Nonlinearly perturbed hyperbolic conservation laws

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Outline

Abstract

In this presentation we attempt to stress two points of view on hyperbolic conservation laws: *modelization* and *analytical theory*. And, how they are sensitively related. While appliers are concerned with reliability, integrity or failure of solutions, mathematcians are concerned with non uniqueness, selection of physically relevant solutions or entropy criteria.
In the modeling process, within simplifications, some "spurious terms" are usually discarded from the equations and so, in order to address uniqueness, a crucial information is lost. We discuss here the relevant dissipative or dispersive effect of some of those small scale terms (zero singular limits).

The perturbed equations under consideration have the form

$$\partial_t u + \operatorname{div} f(u) = \varepsilon \operatorname{div} b(u, \nabla u) + \delta \operatorname{div} \partial_\xi c(u, \nabla u)\,,$$

which include generalized Korteweg-de Vries-Burgers equation when ξ is a space variable and Benjamin-Bona-Mahony-Burgers equation when ξ is the time variable, or

$$\partial_t u + \operatorname{div} f(u) = \delta \operatorname{div} c(u, \nabla \partial_\xi u)\,,$$

which can present unexpected dissipative properties.

Keywords: Hyperbolic conservation law; shock wave; entropy weak solution; measure-valued solution; dissipation; dispersion; diffusion; capilllarity; Burgers equation; KdV-type equation

About the Title:

The class of equations $\left({\color{red}\xi} \in \{t, x_1, \ldots, x_n\}\right)$:

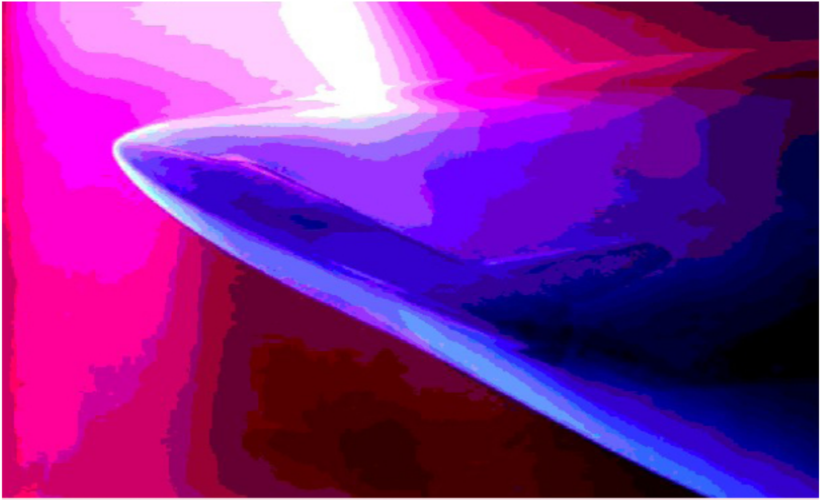
$$\overbrace{\partial_t u + \overbrace{\operatorname{div} f(u)}^{\text{transport}} = \underbrace{\overbrace{\varepsilon \operatorname{div} b(u, \nabla u)}^{\text{viscosity}} + \overbrace{\delta \operatorname{div} \partial_{\color{red}\xi} c(u, \nabla u)}^{\text{capillarity}}}_{\text{pertubation} \quad \mathcal{P}_{\varepsilon,\delta}(u;{\color{red}f},b,c)}}^{(*) \quad \text{div}(\,\ldots\,)}$$

a) 'Nonlinearly perturbed': as $\varepsilon, |\delta| \to 0^+$, $({\color{red}b, c = ?})$,

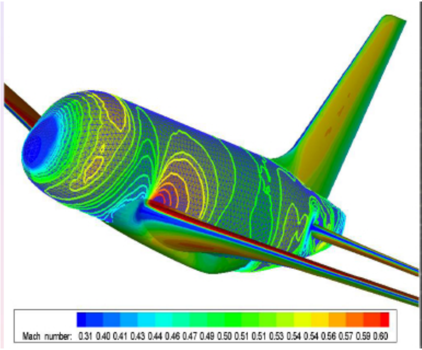
$$\partial_t u + \operatorname{div} f(u) = 0 \qquad \text{(simplified eq.)};$$

