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MULTIPLICITY AND LOCATION RESULTS FOR SECOND ORDER FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this work it is presented some existence, non-existence and location results for the problem composed by the second order fully nonlinear equation

$$u''(x) + f(x, u(x), u'(x)) = s\ p(x) \tag{E}$$

for $x \in [a,b]$, where $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$, $p : [a,b] \to \mathbb{R}^+$ continuous functions and s a real parameter, with the boundary conditions

$$\begin{aligned} L_0(u, u(a), u'(a)) &= 0, \\ L_1(u, u(b), u'(b)) &= 0, \end{aligned} \tag{BC}$$

where L_0 and L_1 are contiuous functions satisfying some adequate monotonicity assumptions.

It will be done a discussion on s about the existence and non-existence of solutions for problem (E)-(BC). More precisely, there are $s_0, s_1 \in \mathbb{R}$ such that:

· for $s < s_0$ or $(s > s_0)$ there is no solution of (E)-(BC).

· for $s = s_0$ problem (E)-(BC) has one solution.

The arguments used apply lower and upper solutions technique, a Nagumo condition and *a priori* estimations.

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1. INTRODUCTION

Consider the problem composed by the equation

$$u''(x) + f(x, u(x), u'(x)) = sp(x) \tag{1.1}$$

with $x \in [a,b]$, where $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ and $p : [a,b] \to \mathbb{R}^+ : (0, +\infty)$ are continuous functions and s a real parameter, and the functional boundary conditions given by

$$\begin{aligned} L_0(u, u(a), u'(a)) &= 0, \\ L_1(u, u(b), u'(b)) &= 0, \end{aligned} \tag{1.2}$$

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where $L_0, L_1 : C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ satisfy some adequate monotone conditions.

This type of problem with a discussion on the number of solutions of the boundary value problem was introduced in [1] and since then are known as Ambrosetti-Prodi problems. Several authors apply this type of discussion to different problems using variational methods, such as [2, 7, 18], or topological techniques, as in [5, 9, 11, 13, 14, 16, 17].

However, as far as we know, an analysis of parameter dependence has never been applied to problems with functional boundary conditions, as in 1.2. The functional dependence in the solution of the equation on the boundary data, allows a huge generality, including, for instance, cases of multipoint, deviated arguments, advances or delays, nonlocal, integro-differential, with maxima or minima arguments,.... These potentialities can be seen, for example, in [3, 4, 6, 10] and the references therein.

The main arguments used in this paper make use of a Nagumo condition, [15], to obtain an *a priori* estimate on the first derivative, and lower and upper solutions method.

At the best of our knowledge, sufficient conditions to guarantee the multiplicity of solution for second order Ambrosetti-Prodi functional boundary value problems, is still an open problem.

2. DEFINITIONS AND AUXILIARY RESULTS

In this section we introduce the notations and definitions needed moving forward, together with some useful results.

In the following, $C^k([a,b])$ denotes the space of real valued functions with continuous i-derivative in $[a,b]$, for $i = 1, ..., k$, equipped with the norm

$$\|y\|_{C^k} = \max_{0 \le i \le k} \left\{ \left| y^{(i)}(x) \right| : x \in [a,b] \right\}.$$

By $C([a,b])$ we denote the space of continuous functions with the norm

$$\|y\| = \max_{x \in [a,b]} |y(x)|.$$

Throughout this paper the following hypotheses will be assumed:

(H1) $L_0 : C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function nondecreasing in the first and third variable.

(H2) $L_1 : C([a,b]) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function nondecreasing in the first and nonincreasing in the third variable.

A Nagumo-type growth condition will be assumed on the nonlinear part of the differential equation. This will be an important tool to prove an *a priori* bound for the derivative of the corresponding solutions. The condition is given by the following definition:

Definition 2.1. Consider $\Gamma, \gamma \in C([a,b])$, such that, $\Gamma(x) \geq \gamma(x)$, $\forall x \in [a,b]$, and the set

$$E = \left\{(x,y,z) \in [a,b] \times \mathbb{R}^2 : \gamma(x) \leq y \leq \Gamma(x)\right\}.$$

