''(+\infty)|, |\Gamma''(+\infty)|\right\}. \tag{7.1.7}$$

By the previous inequality it cannot happen $|u''(t)| > r, \forall t \geq 0$, because

$$|u''(+\infty)| < r.$$

If $|u''(t)| \leq r, \forall t \geq 0$, taking $R > r$ the proof is complete as

$$\|u''\|_2 = \sup_{t\geq 0} |u''(t)| \leq r < R.$$

In the following it will be proved that even when there exists $t \geq 0$ such that $|u''(t)| > r$, the norm $\|u''\|_2$ remains bounded.

Suppose there exists $t_0 > 0$ such that $|u''(t_0)| > r$, that is $u''(t_0) > r$ or $u''(t_0) < -r$.

In the first case, by (7.1.4), one can take $R > r$ such that

$$\int_r^R \frac{s}{h(s)} ds > M \max\left\{M_1 + \sup_{t\geq 0} \frac{\Gamma'(t)}{1+t}\frac{\nu}{\nu-1}, M_1 - \inf_{t\geq 0} \frac{\gamma'(t)}{1+t}\frac{\nu}{\nu-1}\right\}$$

with $M := \sup_{t\geq 0} \psi(t)(1+t)^\nu$ and $M_1 := \sup_{t\geq 0} \frac{\Gamma'(t)}{(1+t)^\nu} - \inf_{t\geq 0} \frac{\gamma'(t)}{(1+t)^\nu}$.

If condition (7.1.5) holds, then by (7.1.7) there are $t_*, t_+ \geq 0$ such that $t_* < t_+, u''(t_*) = r$ and $u''(t) > r, \forall t \in (t_*, t_+]$. Therefore

$$\begin{aligned}
\int_{u''(t_*)}^{u''(t_+)} \frac{s}{h(s)} ds = & \int_{t_*}^{t_+} \frac{u''(s)}{h(u''(s))} u'''(s) ds \leq \int_{t_*}^{t_+} \psi(s) u''(s) ds \\
\leq & \; M \int_{t_*}^{t_+} \frac{u''(s)}{(1+s)^\nu} ds \\
= & \; M \int_{t_*}^{t_+} \left[\left(\frac{u'(s)}{(1+s)^\nu}\right)' + \frac{\nu u'(s)}{(1+s)^{1+\nu}}\right] ds \\
\leq & \; M\left(M_1 + \sup_{t\geq 0} \frac{\Gamma'(t)}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^\nu} ds\right) < \int_r^R \frac{s}{h(s)} ds.
\end{aligned}$$

So $u''(t_+) < R$ and as t_* and t_+ are arbitrary in $\mathbb{R}_0^+$, one has that $u''(t) < R, \forall t \geq 0$.

Similarly, it can be proved the case where there are $t_-, t_* \geq 0$ such that $t_- < t_*$ and $u''(t_*) = -r,\ u''(t) < -r, \forall t \in [t_-, t_*)$.

Therefore $\|u''\|_2 < R, \forall t \geq 0$. □

The lack of compactness of X_{F3} is overcome by the following lemma which gives a general criterion for relative compactness, suggested in [3] or [37] :

Lemma 7.1.5. *A set $Z \subset X_{F3}$ is relatively compact if the following conditions hold:*

i) all functions from Z are uniformly bounded;

ii) all functions from Z are equicontinuous on any compact interval of $\mathbb{R}_0^+$;

iii) all functions from Z are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that

$$\left|\frac{x(t)}{1+t^2} - \lim_{t\to+\infty}\frac{x(t)}{1+t^2}\right| < \epsilon,$$
$$\left|\frac{x'(t)}{1+t} - \lim_{t\to+\infty}\frac{x'(t)}{1+t}\right| < \epsilon,$$
$$\left|x''(t) - \lim_{t\to+\infty} x''(t)\right| < \epsilon \text{ for all } t > t_\epsilon, x \in Z.$$

The functions considered as lower and upper solutions for the initial problem are defined as it follows, with $W^{3,1}\left(\mathbb{R}_0^+\right)$ the usual Sobolev space:

Definition 7.1.6. *A function $\alpha \in X_{F3} \cap W^{3,1}\left(\mathbb{R}_0^+\right)$ is a lower solution of problem (7.0.1),(7.0.2) if*

$$\begin{cases} \alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t)),\ t \geq 0, \\ L_0(\alpha, \alpha(0)) \geq 0, \\ L_1(\alpha, \alpha'(0)) \geq 0, \\ L_2(\alpha, \alpha''(+\infty)) > 0. \end{cases}$$

A function $\beta \in X_{F3} \cap W^{3,1}\left(\mathbb{R}_0^+\right)$ is an upper solution if it satisfies the reverse inequalities.

Remark 7.1.7. *If* $\alpha'(t) \leq \beta'(t)$, *a.e.* $t \geq 0$ *and* $\alpha(0) \leq \beta(0)$, *by integration on* $[0,t]$ *one has* $\alpha(t) \leq \beta(t), \forall t \geq 0$.

The following Lemma, suggested by [104], will ensure the existence and convergence of the derivative of some truncature-function to be used forward:

Lemma 7.1.8 ([104]). *For* $y_1, y_2 \in C^1\left(\mathbb{R}_0^+\right)$ *such that* $y_1(t) \leq y_2(t), \forall t \geq 0$, *define*

$$p(t,v) = \begin{cases} y_2(t) & , v > y_2(t) \\ v & , y_1(t) \leq v \leq y_2(t) \\ y_1(t) & , v < y_1(t). \end{cases}$$

Then, for each $v \in C^1\left(\mathbb{R}_0^+\right)$ *the next two properties hold:*

i) $\frac{d}{dt}p(t,v(t))$ *exists for a.e.* $t \geq 0$;

ii) If $v, v_m \in C^1\left(\mathbb{R}_0^+\right)$ *and* $v_m \to v$ *in* $C^1\left(\mathbb{R}_0^+\right)$ *then*

$$\frac{d}{dt}p(t,v_m(t)) \to \frac{d}{dt}p(t,v(t)) \text{ for a.e. } t \geq 0.$$

7.2 EXISTENCE AND LOCALIZATION RESULTS

In this section it is proved the existence and localization of at least one solution for the problem (7.0.1), (7.0.2).

The following assumptions are needed:

(H_1) There are α, β lower and upper solutions of (7.0.1), (7.0.2), respectively, with $\alpha'(t) \leq \beta'(t)$, $t \geq 0$, $\alpha(0) \leq \beta(0)$ and $\alpha''(+\infty) \leq \beta''(+\infty)$;

(H_2) f satisfies the Nagumo condition on E_{F3} defined with $\gamma = \alpha$ and $\Gamma = \beta$;

$$E_* := \left\{ (t,x,y,z) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \begin{array}{c} \alpha(t) \leq x \leq \beta(t), \\ \alpha'(t) \leq y \leq \beta'(t), \\ \alpha''(+\infty) \leq z(+\infty) \leq \beta''(+\infty) \end{array} \right\};$$

(H_3) $f(t,x,y,z)$ verifies the growth condition

$$\begin{aligned} f(t,\alpha(t),\alpha'(t),\alpha''(t)) &\geq f(t,x,\alpha'(t),\alpha''(t)), \\ f(t,\beta(t),\beta'(t),\beta''(t)) &\leq f(t,x,\beta'(t),\beta''(t)), \end{aligned}$$

for $t \geq 0$ fixed and $\alpha(t) \leq x \leq \beta(t)$;

(H_4) The continuous functions $L_i : C(\mathbb{R}_0^+) \times \mathbb{R} \to \mathbb{R}, i = 0,1,2,$ are such that, for $\alpha \leq v \leq \beta$,

$$\begin{cases} L_i(\alpha, \alpha^{(i)}(0)) \leq L_i(v, \alpha^{(i)}(0)) \text{ and} \\ \quad L_i(\beta, \beta^{(i)}(0)) \geq L_i(v, \beta^{(i)}(0)), \quad \text{for } i = 0,1; \\ L_2(\alpha, \alpha''(+\infty)) \leq L_2(v, \alpha''(+\infty)) \text{ and} \\ \quad L_2(\beta, \beta''(+\infty)) \geq L_2(v, \beta''(+\infty)), \\ \lim\limits_{t\to+\infty} L_2(v,w) \in \mathbb{R}, \text{ and } \alpha''(+\infty) \leq w \leq \beta''(+\infty). \end{cases}$$

Theorem 7.2.1. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1-Carathéodory function. If hypotheses (H_1)-(H_4) are verified then problem (7.0.1), (7.0.2) has at least a solution $u \in X_{F3} \cap W^{3,1}\left(\mathbb{R}_0^+\right)$ and there exists $R > 0$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \;\; \alpha'(t) \leq u'(t) \leq \beta'(t), \;\; -R \leq u''(t) \leq R, \; t \geq 0,$$

and

$$\alpha''(+\infty) \leq u''(+\infty) \leq \beta''(+\infty).$$

Proof.

Let $\alpha, \beta \in X_{F3} \cap W^{3,1}\left(\mathbb{R}_0^+\right)$ verifying (H_1).