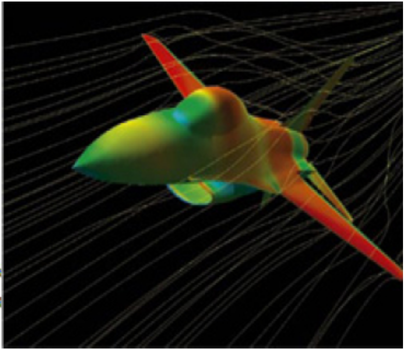
b) 'hyperbolic': finite speed of propagation;

c) 'conservation laws': they have 'divergence form' $(*)$, via modelization of "phys. closed syst.s" (sources; anisotropy).



NASA: Visible shocks at the nose in the windtunnel test





The transonic regime issues:

- control of vibrations and
- shocks strength magnitude

Singular limits

Linear perturbations of the inviscid Burgers equation (quasilinear)

$$u_t + uu_x = 0 :$$

"Vanishing viscosity method" [Kružkov, 1970]

$$u_t + \left(u^2/2\right)_x = \varepsilon u_{xx} \quad \text{[conv.; Burgers dissip. model];}$$

"Zero-dispersion limit" [Lax-Levermore, 1979, 1983]

$$u_t + \left(u^2/2\right)_x = \delta u_{xxx} \quad \text{[}\neg\text{ conv.; KdV disper. model];}$$

"Vanishing viscosity-capillarity method" [Schonbek, 1982]

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta u_{xxx} \quad \text{[KdVB dissip.-disper. model].}$$

Nonlinear perturbations & the Tartar-Schonbek-DiPerna-Szepessy setting:

[LeFloch-Natalini, 1999], [C-LeFloch, 1998], [C, 2008 (2006)], (C, 2012)

$$u_t + f(u)_x = \varepsilon\, b(u_x)_x + \delta\, c(u_x)_{xx}\,;$$

[C-LeFloch, 1999], (Bedjaoui-C, 2011)

$$u_t + f(u)_x = \varepsilon\, (|u_x|^{p-2}u_x)_x + \delta\, g(u_{xx})_x\,;$$

[Bedjaoui-C, 2013 (2012); 2013?][1]

$$u_t + f(u)_x = \delta\, g(u_{xx})_x \quad \text{[conv.; KdV disper. model].}$$

[1] Conjectured by [Brenier-Levy, 2000], with $g(v) = -|v|$ or $-v^2$ and convex f.

Paradox

$$\partial_t u + \text{div}\, f(u) = \varepsilon\ \text{div}\, b(u, \nabla u) + \delta\ \text{div}\, \partial_\xi\ c(u, \nabla u)$$

small scale mechanisms of

ε,b-dissipation (heat conduction... with diffusive effect),

δ,c-dispersion (capillarity... with oscillatory effect).

- $\xi = t$ (the time): **gBBM-Burgers**;
- $\xi = x_k$ (one space variable): **gKdV-Burgers**.

Remark: "Whitham"... travelling-waves... and convex flux...

Nonlinear hyperbolic conservation laws

Formally as $\varepsilon, \delta \to 0$, all the equations have the, same, zero diffusion-dispersion limit (conservation law)

$$\partial_t u + \operatorname{div} f(u) = 0.$$

Well-posedness of the (time-evolution) Cauchy problem means that this equation must be hyperbolic and, because it is nonlinear, it develops discontinuities ("shocks") in finite time: the global (in time, weak-) solutions are not unique.

Question: how can we select the physically relevant solution?

Classification

As the ε,δ-parameters tend to zero, we can:

- have no limit at all (**Failure**; $\delta > \mathcal{O}\left(\varepsilon^{\gamma}\right)$),
- obtain different limits (**Reliability**):
  - *classical-entropy* weak-solutions (**Integrity**; $\delta < \mathcal{O}\left(\varepsilon^{\gamma}\right)$),
  - *nonclassical-entropy* weak-solutions ($\delta = \mathcal{O}\left(\varepsilon^{\gamma}\right)$),

according to the γ-balance of strengths and a growth ratio of diffusion and dispersion.

While our proofs establish **integrity**, major emphasis is on reliability (and failure):

we look to all known physically relevant solutions, both classical and nonclassical;

we are aiming for a realistic multi-space dimensional framework (analytical setting on measure-valued function theory).