Consider the modified and perturbed problem composed by the third order differential equation

$$\begin{aligned} u'''(t) &= f\left(t, \delta_0(t,u(t)), \delta_1(t,u'(t)), \frac{d}{dt}\left(\delta_1(t,u'(t))\right)\right) \\ &+ \frac{1}{1+t^4}\frac{u'(t)-\delta_1(t,u'(t))}{1+|u'(t)-\delta_1(t,u'(t))|}, t \geq 0, \end{aligned} \tag{7.2.1}$$

and the functional boundary equations

$$\begin{cases} u(0) = \delta_0(0, u(0) + L_0(\delta_F(u), u(0))) \\ u'(0) = \delta_1(0, u'(0) + L_1(\delta_F(u), u'(0))) \\ u''(+\infty) = \delta_\infty(u''(+\infty)) + L_2(\delta_F(u), \delta_\infty(u''(+\infty))), \end{cases} \tag{7.2.2}$$

where functions $\delta_i : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ are given by

$$\delta_i(t,x) = \begin{cases} \beta^{(i)}(t) & , x > \beta^{(i)}(t) \\ x & , \alpha^{(i)}(t) \leq x \leq \beta^{(i)}(t), i = 0, 1, \\ \alpha^{(i)}(t) & , x < \alpha^{(i)}(t) \end{cases}$$

$$\delta_\infty(x(+\infty)) = \begin{cases} \beta''(+\infty) & , x(+\infty) > \beta''(+\infty) \\ x(+\infty) & , \alpha''(+\infty) \leq x(+\infty) \leq \beta''(+\infty), \\ \alpha''(+\infty) & , x(+\infty) < \alpha''(+\infty) \end{cases}$$

$$\delta_F(v) = \begin{cases} \beta & , v > \beta \\ v & , \alpha \leq v \leq \beta. \\ \alpha & , v < \alpha \end{cases}$$

For clearness, the proof follows several steps:

Step 1: *If u is a solution of (7.2.1), (7.2.2) then*

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \ \ \alpha(t) \leq u(t) \leq \beta(t), \ \ -R \leq u''(t) \leq R, \ \forall t \geq 0$$

$$\text{and} \quad \alpha''(+\infty) \leq u''(+\infty) \leq \beta''(+\infty).$$

Let u be a solution of the modified problem (7.2.1), (7.2.2) and suppose, by contradiction, that there exists $t \geq 0$ such that $\alpha'(t) > u'(t)$. Therefore,

$$\inf_{t \geq 0}(u'(t) - \alpha'(t)) < 0.$$

- If the infimum is attained at $t = 0$, then

$$\min_{t \geq 0}(u'(t) - \alpha'(t)) = u'(0) - \alpha'(0) < 0,$$

therefore the contradiction holds

$$\begin{aligned}0 &> u'(0)-\alpha'(0)=\delta_1(0,u'(0)+L_1\left(\delta_F(u),u'(0)\right))-\alpha'(0)\\ &\geq \alpha'(0)-\alpha'(0)=0.\end{aligned}$$

- If the infimum occurs at $t=+\infty$, then

$$\inf_{t\geq 0}(u'(t)-\alpha'(t))=u'(+\infty)-\alpha'(+\infty)<0.$$

Therefore $u''(+\infty)-\alpha''(+\infty)\leq 0$ and by (H_4) and Definition 7.1.6 the contradiction holds

$$\begin{aligned}0 &\geq u''(+\infty)-\alpha''(+\infty)=\delta_\infty(u''(+\infty))+L_2\left(\delta_F(u),\delta_\infty(u''(+\infty))\right)\\ &\geq L_2(\delta_F(u),\alpha''(+\infty))\geq L_2(\alpha,\alpha''(+\infty))>0. \qquad (7.2.3)\end{aligned}$$

- If there is an interior point $t_*\in\mathbb{R}^+$ such that

$$\min_{t\geq 0}(u'(t)-\alpha'(t)):=u'(t_*)-\alpha'(t_*)<0,$$

then there exists $0\leq t_1<t_*$ where

$$\begin{aligned}u'(t)-\alpha'(t) &< 0,\ \ u''(t)-\alpha''(t)\leq 0,\ \ \forall t\in[t_1,t_*],\\ u'''(t)-\alpha'''(t) &\geq 0,\ \ a.e.t\in[t_1,t_*].\end{aligned}$$

Therefore, for $t\in[t_1,t_*]$ by (H_3) and Definition 7.1.6 the contradiction holds

$$\begin{aligned}0 &\leq \int_{t_1}^{t}\left[u'''(s)-\alpha'''(s)\right]ds\\ &= \int_{t_1}^{t}\left[f\left((s,\delta_0(s,u(s)),\delta_1(s,u'(s)),\frac{d}{ds}(\delta_1(s,u'(s)))\right)\right.\\ &\quad\left.+\frac{1}{1+s^4}\frac{u'(s)-\delta_1(s,u'(s))}{1+|u'(s)-\delta_1(s,u'(s))|}-\alpha'''(s)\right]ds\\ &\leq \int_{t_1}^{t}\left[f(s,\alpha(s),\alpha'(s),\alpha''(s))+\frac{u'(s)-\alpha'(s)}{1+|u'(s)-\alpha'(s)|}-\alpha'''(s)\right]ds\\ &\leq \int_{t_1}^{t}\left[\frac{u'(s)-\alpha'(s)}{1+|u'(s)-\alpha'(s)|}\right]ds<0.\end{aligned}$$

So $u'(t) \geq \alpha'(t)$ for $t > 0$.

In a similar way it can be proved that $u'(t) \leq \beta'(t)$, and, therefore,

$$\alpha'(t) \leq u'(t) \leq \beta'(t), \forall t \geq 0. \tag{7.2.4}$$

Remark that $\alpha(0) \leq u(0)$, otherwise, by (H_4) and Definition 7.1.6, it will happen the contradiction

$$\begin{aligned} 0 &> u(0) - \alpha(0) = \delta_0(0, u(0) + L_0(\delta_F(u), u(0))) - \alpha(0) \\ &\geq L_0(\delta_F(u), u(0))) \geq L_0(\alpha, \alpha(0))) \geq 0. \end{aligned}$$

Analogously, it can be proved that $u(0) \leq \beta(0)$. So, integrating (7.2.4) in $[0, t]$, it is easily obtained that $\alpha(t) \leq u(t) \leq \beta(t), \forall t \geq 0$.

Arguing like in (7.2.3) one can prove that $u''(+\infty) \geq \alpha''(+\infty)$ and, similarly, that $u''(+\infty) \leq \beta''(+\infty)$.

Therefore, $(t, u(t), u'(t), u''(t)) \in E_*$ and the inequality $-R \leq u''(t) \leq R$ is a direct consequence of Lemma 7.1.4.

Step 2: *The problem (7.2.1), (7.2.2) has at least one solution.*

Define the operator $T : X_{F3} \to X_{F3}$

$$Tu(t) = \Delta + \Gamma t + \frac{\Psi t^2}{2} + \int_0^{+\infty} G(t,s) F_u(s) ds,$$

where

$$\Delta := \delta_0\left(0, u(0) + L_0 \delta_F(u), u(0))\right),$$

$$\Gamma := \delta_1\left(0, u'(0) + L_0\left(\delta_F(u), u'(0)\right)\right),$$

$$\Psi := \delta_\infty(u''(+\infty)) + L_2\left(\delta_F(u), \delta_\infty(u''(+\infty))\right),$$

$G(t, s)$ is the Green function given by (7.1.3) associated with the problem

$$\begin{cases} u'''(t) = F_u(t), t \geq 0, \\ u(0) = \Delta, \\ u'(0) = \Gamma, \\ u''(+\infty) = \Psi, \end{cases} \tag{7.2.5}$$

and

$$\begin{aligned} F_u(t) &:= f\left(t, \delta_0(t,u(t)), \delta_1(t,u'(t)), \frac{d}{dt}(\delta_1(t,u'(t)))\right) \\ &+ \frac{1}{1+t^4}\frac{u'(t)-\delta_1(t,u'(t))}{1+|u'(t)-\delta_1(t,u'(t))|}. \end{aligned}$$

By Lemma 7.1.1 the fixed points of T are solutions of (7.2.5) and, therefore, of problem (7.2.1), (7.2.2).

So it is enough to prove that T has a fixed point.

Step 2.1: *T is well defined and , for a compact $D \subset X_{F3}$, $TD \subset D$.*

As f is a L^1-Carathéodory function, $Tu \in C^2\left(\mathbb{R}_0^+\right)$ and for any $u \in X_{F3}$ with

$$\rho > \max\left\{\|u\|_{X_{F3}}, \|\alpha\|_{X_{F3}}, \|\beta\|_{X_{F3}}, R\right\}$$

there exists a positive function $\phi_\rho(t)$ such that $t^2\phi_\rho(t), t\phi_\rho(t), \phi_\rho(t) \in L^1\left(\mathbb{R}_0^+\right)$ and

$$\begin{aligned} \int_0^{+\infty} |F_u(s)|\,ds &\leq \int_0^{+\infty}\left(\phi_\rho(s) + \frac{1}{1+s^4}\right)ds < +\infty, \\ \int_0^{+\infty} |sF_u(s)|\,ds &\leq \int_0^{+\infty}\left(s\phi_\rho(s) + \frac{s}{1+s^4}\right)ds < +\infty, \\ \int_0^{+\infty} \left|s^2F_u(s)\right|ds &\leq \int_0^{+\infty}\left(s^2\phi_\rho(s) + \frac{s^2}{1+s^4}\right)ds < +\infty. \end{aligned}$$

That is $F_u, tF_u, t^2F_u \in L^1\left(\mathbb{R}_0^+\right)$.

By Lebesgue Dominated Convergence Theorem, Lemma 7.1.3 and (H_4), setting

$$\lim_{t\to+\infty} L_2\left(\delta_F(u), \delta_\infty(u''(+\infty))\right) := L,$$

$$\text{and } \; M_\infty := \max\left\{\left|\alpha''(+\infty)\right| + |L|, \left|\beta''(+\infty)\right| + |L|\right\},$$

one has

$$\begin{aligned}
\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^2} &= \lim_{t\to+\infty}\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}+\int_0^{+\infty}\lim_{t\to+\infty}\frac{G(t,s)}{1+t^2}F_u(s)ds \\
&\leq \frac{M_\infty}{2}+\frac{1}{2}\int_0^{+\infty}\left(\phi_\rho(s)+\frac{1}{1+s^4}\right)ds<+\infty, \\
\lim_{t\to+\infty}\frac{(Tu)'(t)}{1+t} &= \lim_{t\to+\infty}\frac{\Gamma+\Psi t}{1+t}+\int_0^{+\infty}\lim_{t\to+\infty}\frac{G_1(t,s)}{1+t}F_u(s)ds \\
&\leq M_\infty+\int_0^{+\infty}\left(\phi_\rho(s)+\frac{1}{1+s^4}\right)ds<+\infty, \\
\lim_{t\to+\infty}(Tu)''(t) &= M_\infty+\lim_{t\to+\infty}\int_t^{+\infty}F_u(s)ds<+\infty.
\end{aligned}$$

Therefore $Tu\in X_{F3}$.

Consider now the subset $D\subset X_{F3}$ given by

$$D:=\left\{x\in X_{F3}:\|u\|_{X_{F3}}<\rho_0\right\},$$

with $\rho_0>0$ such that

$$\begin{aligned}
\rho_0 &> \max\{|\alpha(0)|,|\beta(0)|\}+\max\{|\alpha'(0)|,|\beta'(0)|\}+|k_0| \\
&+\int_0^{+\infty}M(s)\left(\phi_\rho(s)+\frac{1}{1+s^4}\right)ds,
\end{aligned}$$

where

$$k_0:=\max\{|\alpha''(+\infty)|,|\beta''(+\infty)|\}+\sup_{t\geq 0}L_2(v,w),$$

for $\alpha\leq v\leq\beta$ and $\alpha''(+\infty)\leq w\leq\beta''(+\infty)$ and

$$M(s):=\max\left\{\sup_{t\geq 0}\frac{|G(t,s)|}{1+t^2},\sup_{t\geq 0}\frac{|G_1(t,s)|}{1+t},1\right\}.$$

So, for $t\geq 0$,

$$\begin{aligned}\|Tu\|_0 &= \sup_{t\geq 0}\frac{|Tu(t)|}{1+t^2} \leq \sup_{t\geq 0}\left(\frac{\left|\Delta+\Gamma t+\frac{\Psi t^2}{2}\right|}{1+t^2}\right)\\ &+\sup_{t\geq 0}\left(\int_0^{+\infty}\frac{|G(t,s)|}{1+t^2}\,|F_u(s)|\,ds\right)\\ &\leq |\Delta|+|\Gamma|+\frac{|\Psi|}{2}+\int_0^{+\infty}M(s)\left(\phi_{\rho_0}(s)+\frac{1}{1+s^4}\right)ds<\rho_0,\end{aligned}$$

$$\begin{aligned}\left\|(Tu)'\right\|_1 &= \sup_{t\geq 0}\frac{|(Tu)'|}{1+t} \leq \sup_{t\geq 0}\left(\frac{|\Gamma+\Psi t|}{1+t}+\int_0^{+\infty}\frac{|G_1(t,s)|}{1+t}\,|F_u(s)|\,ds\right)\\ &\leq |\Gamma|+|\Psi|+\int_0^{+\infty}M(s)\left(\phi_{r_1}(s)+\frac{1}{1+s^4}\right)ds<\rho_0,\end{aligned}$$

and

$$\begin{aligned}\left\|(Tu)''\right\|_2 &= \sup_{t\geq 0}\left|(Tu)''\right| \leq \sup_{t\geq 0}\left(|\Psi|+\int_t^{+\infty}|F_u(s)|\,ds\right)\\ &\leq \sup_{t\geq 0}\left(|\Psi|+\int_t^{+\infty}\phi_{r_1}(s)+\frac{1}{1+s^4}ds\right)<\rho_0.\end{aligned}$$

So, $TD\subset D$.

Step 2.2: *T is continuous.*

Consider a convergent sequence $u_n\to u$ in X_{F3}, there exists $\rho_1>0$ such that $\max\left\{\sup_n\|u_n\|_{X_{F3}},\|\alpha\|_{X_{F3}},\|\beta\|_{X_{F3}},R\right\}<\rho_1$. By Lemma 7.1.8 one has

$$\begin{aligned}\|Tu_n-Tu\|_X &= \max\left\{\|Tu_n-Tu\|_0,\left\|(Tu_n)'-(Tu)'\right\|_1,\left\|(Tu_n)''-(Tu)''\right\|_2\right\}\\ &\leq \int_0^{+\infty}M(s)\,|F_{u_n}(s)-F_u(s)|\,ds\longrightarrow 0,\ \text{as } n\to+\infty\end{aligned}$$

Step 2.3: *T is compact.*

Let $B\subset X_{F3}$ be any bounded subset. Therefore there is $r>0$ such that $\|u\|_{X_{F3}}<r$, $\forall u\in B$.

For each $u\in B$, and for $\max\left\{r,R,\|\alpha\|_{X_{F3}},\|\beta\|_{X_{F3}}\right\}<r_1$, it can be applied similar arguments to Step 2.1 and prove that $\|Tu\|_0$, $\|(Tu)'\|_1$ and $\|(Tu)''\|_2$ are finite.

So $\|Tu\|_{X_{F3}} = \max\{\|Tu\|_0, \|(Tu)'\|_1, \|(Tu)''\|_2\} < +\infty$, that is, TB is uniformly bounded in X_{F3}.

TB is equicontinuous, because, for $L > 0$ and $t_1, t_2 \in [0, L]$, one has, as $t_1 \to t_2$,

$$\begin{aligned} \left|\frac{Tu(t_1)}{1+t_1^2} - \frac{Tu(t_2)}{1+t_2^2}\right| &\leq \left|\frac{\Delta + \Gamma t_1 + \frac{\Psi t_1}{2}}{1+t_1^2} - \frac{\Delta + \Gamma t_2 + \frac{\Psi t_2}{2}}{1+t_2^2}\right| \\ &+ \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right| |F(u(s))|\,ds \\ &\leq \left|\frac{\Delta + \Gamma t_1 + \frac{\Psi t_1}{2}}{1+t_1^2} - \frac{\Delta + \Gamma t_2 + \frac{\Psi t_2}{2}}{1+t_2^2}\right| \\ &+ \int_0^{+\infty}\left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right| \left(\phi_{r_1}(s) + \frac{1}{1+s^4}\right) ds \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} \left|\frac{(Tu)'(t_1)}{1+t_1} - \frac{(Tu)'(t_2)}{1+t_2}\right| &\leq \left|\frac{\Gamma + \Psi t_1}{1+t_1} - \frac{\Gamma + \Psi t_2}{1+t_2}\right| \\ &+ \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1} - \frac{G_1(t_2,s)}{1+t_2}\right| |F(u(s))|\,ds \\ &\leq \left|\frac{\Gamma + \Psi t_1}{1+t_1} - \frac{\Gamma + \Psi t_2}{1+t_2}\right| \\ &+ \int_0^{+\infty}\left|\frac{G_1(t_1,s)}{1+t_1} - \frac{G_1(t_2,s)}{1+t_2}\right| \left(\phi_{r_1}(s) + \frac{1}{1+s^4}\right) ds \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} \left|(Tu)''(t_1) - (Tu)''(t_2)\right| &= \left|\int_{t_1}^{+\infty} F_u(s)ds - \int_{t_2}^{+\infty} F_u(s)ds\right| \\ &\leq \int_{t_1}^{t_2} |F_u(s)|\,ds \\ &\leq \int_{t_1}^{t_2}\left(\phi_{r_1}(s) + \frac{1}{1+s^4}\right) ds \longrightarrow 0. \end{aligned}$$

Moreover TB is equiconvergent at infinity, because, as $t \to +\infty$,

$$\begin{aligned}\left|\frac{Tu(t)}{1+t^2}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t^2}\right| &\leq \left|\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}-\frac{\Psi}{2}\right| \\ &+\int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2}+\frac{1}{2}\right||F_u(s)|\,ds \\ &\leq \left|\frac{\Delta+\Gamma t+\frac{\Psi t^2}{2}}{1+t^2}-\frac{\Psi}{2}\right| \\ &+\int_0^{+\infty}\left|\frac{G(t,s)}{1+t^2}+\frac{1}{2}\right|\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\to 0,\end{aligned}$$

$$\begin{aligned}\left|\frac{(Tu)'(t)}{1+t}-\lim_{t\to+\infty}\frac{Tu(t)}{1+t}\right| &\leq \left|\frac{\Gamma+\Psi t}{1+t}-\Psi\right| \\ &+\int_0^{+\infty}\left|\frac{G_1(t,s)}{1+t}+1\right||F_u(s)|\,ds \\ &\leq \left|\frac{\Gamma+\Psi t}{1+t}-\Psi\right| \\ &+\int_0^{+\infty}\left|\frac{G_1(t,s)}{1+t}+1\right|\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\to 0,\end{aligned}$$

and

$$\begin{aligned}\left|(Tu)''(t)-\lim_{t\to+\infty}(Tu)''(t)\right| &= \int_t^{+\infty}|F_u(s)|\,ds \\ &\leq \int_t^{+\infty}\left(\phi_{\rho_1}+\frac{1}{1+s^4}\right)ds\longrightarrow 0.\end{aligned}$$

So, by Lemma 7.1.5, TB is relatively compact.