We use energy based methods,

without, until now, dimensional arguments.

Remark: numerics is hopeless.

Issues concern:

the behaviour and selection of the right models/solutions;

the proof of a "vanishing viscosity-capillarity method".

[C, 2008 (2007)], [C-Sasportes, 2009 (2012)], (C, 2012)

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div} \left(\varepsilon\, b_j(u, \nabla u) + \delta\, g(u) \sum_{k=1}^{d} \partial_{x_k} c_{jk}\big(g(u)\, \nabla u\big) \right)_{1 \le j \le d}$$

$$u(x,0) = u_0^{\varepsilon,\delta}(x)\,.$$

(A_2) for some $\mu \geq 0$, $r > 0$, $|b(u,\lambda)| = \mathcal{O}\left(|u|^{\mu}\right) \mathcal{O}\left(|\lambda|^{r}\right)$ as $|u|, |\lambda| \to \infty$,

(A_3) for some $\varphi \geq 0$, $\vartheta < 1$, $D > 0$,
$\lambda \cdot b(u,\lambda) \geq D\,|u|^{\mu\varphi}\,|\lambda|^{r+1-\vartheta}$, $\forall u \in \mathbb{R}$, $\lambda \in \mathbb{R}^d$.

(A_4) for some $\rho > 0$, $K_c > 0$: $\forall \lambda \in \mathbb{R}^d$ $\|[c_{jk}(\lambda)]\| \leq K_c\,|\lambda|^{\rho}$.

We proved convergence if $\delta = o(\varepsilon^{\frac{\rho+2}{r+1-\vartheta}})$ with $r + 1 - \vartheta \geq \rho + 2$.

Entropy m.-v. solution

Assume that $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $f \in C(\mathbb{R})^d$ satisfies the growth condition

$$(A_1) \;\; |f(u)| = \mathcal{O}(|u|^m) \text{ as } |u| \to \infty, \qquad \text{for some } m \in [0, q).$$

A Young measure ν associated with $\{u_n\}_{n\in\mathbb{N}}$, a *bounded sequence* in $L^\infty((0,T); L^q(\mathbb{R}^d))$, is called an entropy measure-valued solution if

$$\partial_t \langle \nu_{(\cdot)}, |u-k| \rangle + \mathrm{div} \langle \nu_{(\cdot)}, \mathrm{sgn}(u-k)(f(u)-f(k)) \rangle \leq 0, \qquad \text{for all } k \in \mathbb{R},$$

in the sense of distributions on $\mathbb{R}^d \times (0,T)$;

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle \, dxds = 0,$$

for all compact set $K \subseteq \mathbb{R}^d$.

A priori estimates

$$u_t + f(u)_x = \varepsilon \left(|u_x|^{p-2} u_x\right)_x + \delta\, g(u_{xx})_x$$

Multiply by a function $\eta'(u)$ and let $q' = \eta' f'$ be the derivative of a new flux function:

$$\begin{aligned} \eta(u)_t + q(u)_x &= \varepsilon \left(\eta'(u)\, |u_x|^{p-2} u_x\right)_x - \varepsilon\, \eta''(u)\, |u_x|^p \\ &\quad + \delta \left(\eta'(u)\, g(u_{xx})\right)_x - \delta\, \eta''(u)\, u_x\, g(u_{xx})\,. \end{aligned}$$

Integrate over $\mathbb{R} \times [0, t]$ with $\eta(u) = |u|^{\alpha+1}$. The conservative terms vanish and we obtain the

Lemma

Let $\alpha \geq 1$ and $g : \mathbb{R} \to \mathbb{R}$ be any dispersion function. Each solution satisfies for $t \in [0, T]$

$$\begin{aligned}
\int_{\mathbf{R}} |u(t)|^{\alpha+1}\, dx \quad &+ \quad (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathbf{R}} |u|^{\alpha-1}\, |u_x|^p\, dxds \\
&+ \quad (\alpha+1)\,\alpha\,\delta \int_0^t \int_{\mathbf{R}} |u|^{\alpha-1}\, u_x\, g(u_{xx})\, dxds \\
&= \quad \int_{\mathbf{R}} |u_0|^{\alpha+1}\, dx\,.
\end{aligned}$$