Then by Schauder's Fixed Point Theorem 1.1.6, T has at least one fixed point $u_1 \in X_{F3}$.

Step 3: *u_1 is a solution of (7.0.1), (7.0.2).*

Suppose, by contradiction, that

$$\alpha(0) > u_1(0) + L_0(\delta_F, u_1(0)).$$

Then, by (7.2.2), $u_1(0) = \alpha(0)$ and, by (H_4), and Definition 7.1.6, the following contradiction holds

$$\begin{aligned} u_1(0) + L_0(\delta_F(u_1), u_1(0)) &= \alpha(0) + L_0(\delta_F(u_1), \alpha(0)) \\ &\geq \alpha(0) + L_0(\alpha, \alpha(0)) \geq \alpha(0). \end{aligned}$$

So $\alpha(0) \leq u_1(0) + L_0(\delta_F, u_1(0))$. In a similar way it can be proved that $u_1(0) + L_0(\delta_F(u_1), u_1(0)) \leq \beta(0)$.

Assuming, by contradiction, that $\alpha'(0) > u_1'(0) + L_1(\delta_F(u_1), u_1'(0))$, then $u_1'(0) = \alpha'(0)$ and, by (H_4) and Definition 7.1.6, this contradiction is achieved:

$$\begin{aligned} u_1'(0) + L_1(\delta_F(u_1), u_1'(0)) &= \alpha'(0) + L_1(\delta_F(u_1), \alpha'(0)) \\ &\geq \alpha'(0) + L_1(\alpha, \alpha'(0)) \geq \alpha'(0). \end{aligned}$$

So $\alpha'(0) \leq u_1'(0) + L_1(\delta_F(u_1), u_1'(0))$. By similar arguments it can be proved that $u_1'(0) + L_1(\delta_F(u_1), u_1'(0)) \leq \beta'(0)$.

By Step 1, $\alpha(0) \leq u_1(0) \leq \beta(0), \alpha'(0) \leq u_1'(0) \leq \beta'(0)$ and $-R \leq u_1''(+\infty) \leq R$, and therefore, $u_1(t)$ verifies the differential equation (7.0.1) and boundary conditions (7.0.2), that is, u_1 is a solution of (7.0.1), (7.0.2). □

7.3 FALKNER–SKAN EQUATION

A classical third order differential equation, known as the Falkner-Skan equation, is at the form

$$u'''(t) + au(t)u''(t) + b(1 - (u'(t))^2) = 0, t \geq 0. \tag{7.3.1}$$

This general equation is obtained from partial differential equations, by some transformation technique (see [115]).

When $b = 0$, (7.3.1) it is known as the Blasius equation, and it models the behavior of a viscous flow over a flat plate. A boundary layer is created by a two-dimensional flow over a fixed impenetrable surface, and particles move more slowly near the surface than near the free stream. In this way equation (7.3.1) can be subject to the following boundary conditions on the half line

$$u(0) = 0,\ u'(0) = 0,\ u'(+\infty) = 1. \tag{7.3.2}$$

In the literature, only numerical techniques are applied to deal with these type of problems, (7.3.1), (7.3.2), with general a, b (see, for instance, [117]).

To illustrated the main result let consider a boundary value problem of this family, composed by the third order fully differential equation

$$u'''(t) = \frac{(u'(t))^2 - 1}{1+t^6} - \frac{u(t)|u''(t)|}{e^{3t}} + \frac{u''(t)}{1+t^4}, t \geq 0, \tag{7.3.3}$$

and the functional boundary conditions on the half-line:

$$\begin{aligned} \int_0^{+\infty} \frac{|u(t)|}{(t^2+t+1)(t^2+1)} dt - 2u(0) &= 0, \\ u'(0) &= 1, \\ \inf_{t\geq 0} \frac{u(t)}{1+t^2} - u''(+\infty) &= -0.5. \end{aligned} \tag{7.3.4}$$

Remark that the above problem is a particular case of (7.0.1), (7.0.2) with

$$\begin{aligned} f(t,x,y,z) &= \frac{y^2-1}{1+t^6} - \frac{x|z|}{e^{3t}} + \frac{z}{1+t^4}, \\ L_0(a,b) &= \int_0^{+\infty} \frac{|a(t)|}{(t^2+t+1)(t^2+1)} dt - 2b \\ L_1(a,c) &= c-1 \\ L_2(a,d) &= \inf_{t\geq 0} \frac{a(t)}{1+t^2} - d + 0.5. \end{aligned} \tag{7.3.5}$$

Functions $\beta(t) = t^2+t+1$ and $\alpha(t) = t$ are, respectively, upper and lower solutions of the problem (7.3.3), (7.3.4), verifying (H_1).

The nonlinear function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ verifies the assumptions of Theorem 7.2.1. In fact:

- f is a L^1-Carathéodory function as for $|x| < \rho(1+t^2)$, $|y| < \rho(1+t)$ and $|z| < \rho$, one has

$$|f(t,x,y,z)| \leq \frac{\rho^2(1+t)^2+1}{1+t^6} + \frac{\rho^2(1+t^2)}{e^{3t}} + \frac{\rho}{1+t^4} := \phi_\rho(t),$$

  with ϕ_ρ, $t\phi_\rho$, $t^2\phi_\rho \in L^1\left(\mathbb{R}_0^+\right)$;

- f verifies the Nagumo condition on the set

$$E_* = \left\{ (t,x,y,z) \in \mathbb{R}_0^+ \times \mathbb{R}^3 : \begin{array}{c} t \leq x \leq t^2+t+1, \\ 1 \leq y \leq 2t+1, \\ 0 \leq z(+\infty) \leq 2 \end{array} \right\},$$

  with $\psi(t) = \frac{k}{1+t^4}$ and $h = 1$, where $k > 0$ is a real constant;

- $f(t,x,y,z)$ is non-increasing in x, therefore it satisfies.(H_3);

As the functions L_i, $i = 0,1,2$, given by (7.3.5) verify (H_4), then, by Theorem 7.2.1, there is at least a solution u of (7.3.3), (7.3.4) such that

$$t \leq u(t) \leq t^2+t+1,\ 1 \leq u'(t) \leq 2t+1,\ 0 \leq u''(t) \leq 2,\ \text{for } t \geq 0.$$

This localization part shows that this solution is unbounded, nonnegative, increasing and convex.

8

PHI-LAPLACIAN EQUATIONS WITH FUNCTIONAL BOUNDARY CONDITIONS

This chapter is concerned with the study of ϕ- Laplacian equations, sometimes called in the literature as half-linear equations. More precisely, we consider a fully nonlinear equation on the half line

$$(\phi(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, \ t \geq 0, \qquad (8.0.1)$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^+ \to \mathbb{R}_0^+$ are both continuous functions, verifying adequate assumptions, but q is allowed to have a singularity when $t = 0$, coupled with the functional boundary conditions

$$L(u, u(0), u'(0)) = 0, \ \ u'(+\infty) := \lim_{t \to +\infty} u'(t) = B, \qquad (8.0.2)$$

where $L : C(\mathbb{R}_0^+) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with properties to be precise later and $B \in \mathbb{R}$. Remark that if $B \neq 0$ the solution of (8.0.1), (8.0.2) is unbounded and for $B = 0$ the corresponding solution must be bounded.

Boundary value problems, usually, are considered on compact domains. However, problems on the half-line are becoming increasingly more popular on the literature, due to their applications to fields like Engineering, Chemistry and Biology (see, for instance, [86, 109, 111]). Moreover, if equation (8.0.1) is considered on the whole real line, some techniques to guarantee the existence of homoclinic and heteroclinic solutions have been developed in last years, as it can be seen in [78, 79, 80, 88].

Problems defined on unbounded domains require more delicate procedures to deal with the lack of compactness. In this chapter, this is overcome by applying the so-called Bielecki norm and the equiconvergence at ∞, as in [37].

It is important to note that, in this chapter, it is introduced two types of new features:

- The homeomorphism ϕ does not need to be surjective, that is, $\phi(\mathbb{R})$ can be different from $\mathbb{R}$. This is overcome by an auxiliary surjective homeomorphism that extends, eventually, ϕ;

- A new and more general type of boundary conditions, given by a functional that can depend globally on the unknown function.

Moreover, this method can be applied to classical or singular ϕ-Laplacian, that is, even for homeomorphism $\phi : (-a, a) \to \mathbb{R}.$, with $0 < a < +\infty$ (for more details see [17, 28]).

In general, the lower and upper solutions method is a very adequate and useful technique to deal with functional boundary value problems as it provides not only the existence of bounded or unbounded solutions but also their localization and, from that, some qualitative data about solutions, their variation and behavior (see [26, 49, 50, 73, 74, 82]).

The technique used in this paper follows the work [47], and apply some arguments suggested in [40], combined with the upper and lower solution and a Nagumo condition to control the first derivative. The usage of such tool allows to improve the existent, namely the introduction of functional boundary conditions in the problem. These boundary conditions are very general in nature. They not only generalize most of the classical boundary conditions as they also cover the separated and multipoint cases, nonlocal or integral conditions or other boundary conditions with maximum/minimum arguments, that is, for example, of the type

$$u(0) = \max_{t \geq 0} u(t) \;\text{ or } u'(\tau) = \min_{t \geq 0} u'(t), \text{ with } \tau \geq 0,$$

provided that the assumptions on L are satisfied.

The chapter is organized as it follows: in the first section some auxiliary result are defined such as the space, the weighted norms, lower and upper solutions to be used and the necessary lemmas to proceed. The second section contains new results of existence and localization of solutions. Finally, two examples, which are not covered by the existent literature, show the applicability of the main theorems. In the first one the Nagumo conditions are verified. On the other hand, in the second one, these assumptions are replaced by a stronger condition on lower and upper solutions together with a local monotone growth on f.

8.1 PRELIMINARY RESULTS

In this section, it is presented some definitions and auxiliary results needed for the proof of the main result. Consider the following space

$$X_\phi = \left\{ x \in C^1(\mathbb{R}_0^+) : \lim_{t\to+\infty} \frac{x(t)}{e^{\theta t}} \in \mathbb{R} \right\}$$

equipped with a Bielecki norm type in $C^1(\mathbb{R}_0^+)$,

$$\|x\|_{X_\phi} := \max\left\{ \|x\|_0, \|x'\|_1 \right\},$$

where

$$\|w\|_0 = \sup_{t\geq 0} \frac{|w(t)|}{e^{\theta t}} \text{ and } \|w\|_1 = \sup_{t\geq 0} |w(t)|.$$

In this way, it is clear that $(X_\phi, \|.\|_{X_\phi})$ is a Banach space.

In addition, the following conditions must hold:

(H1) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$;

(H2) The function $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(t, x, y)$ is uniformly bounded for $t > 0$ when x and y are bounded.

(H3) The function $q : \mathbb{R}^+ \to \mathbb{R}_0^+$ is integrable, not identically to 0 in a subinterval of $\mathbb{R}^+$.

(H4) $L : C(\mathbb{R}^+) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in the first and third variables.

The approach on problem (8.0.1), (8.0.2), will be from the perspective of a fixed point problem. In this order, next lemmas will establish the link between problem (8.0.1), (8.0.2) and its integral formulation.

Let $\gamma, \Gamma \in X_\phi$ be such that $\gamma(t) \le \Gamma(t), \forall t \ge 0$. Consider the set, for $\theta > 0$,

$$E_\theta = \left\{ (t, x, y) \in \mathbb{R}_0^+ \times \mathbb{R}^2 : \frac{\gamma(t)}{e^{\theta t}} \le x \le \frac{\Gamma(t)}{e^{\theta t}} \right\}.$$

The following Nagumo condition allows some *a priori* bounds on the first derivative of the solution:

Definition 8.1.1. *A function $f : E_\theta \to \mathbb{R}$ is said to satisfy a Nagumo-type growth condition in E_θ if, for some positive and continuous functions ψ, h, such that*

$$\sup_{t \ge 0} \psi(t) < +\infty, \quad \int_0^{+\infty} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)} ds = +\infty, \tag{8.1.1}$$

it verifies

$$|q(t) f(t, x, y)| \le \psi(t) h(|y|), \ \forall (t, x, y) \in E_\theta. \tag{8.1.2}$$

Lemma 8.1.2. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying a Nagumo-type growth condition in E_θ. Then there exists $N > 0$ (not depending on u) such that every solution u of (8.0.1), (8.0.2) with*

$$\frac{\gamma(t)}{e^{\theta t}} \le u(t) \le \frac{\Gamma(t)}{e^{\theta t}}, \text{ for } t \ge 0,\ \theta > 0,$$

one has

$$\|u'\|_1 < N. \tag{8.1.3}$$

Proof.

Let u be a solution of (8.0.1), (8.0.2) with $(t, u(t), u'(t)) \in E_\theta$. Consider $r > 0$ such that

$$r > |B|. \tag{8.1.4}$$

If $|u'(t)| \le r, \forall t \ge 0$, taking $N > r$ the proof is complete as

$$\|u'\|_1 = \sup_{t \ge 0} |u'(t)| \le r < N.$$

Suppose there exists $t_0 \geq 0$ such that $|u'(t_0)| > N$, that is $u'(t_0) > N$ or $u'(t_0) < -N$.

In the first case, by (8.1.1), one can take $N > r$ such that

$$\int_{\phi(r)}^{\phi(N)} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)} ds > M \left(\sup_{t\geq 0} \frac{\Gamma(t)}{e^{\theta t}} - \inf_{t\geq 0} \frac{\gamma(t)}{e^{\theta t}} \right) \tag{8.1.5}$$

with $M := \sup_{t\geq 0} \psi(t)$.

Consider $t_1, t_2 \in [t_0, +\infty)$ such that $t_1 < t_2,\ u'(t_1) = N,\ u'(t_2) = r$ and $r \leq u'(t) \leq N, \forall t \in [t_1, t_2]$. Therefore, the following contradiction with (8.1.5) is achieved:

$$\begin{aligned}
\int_{\phi(r)}^{\phi(N)} \frac{|\phi^{-1}(s)|}{h(|\phi^{-1}(s)|)} ds = &\int_{\phi(u'(t_2))}^{\phi(u'(t_1))} \frac{\phi^{-1}(s)}{h(\phi^{-1}(s))} ds \\
= &\int_{t_2}^{t_1} \frac{u'(s)}{h(u'(s))} (\phi(u'(s)))' ds \\
= &-\int_{t_1}^{t_2} \frac{q(s)f(s,u(s),u'(s))}{h(u'(s))}\, u'(s) ds \\
\leq &\int_{t_1}^{t_2} \frac{|q(s)f(s,u(s),u'(s))|}{h(u'(s))}\, u'(s)\, ds \\
\leq &\int_{t_1}^{t_2} \psi(s)u'(s)\, ds \leq M \int_{t_1}^{t_2} u'(s)\, ds \\
\leq &M(u(t_2) - u(t_1)) \leq M \left(\sup_{t\geq 0} \frac{\Gamma(t)}{e^{\theta t}} - \inf_{t\geq 0} \frac{\gamma(t)}{e^{\theta t}} \right).
\end{aligned}$$

So $u'(t) < N, \forall t \geq 0$.

Similarly, it can be proved that $u'(t) > -N, \forall t \geq 0$ and, therefore, $\|u'\|_1 < N, \forall t \geq 0$. □

Define a surjective homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(y) = \begin{cases} \phi(y) & \text{if } |y| \leq R \\ \frac{\phi(R)-\phi(-R)}{2R} y + \frac{\phi(R)+\phi(-R)}{2} & \text{if } |y| > R \end{cases} \tag{8.1.6}$$

with $R > 0$ to be defined later.

Lemma 8.1.3. *Let $v \in L^1\left(\mathbb{R}_0^+\right)$. Then $u \in X_\phi$ such that $(\varphi(u'(t))) \in AC\left(\mathbb{R}_0^+\right)$ is the unique solution of*

$$\begin{aligned} (\varphi(u'(t)))' + v(t) &= 0, \; t \geq 0 \\ u(0) &= A \\ u'(+\infty) &= B, \end{aligned} \tag{8.1.7}$$

with $A, B \in \mathbb{R}$, if and only if

$$u(t) = A + \int_0^t \varphi^{-1}\left(\varphi\left(B\right) + \int_s^{+\infty} v\left(\tau\right) d\tau\right) ds \tag{8.1.8}$$

Proof.

Let $u \in X_\phi$ be a solution of (8.1.7). Then

$$(\varphi(u'(t)))' = -v(t),$$

by integration one has

$$\varphi(u'(t)) = \varphi(B) + \int_t^{+\infty} v(s)ds.$$

As φ is continuous and $\varphi(\mathbb{R}) = \mathbb{R}$, then

$$u'(t) = \varphi^{-1}\left(\varphi(B) + \int_t^{+\infty} v(s)ds\right)$$

and by integration again,

$$u(t) = A + \int_0^t \varphi^{-1}\left(\varphi(B) + \int_s^{+\infty} v\left(\tau\right) d\tau\right) ds.$$

□

The lack of compactness is overcome by the following lemma, which will provide a general criteria for relative compactness.

Lemma 8.1.4. *([37]) Let $M \subset X_\phi$. The set M is said to be relatively compact if the following conditions hold:*

a) M is uniformly bounded in X_ϕ;

b) the functions belonging to M are equicontinuous on any compact interval of $\mathbb{R}_0^+$;

c) *the functions f from M are equiconvergent at $+\infty$, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $\|f(t) - f(+\infty)\|_{X_\phi} < \varepsilon$ for any $t > T(\varepsilon)$ and $f \in M$.*

The adaptation of the euclidean norm of $\mathbb{R}^n$ to the weighted norms of X_ϕ is a scholar exercise and, by this reason, was omitted.