Usually, taking $\alpha = 1$, we deduce the a priori L^2 first energy estimates. It is not the case here, unless the factor δu_x of $g(u_{xx})$ was always negative (g will be always negative and $\varepsilon \geq 0$):

Corollary

Let the dispersion $g : \mathbb{R} \to \mathbb{R}$ be a negative function, then any solution verifies for $t \in [0, T]$

$$\int_{\mathbf{R}} u(t)^2 \, dx + 2\,\varepsilon \int_0^t \int_{\mathbf{R}} |u_x|^p \, dxds = \|u_0\|_2^2 + 2\,\delta \int_0^t \int_{\mathbf{R}} u_x \, |g(u_{xx})| \, dxds.$$

Remark: The Brenier-Levy case corresponds to $\varepsilon = 0$...

We use now the multipliers $(q+2)(|u_x|^q u_x)_x$ and $(q+2)(u_x^{q+1})_x$, with the assumption that $\mathcal{G}$ ($\mathcal{G}'' = G' = g$) is a homogeneous function of degree $n+2$:

$$
\begin{aligned}
\big((q+2)\,u_t\,|u_x|^q\,u_x\big)_x \;-\; (|u_x|^{q+2})_t &= \cdots \\
&= -\big((q+1)|u_x|^{q+2} f'(u)\big)_x \\
&\quad +(q+1)|u_x|^{q+2}\,u_x\,f''(u) \\
&\quad +\varepsilon\,(q+2)(q+1)(p-1)\,|u_x|^{p+q-2}\,u_{xx}^2 \\
&\quad +\big(\delta\,(q+2)(q+1)\,n\,|u_x|^q\,G(u_{xx})\big)_x \\
&\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,|u_x|^{q-2}\,u_x\,\mathcal{G}(u_{xx})\,,
\end{aligned}
$$

$$
\begin{aligned}
\big((q+2)\,u_t\,u_x^{q+1}\big)_x \;-\; (u_x^{q+2})_t &= \cdots \\
&= -\big((q+1)\,u_x^{q+2} f'(u)\big)_x \\
&\quad +(q+1)\,u_x^{q+3}\,f''(u) \\
&\quad +\varepsilon\,(q+2)(q+1)(p-1)\,u_x^{q}\,|u_x|^{p-2}\,u_{xx}^2 \\
&\quad +\big(\delta\,(q+2)(q+1)\,n\,u_x^{q}\,G(u_{xx})\big)_x \\
&\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,u_x^{q-1}\,\mathcal{G}(u_{xx}).
\end{aligned}
$$

Integrate over $\mathbb{R}\times[0,t]$, restrict to odd q and add... If we abbreviate as $\mathcal{U}^+$ the $\{(x,t)\in\mathbb{R}\times[0,T]:\ \delta\,u_x>0\}$ or their section by $t=s$ as ${\mathcal{U}_s}^+$, then we proved the:

Lemma

Let q be an odd number and $g : \mathbb{R} \to \mathbb{R}$ be the negative positively homogeneous dispersion function (of degree $n \geq 1$) with homogeneous $\mathcal{G}$ such that $\mathcal{G}'' = G' = g$. Each solution satisfies for $t \in [0, T]$

$$\begin{aligned}
\int_{\mathcal{U}_{\mathbf{t}}{}^+} |u_x(t)|^{q+2}\, dx + \varepsilon\,(q+2)(q+1)(p-1) \int_0^t \int_{\mathcal{U}_{\boldsymbol{s}}{}^+} |u_x|^{p+q-2}\, u_{xx}^2\, dxds \\
+|\delta|\,(q+2)(q+1)\,q\,(n+2)\,n \int_0^t \int_{\mathcal{U}_{\boldsymbol{s}}{}^+} |u_x|^{q-1}\, |\mathcal{G}(u_{xx})|\, dxds \\
+\operatorname{sgn}(\delta)\,(q+1) \int_0^t \int_{\mathcal{U}_{\boldsymbol{s}}{}^+} |u_x|^{q+3}\, f''(u)\, dxds \\
= \int_{\mathcal{U}_{\mathbf{0}}{}^+} |u_0'|^{q+2}\, dx\,.
\end{aligned}$$

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A+

Suggestions !?...

Obrigado!