To prove the main result it is important to rely on the upper and lower solution method. The functions to be considered as upper and lower solutions are defined as it follows:

Definition 8.1.5. *A function $\alpha \in X_\phi \cap C^2(\mathbb{R}^+)$ such that $(\phi(\alpha')) \in AC(\mathbb{R}_0^+)$ is said to be a lower solution of problem (8.0.1), (8.0.2) if*

$$(\phi(\alpha'))'(t) + q(t)f(t, \alpha(t), \alpha'(t)) \geq 0$$

and

$$L(\alpha, \alpha(0), \alpha'(0)) \geq 0, \quad \alpha'(+\infty) < B \tag{8.1.9}$$

where $B \in \mathbb{R}$.

A function $\beta \in X_\phi \cap C^2(\mathbb{R}^+)$ is an upper solution if it satisfies the reversed inequalities.

The following condition is applied for well ordered lower and upper solutions of problem (8.0.1), (8.0.2):

(H5) There are α and β lower and upper solutions of (8.0.1), (8.0.2), respectively, such that

$$\alpha(t) \leq \beta(t), \forall t \geq 0. \tag{8.1.10}$$

Throughout the proof of the main result a modified and perturbed problem will be considered. It is given by

$$\begin{cases} (\varphi(u'(t)))' + q(t)f(t, \delta_0(t,u), \delta_1(t,u')) = 0 \\ u(0) = \delta_0(0, u(0) + L(u, u(0), u'(0))) \\ u'(+\infty) = B \end{cases} \tag{8.1.11}$$

with the truncature $\delta_0 : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is given by

$$\delta_0(t,y) = \begin{cases} \beta(t) & , y > \beta(t) \\ y & , \alpha(t) \leq y \leq \beta(t) \\ \alpha(t) & , y < \alpha(t), \end{cases} \tag{8.1.12}$$

and $\delta_1 : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ by

$$\delta_1(t,w) = \begin{cases} N & , w > N \\ w & , -N \leq w \leq N \\ -N & , w < -N, \end{cases} \tag{8.1.13}$$

where N is defined in (8.1.3), for functions f satisfying Nagumo's condition.

Consider $\varphi : \mathbb{R} \to \mathbb{R}$ given by (8.1.6) where $R := \max\{N, \|\alpha'\|_1, \|\beta'\|_1\}$, with N given by (8.1.3).

The operator $T : X_\phi \to X_\phi$, associated to (8.1.11) can then be defined as

$$\begin{aligned}(Tu)(t) \quad : \quad &= \delta_0(0, u(0) + L(u, u(0), u'(0))) + \\ &\int_0^t \varphi^{-1}\left(\varphi(B) + \int_s^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\right) ds.\end{aligned} \tag{8.1.14}$$

One of the essential step is to prove that the operator T has a fixed point. However, the function q may, or may not, be singular at the origin. In this way two results are presented: one for the regular case, where q is not singular when $t = 0$, and another result for the singular case.

First, let us start by presenting some lemmas for the regular case.

Lemma 8.1.6. *(Regular case) Assume that $q : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and that conditions (H1), (H2), (H3) and (H5) hold. Then the operator T is well defined.*

Proof.

For any $u \in X_\phi$ there is $K > 0$, such that $\|u\|_{X_\phi} < K$.

From (8.1.11) and (8.1.12)

$$\begin{aligned}\lim_{t\to+\infty}\frac{(Tu)(t)}{e^{\theta t}} &\leq \lim_{t\to+\infty}\frac{\beta(0)}{e^{\theta t}}+\\ &\lim_{t\to+\infty}\frac{\int_0^t \varphi^{-1}\left(\varphi(B)+\int_s^{+\infty}q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)ds}{e^{\theta t}}\\ &\leq \lim_{t\to+\infty}\frac{\int_0^t \varphi^{-1}\left(\varphi(B)+\int_s^{+\infty}q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)ds}{e^{\theta t}}.\end{aligned}$$

As $\delta_0(\tau,u)$ and $\delta_1(\tau,u')$ are bounded, by (H2), then

$$f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))$$

is uniformly bounded. Let us define

$$S_K := \sup_{t\geq 0}\left\{f(t,x,y), t\geq 0, |x|\in(0,K_0), |y|\in(0,N)\right\}, \qquad (8.1.15)$$

with $K_0=\max\{\|\alpha\|_0,\|\beta\|_0\}$ and N given by (8.1.3).

Remark that S_K does not depend on u.

From (H3) it can be defined k_1 a real number such that

$$\int_s^{+\infty} q(\tau)S_K d\tau := k_1. \qquad (8.1.16)$$

As φ is nondecreasing, the previous inequality now becomes

$$\begin{aligned}\lim_{t\to+\infty}\frac{(Tu)(t)}{e^{\theta t}} &\leq \lim_{t\to+\infty}\frac{\int_0^t \varphi^{-1}\left(\varphi(B)+S_K\int_s^{+\infty}q(\tau)d\tau\right)ds}{e^{\theta t}}\\ &\leq \lim_{t\to+\infty}\frac{\int_0^t \varphi^{-1}\left(\varphi(B)+k_1\right)ds}{e^{\theta t}}\\ &\leq \lim_{t\to+\infty}\frac{\varphi^{-1}\left(\varphi(B)+k_1\right)t}{e^{\theta t}}=0. \qquad (8.1.17)\end{aligned}$$

For

$$\begin{aligned}\lim_{t\to+\infty}(Tu)'(t) &= \varphi^{-1}\left(\varphi(B)+\int_t^{+\infty}q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))d\tau\right)\\ &= B<+\infty.\end{aligned}$$

Therefore T is well defined. □

Lemma 8.1.7. *(Regular case) Assume that* $q : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is continuous and that conditions (H1), (H2), (H3), (H4) and (H5) hold. Then the operator T is continuous.*

Proof.

Consider a convergent sequence $u_n \to u \in X_\phi$.

By the arguments used in the previous lemma, the upper bounds are uniform and, therefore, do not depend on n.

Defining

$$\Theta := \varphi(B) + \int_s^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u_n), \delta_1(\tau, u_n'))d\tau$$

and as φ is continuous, by (H2) and the Lebesgue's Dominated Convergence Theorem, one has

$$\begin{aligned} &\|(Tu_n) - (Tu)\|_0 \\ = &\sup_{t\geq 0} e^{-\theta t} \left| \begin{array}{c} \delta\left(0, u_n\left(0\right) + L\left(u_n, u_n\left(0\right), u_n'\left(0\right)\right)\right) + \int_0^t \varphi^{-1}(\Theta)ds \\ -\delta\left(0, u\left(0\right) + L\left(u, u\left(0\right), u'\left(0\right)\right)\right) - \int_0^t \varphi^{-1}(\Theta)ds \end{array} \right| \to 0, \end{aligned}$$

as $n \to +\infty$, and

$$\begin{aligned} &\left\|(Tu_n)' - (Tu)'\right\|_1 \\ \leq &\sup_{t\geq 0} \left| \begin{array}{c} \varphi^{-1}\left(\varphi(B) + \int_t^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u_n), \delta_1(\tau, u_n'))d\tau\right) \\ -\varphi^{-1}\left(\varphi(B) + \int_t^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\right) \end{array} \right| \to 0, \end{aligned}$$

as $n \to +\infty$.

Therefore T is continuous in X_ϕ. □

Lemma 8.1.8. *The operator T is compact.*

Proof.

The idea in this proof is to apply Lemma 8.1.4. For that it is important to show that the operator T is equicontinuous and equiconvergent at $+\infty$.

Let us consider $t_1, t_2 \in (0, T_0)$, where $T_0 > 0$ and $t_1 < t_2$.

Defining $\Theta := \varphi(B) + \displaystyle\int_s^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau$, then, for $\theta > 0$,

$$\begin{aligned}\left|\frac{(Tu)(t_1)}{e^{\theta t_1}}-\frac{(Tu)(t_2)}{e^{\theta t_2}}\right| &\leq \max\{|\alpha(0)|,|\beta(0)|\}\frac{e^{\theta t_2}-e^{\theta t_1}}{e^{\theta(t_1+t_2)}} \\ &+\left|\frac{e^{\theta t_2}-e^{\theta t_1}}{e^{\theta(t_1+t_2)}}\int_0^{t_1}\varphi^{-1}(\Theta)ds\right|+\left|\frac{e^{\theta t_1}\int_{t_1}^{t_2}\varphi^{-1}(\Theta)ds}{e^{\theta(t_1+t_2)}}\right| \\ &\leq \max\{|\alpha(0)|,|\beta(0)|\}\frac{e^{\theta t_2}-e^{\theta t_1}}{e^{\theta(t_1+t_2)}} \\ &+\left|\frac{e^{\theta t_2}-e^{\theta t_1}\int_0^{t_1}\varphi^{-1}\left(\varphi(B)+S_K\int_s^{+\infty}q(\tau)d\tau\right)}{e^{\theta(t_1+t_2)}}\right| \\ &+\left|\frac{e^{\theta t_1}\int_{t_1}^{t_2}\varphi^{-1}\left(\varphi(B)+S_K\int_s^{+\infty}q(\tau)d\tau\right)}{e^{\theta(t_1+t_2)}}\right|\to 0,\end{aligned}$$

as $t_1 \to t_2$.

Also, as φ^{-1} is continuous, defining $F := q(\tau)f(\tau,\delta_0(\tau,u),\delta_1(\tau,u'))$, by (8.1.15) and (8.1.16),

$$\left|(Tu)'(t_1)-(Tu)'(t_2)\right| = \left|\varphi^{-1}\left(\int_{t_1}^{+\infty}Fd\tau\right)-\varphi^{-1}\left(\int_{t_2}^{+\infty}Fd\tau\right)\right|\to 0,$$

as $t_1 \to t_2$. Therefore T is equicontinuous.

For the equiconvergence at $+\infty$ of the operator T, one has, by (8.1.17),

$$\left|\frac{(Tu)(t)}{e^{\theta t}}-\lim_{t\to+\infty}\frac{(Tu)(t)}{e^{\theta t}}\right| = \left|e^{-\theta t}\int_0^t\varphi^{-1}(\Theta)ds\right|\to 0,$$

as $t \to +\infty$. For

$$\left|(Tu)'(t)-\lim_{t\to+\infty}(Tu)'(t)\right| = \left|\varphi^{-1}(\Theta)-\lim_{t\to+\infty}\varphi^{-1}(\Theta)\right|$$

that tends to 0 as $t \to +\infty$, from (H3) and the continuity of φ^{-1}.

As T is equicontinuous and equiconvergent, then from Lemma 8.1.4, T is compact. □

Now let us consider the singular case.

Lemma 8.1.9. *(Singular case) Let q be singular at $t = 0$. Then the operator T given by (8.1.14) is completely continuous.*

Proof.

For each $n \geq 1$ and $\Theta := \varphi(B) + \int_s^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau$ let us define the approximating operator $T_n : X_\phi \to X_\phi$ given by

$$(T_n u)(t) := \delta_0(0, u(0) + L(u, u(0), u'(0))) + \int_{\frac{1}{n}}^{t} \varphi^{-1}(\Theta)ds. \quad (8.1.18)$$

In this case it is sufficient to show that T_n tends to T on X_ϕ. In fact, from (H1), (H2), (H3), (8.1.15) and (8.1.16), one has

$$\begin{aligned}\left|\frac{(Tu)(t)}{e^{\theta t}} - \frac{(T_n u)(t)}{e^{\theta t}}\right| &= \left|\frac{\int_0^{\frac{1}{n}} \varphi^{-1}(\Theta)ds}{e^{\theta t}}\right| \\ &\leq \frac{\int_0^{\frac{1}{n}} \varphi^{-1}\left(\varphi(B) + S_K \int_s^{+\infty} q(\tau)d\tau\right)}{e^{\theta t}} \to 0,\end{aligned}$$

as $n \to +\infty$, and,

$$|(Tu)'(t) - (T_n u)'(t)| =$$
$$\left|\begin{array}{c}\varphi^{-1}\left(\varphi(B) + \int_{\frac{1}{n}}^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\right) \\ -\varphi^{-1}\left(\varphi(B) + \int_0^{+\infty} q(\tau) f(\tau, \delta_0(\tau, u), \delta_1(\tau, u'))d\tau\right)\end{array}\right| \to 0,$$

as $n \to +\infty$.

Hence the operator T is completely continuous. □

8.2 EXISTENCE AND LOCALIZATION RESULT

In this section it is proved an existence and location result for (8.0.1), (8.0.2).

Theorem 8.2.1. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}_0^+ \to \mathbb{R}$ be both continuous functions, where q can have a singularity when $t = 0$, and f verifies the Nagumo conditions (8.1.1) and (8.1.2). If conditions (H1), (H2), (H3), (H4) and (H5) are satisfied, then problem (8.0.1), (8.0.2) has at least one solution $u \in X_\phi$ and there exists $N > 0$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \textit{and} \quad -N < u'(t) < N, \ \forall t \geq 0.$$

Proof.
Claim 1 - *Every solution u of (8.1.11) verifies $\alpha(t) \leq u(t) \leq \beta(t)$ and there is $N > 0$ such that $-N < u'(t) < N, \forall t \geq 0$.*

Let $u \in X_\phi$ be a solution of the modified problem (8.1.11) and suppose, by contradiction, that there exists $t > 0$ such that $\alpha(t) > u(t)$. Therefore

$$\inf_{t\geq 0}(u(t) - \alpha(t)) < 0.$$

Suppose that this infimum is attained as $t \to +\infty$. Therefore

$$\lim_{t\to+\infty}(u'(t) - \alpha'(t)) = u'(+\infty) - \alpha'(+\infty) \leq 0.$$

By Definition 8.1.5, one gets the contradiction,

$$0 \geq u'(+\infty) - \alpha'(+\infty) = B - \alpha'(+\infty) > 0.$$

Analogously, the infimum does not happen at $t = 0$, otherwise the following contradiction holds:

$$0 > u(0) - \alpha(0) = \delta(0, u(0) + L(u, u(0), u'(0))) - \alpha(0) \geq 0.$$

Therefore there are $t_* > 0$ and $t_0 < t_*$ such that

$$\begin{aligned} \min_{t\geq 0}(u(t) - \alpha(t))) \quad &: = u(t_*) - \alpha(t_*) < 0, \\ u'(t_*) \quad &= \alpha'(t_*), \\ u(t) < \alpha(t) \quad &, \quad u'(t) < \alpha'(t),\ \forall t \in [t_0, t_*[, \end{aligned}$$

and, by (H1),

$$\varphi(u'(t)) < \varphi(\alpha'(t)),\ \forall t \in [t_0, t_*[. \tag{8.2.1}$$

So, for $t \in [t_0, t_*[$, by (8.1.11), (8.1.12), (8.1.6) and Definition 8.1.5, one has

$$\begin{aligned} (\varphi(u'(t)))' &= -q(t)f(t, \delta_0(t, u), \delta_1(t, u')) = -q(t)f(t, \alpha(t), \alpha'(t)) \\ &\leq (\phi(\alpha'(t)))' = (\varphi(\alpha'(t)))'. \end{aligned}$$

Therefore the function $\varphi(u'(t)) - \varphi(\alpha'(t))$ is non-increasing on $[t_0, t_*[$ and

$$\varphi(u'(t_0)) - \varphi(\alpha'(t_0)) \geq \varphi(u'(t_*)) - \varphi(\alpha'(t_*)) = 0,$$

which is a contradiction with (8.2.1).

So, $u(t) \geq \alpha(t), \forall t \geq 0$.

Analogously it can be shown that $u(t) \leq \beta(t), \forall t \geq 0$.

The first derivatives inequalities are an immediate consequence of Lemma 8.1.2, taking

$$\gamma(t) = \frac{\alpha(t)}{e^{\theta t}} \text{ and } \Gamma(t) = \frac{\beta(t)}{e^{\theta t}}, \text{ for } t \geq 0,\ \theta > 0.$$

From the lemmas in the previous section one has that the operator T is completely continuous, both for the singular and regular cases.

Claim 2 - *The problem (8.1.11) has at least a solution $u \in X_\phi$.*

In order to apply the Schauder's fixed point theorem, we consider a closed and bounded set D defined as

$$D = \left\{ u \in X_\phi : \|u\|_X \leq \rho \right\},$$

with ρ such that

$$\rho := \max \left\{ K_0 + \sup_{t \in [0,+\infty)} \left(\frac{\varphi^{-1}\left(\varphi\left(B\right) + k_1\right) t}{e^{\theta t}} \right), \left| \varphi^{-1}\left(\varphi\left(B\right) + k_1\right) \right| \right\},$$

where K_0 is given by (8.1.15) and k_1 by (8.1.16).

For $u \in D$, arguing as in the proof of Lemma 8.1.6, as φ^{-1} is increasing, we have, for S_K given by (8.1.15),

$$\begin{aligned} \|Tu\|_0 &= \sup_{t \in [0,+\infty)} \frac{\left|\left(Tu\right)(t)\right|}{e^{\theta t}} \\ &\leq \sup_{t \in [0,+\infty)} \left(K_0 + \frac{\int_0^t \varphi^{-1}\left(\varphi\left(B\right) + \int_s^\infty q\left(\tau\right) S_K\right) ds}{e^{\theta t}} \right) \\ &\leq \sup_{t \in [0,+\infty)} \left(K_0 + \frac{\int_0^t \varphi^{-1}\left(\varphi\left(B\right) + k_1\right) ds}{e^{\theta t}} \right) \\ &= \sup_{t \in [0,+\infty)} \left(K_0 + \frac{\varphi^{-1}\left(\varphi\left(B\right) + k_1\right) t}{e^{\theta t}} \right) \leq \rho, \end{aligned}$$

and

$$\begin{aligned}\left\|(Tu)'\right\|_1 &= \sup_{t\in[0,+\infty)}\left|(Tu)'(t)\right| \\ &\le \sup_{t\in[0,+\infty)}\left|\varphi^{-1}\left(\varphi(B)+\int_0^\infty q(\tau)\, f\left(\tau,\delta_0(\tau,u),\delta_1(\tau,u')\right)d\tau\right)\right| \\ &\le \sup_{t\in[0,+\infty)}\left|\varphi^{-1}\left(\varphi(B)+k_1\right)\right| \le \rho.\end{aligned}$$

Therefore $TD \subseteq D$.

Then by Schauder's Fixed Point Theorem 1.1.6, T has at least one fixed point $u \in X_\phi$, that is, the problem (8.1.11) has at least one solution $u \in X_\phi$.

Claim 3 - *Every solution u of the problem (8.1.11) is a solution of problem (8.0.1), (8.0.2).*

Let u be a solution of the modified problem (8.1.11). By last claim, function u verifies equation (8.0.1).

Then, it will be enough to prove the inequalities

$$\alpha(0) \le u(0) + L(u, u(0), u'(0)) \le \beta(0).$$

Suppose, by contradiction, that $\alpha(0) > u(0) + L(u, u(0), u'(0))$.

By (8.1.11) and (8.1.12),

$$u(0) = \delta_0(0, u(0) + L(u, u(0), u'(0))) = \alpha(0).$$

Therefore, by Claim 1, $u'(0) \ge \alpha'(0)$.

By (H4) and Definition 8.1.5, the following contradiction is obtained

$$0 > u(0) + L(u, u(0), u'(0)) - \alpha(0) \ge L(\alpha, \alpha(0), \alpha'(0)) \ge 0.$$

In a similar way one can prove that $u(0) + L(u, u(0), u'(0)) \le \beta(0)$. □

Remark 8.2.2. *Theorem 8.2.1 still remains true for singular $\phi-$Laplacian equations. Indeed, from Nagumo condition and Lemma 8.1.2, for every u* *solution of problem (8.1.11), $\|u'(t)\|_1 < N$, and, therefore, considering in (8.1.6),$R > N$, one has*

$$\phi :]-N, N[\to \mathbb{R} \text{ and } \phi(u'(t)) \equiv \varphi(u'(t)),\ \forall t \in \mathbb{R}_0^+.$$

The control on the first derivative given by Nagumo condition and Lemma 8.1.2, which implies a subquadratic growth on the nonlinearity, can be overcome assuming stronger conditions on lower and upper solutions, as in the next theorem.

Theorem 8.2.3. *Let $f : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}_0^+ \to \mathbb{R}$ be both continuous functions, where q can have a singularity when $t = 0$. Assume that there are α and β lower and upper solutions of (8.0.1), (8.0.2), respectively, such that*

$$\alpha'(t) \leq \beta'(t), \forall t \geq 0, \tag{8.2.2}$$

and

$$\alpha(0) \leq \beta(0). \tag{8.2.3}$$

If conditions (H1), (H2), (H3) and (H4) are satisfied and

$$f(t, \alpha(t), y) \leq f(t, x, y) \leq f(t, \beta(t), y), \tag{8.2.4}$$

for $\alpha(t) \leq x \leq \beta(t)$ and $y \in \mathbb{R}$ fixed, then problem (8.0.1), (8.0.2) has at least a solution $u \in X_\phi$ such that

$$\alpha'(t) \leq u'(t) \leq \beta'(t),\ \forall t \geq 0.$$

Remark 8.2.4. *Condition (8.2.2) together with (8.2.3) imply (H5).*

Proof.

The proof follows analogous steps as in Claims 1 and 2 of Theorem 8.2.1, with φ defined by

$$R := \max\left\{\left\|\alpha'\right\|_1, \left\|\beta'\right\|_1\right\}. \tag{8.2.5}$$

It remains to prove that $\alpha'(t) \leq u'(t) \leq \beta'(t), \forall t \geq 0$.

Assume that there is a $t \geq 0$ such that $u'(t) < \alpha'(t)$, and define $t_0 \geq 0$ as

$$\inf_{t \geq 0}\left(u'(t) - \alpha'(t)\right) := u'(t_0) - \alpha'(t_0) < 0. \tag{8.2.6}$$

By (8.0.2), there is $t_1 \in (t_0, +\infty)$ such that $u'(t_1) = \alpha'(t_1)$.

By (8.2.4), for $t \in [t_0, t_1]$,

$$\begin{aligned}(\varphi(u'(t)))'(t) &= -q(t) f(t, \delta_0(t,u), \delta_1(t,u')) = -q(t) f(t, \delta_0(t,u), \alpha'(t)) \\ &\leq -q(t) f(t, \alpha(t), \alpha'(t)) \leq (\phi(\alpha'(t)))' = (\varphi(\alpha'(t)))'.\end{aligned}$$

Therefore, $\varphi(u'(t)) - \varphi(\alpha'(t))$ is non-increasing on $[t_0, t_1]$ and

$$\varphi(u'(t_0)) - \varphi(\alpha'(t_0)) \geq \varphi(u'(t_1)) - \varphi(\alpha'(t_1)) = 0.$$

So, $\varphi(u'(t_0)) \geq \varphi(\alpha'(t_0))$, and by (H1), $u'(t_0) \geq \alpha'(t_0)$ which contradicts (8.2.6). That is, $\alpha'(t) \leq u'(t)$, $\forall t \geq 0$.

In the same way it can be shown that $u'(t) \leq \beta'(t)$, $\forall t \geq 0$. □

Remark 8.2.5. *Theorem 8.2.3 holds for singular $\phi-$Laplacian equations. If in (8.1.6) it is consider R given by (8.2.5), one has*

$$\phi :]-R, R[\to \mathbb{R} \text{ and } \phi(u'(t)) \equiv \varphi(u'(t)), \forall t \geq 0.$$

8.3 EXAMPLES

In order to demonstrate the applicability of the results in this chapter two examples will follow. In the first one the nonlinearity f satisfies the Nagumo conditions and, in the second one, this assumption is replaced by a monotone behavior in f.

In both cases the null function is not a solution of the referred problem.

EXAMPLE A

Consider for some $\theta > 0$ the nonlinear problem composed by the differential equation

$$\frac{u''(t)}{1+(u'(t))^2} - \frac{1}{1+t^2}\frac{u(t)(u'(t))^2}{1+u^2(t)} = 0,\ t \geq 0, \qquad (8.3.1)$$

and the functional boundary conditions

$$\max_{t\geq 0}\frac{|u(t)|}{e^{\theta t}} + (u'(0))^3 - u(0) = 0, \quad u'(+\infty) = \frac{1}{2}. \qquad (8.3.2)$$

Remark that this problem (8.3.1), (8.3.2) is a particular case of (8.0.1)-(8.0.2) with

- $\phi(v) = \arctan v$;
- $f(t,x,y) = -\frac{xy^2}{1+x^2}$;
- $q(t) = \frac{1}{1+t^2}$;
- $L(u,x,y) = \max_{t\in\mathbb{R}_0^+}\frac{|u(t)|}{e^{\theta t}} + y^3 - x$;
- $B = \frac{1}{2}$.

With these settings it is easy to prove the following statements:

- ϕ is a nonsurjective homeomorphism satisfying (H1);
- $f(t,x,y)$ and $q(t)$ verify (H2), (H3) and the Nagumo conditions (8.1.1) and (8.1.2) with $\psi(t) \equiv 1$ and $h(|y|) = y^2$;
- $L(u,x,y)$ satisfies (H4);
- $\alpha(t) = 0,5$ and $\beta(t) = t+2$ are, respectively, lower and upper solutions of (8.3.1), (8.3.2) verifying (H5).

So, by Theorem 8.2.1, there is, at least, a solution u of (8.3.1), (8.3.2) such that

$$0,5 \leq u(t) \leq t+2,\ \forall t \geq 0.$$

Moreover, this solution is unbounded and, from the location part, strictly positive in $\mathbb{R}_0^+$.

EXAMPLE B

The functional problem

$$\begin{cases} 3(u'(t))^2u''(t) + \frac{1}{1+t^3}\left(\arctan\left((u(t))^3\right) - 2\frac{(u'(t))^5}{1+|u'(t)|^5}\right) = 0, \ t \geq 0, \\ \int_0^1 \frac{u(t)}{e^{\theta t}}dt - 5u(0) + u'(0) = 1, \\ u'(+\infty) = B, \end{cases} \tag{8.3.3}$$

for some $\theta > 0$ and $B > -1$, is a particular case of (8.0.1), (8.0.2) with

- $\phi(v) = v^3$;
- $f(t,x,y) = \arctan\left(x^3\right) - 2\frac{y^5}{1+|y|^5}$;
- $q(t) = \frac{1}{1+t^3}$;
- $L(u,x,y) = \int_0^1 \frac{u(t)}{e^{\theta t}}dt - 5x + y - 1.$

Remark that, in this case, ϕ is a surjective homeomorphism and f does not satisfy the Nagumo conditions but it verifies (8.2.4).

As the functions $\alpha(t) = -t-1$ and $\beta(t) \equiv 0$ are, respectively, lower and upper solutions of (8.3.3), satisfying assumptions (8.2.2) and (8.2.3), then, by Theorem 8.2.3, there is, at least, a solution u of (8.3.3), such that

$$-t-1 \leq u(t) \leq 0, \ \forall t \geq 0.$$

Indeed, this solution is unbounded if $B \neq 0$ and bounded if $B = 0$, and, in any case, non-positive in $\mathbb{R}_0^+$.

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