A function $g : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is said to verify a Nagumo-type condition in E if there exists $\varphi \in C([0,+\infty), (0,+\infty))$ such that

$$|g(x,y,z)| \leq \varphi(|z|), \tag{2.1}$$

for every $(x,y,z) \in E$, and

$$\int_k^{+\infty} \frac{s}{\varphi(s)} ds > \max_{x\in[a,b]} \Gamma(x) - \min_{x\in[a,b]} \gamma(x) \tag{2.2}$$

where k is given by

$$k := \max\left\{\frac{\Gamma(b) - \gamma(a)}{b-a}, \frac{\Gamma(a) - \gamma(b)}{b-a}\right\}.$$

Lemma 2.2. *Let $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, verifying Nagumo-type conditions (2.1) and (2.2) in*

$$E = \left\{(x,y,z) \in [a,b] \times \mathbb{R}^2 : \gamma(x) \leq y \leq \Gamma(x)\right\}$$

where $\gamma(x)$ and $\Gamma(x)$ are continuous functions such that, $\gamma(x) \leq \Gamma(x)$, for every $x \in [a,b]$.

Then there is $R > 0$ such that for every solution $u(x)$ of equation (1.1) satisfying

$$\gamma(x) \leq u(x) \leq \Gamma(x), \ \forall x \in [a,b], \tag{2.3}$$

we have $\|u\| < R$.

Proof. The arguments considered for this proof are similar to standard ones presented in [13], considering

$$g(x,y,z) = sp(x) - f(x,y,z) \tag{2.4}$$

and $\bar{\varphi}(|z|) := |s| \|p\| + \varphi(|z|)$, as the integrals

$$\int_k^{+\infty} \frac{s}{\varphi(s)} ds \text{ and } \int_k^{+\infty} \frac{s}{|s| \|p\| + \varphi(|z|)} ds,$$

are of the same kind. $\square$

The main tool used throughout this paper is the lower and upper solution method. Consider the definition:

Definition 2.3. A function $\alpha \in C^2([a,b])$ is said to be a lower solution of the problem (1.1)-(1.2) if:

$$\alpha''(x) \geq sp(x) - f(x, \alpha(x), \alpha'(x)), \tag{2.5}$$

and

$$\begin{aligned} L_0(\alpha, \alpha(a), \alpha'(a)) &\geq 0, \\ L_1(\alpha, \alpha(b), \alpha'(b)) &\geq 0, \end{aligned} \tag{2.6}$$

A function $\beta \in C^2([a,b])$ is said to be an upper solution of the problem (1.1)-(1.2) if the reversed inequalities hold.

3. General existence and localization result

The arguments used in the proof require the following lemma, given in [19]:

Lemma 3.1. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = \max\{v, \min\{u, w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ *exists for a.e. $x \in I$.*
(b) *If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then*

$$\frac{d}{dx}q(x,u_m(x)) \to \frac{d}{dx}q(x,u(x)) \text{ for a.e. } x \in I.$$

We are now in a position to introduce the existence result.

Theorem 3.2. *Let $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that there exist upper and lower solutions of the problem (1.1)-(1.2), respectively, $\alpha(x)$ and $\beta(x)$, such that,*

$$\alpha(x) \leq \beta(x), \forall x \in [a,b],$$

f satisfies Nagumo conditions (2.1) and (2.2) in

$$E_* = \left\{(x, y_0, y_1) \in [a,b] \times \mathbb{R}^2 : \alpha(x) \leq y_0 \leq \beta(x)\right\}.$$

If conditions (H1) and (H2) hold then the problem (1.1)-(1.2) has at least a solution $u(x) \in C^2([a,b])$, satisfying

$$\alpha(x) \leq u(x) \leq \beta(x), \ \forall\, x \in [a,b],$$

and $|u'(x)| \leq K$, where

$$K = \max\{k, |\alpha'(x)|, |\beta'(x)|\}. \tag{3.1}$$

Remark 3.3. If there exist functions, $\alpha(x)$ and $\beta(x)$, lower and upper solutions of the problem (1.1)-(1.2) for some values of s, then s belongs to a bounded set, as

$$\alpha''(x) + f(x, \alpha(x), \alpha'(x)) \leq sp(x) \leq \beta''(x) + f(x, \beta(x), \beta'(x)),$$

for every $x \in [a, b]$.

Proof. Define the continuous functions

$$\delta(x, y) = \max\{\alpha(x), \min\{y, \beta(x)\}\} \tag{3.2}$$

and

$$\eta(v(x)) = \max\left\{-K, \min\left\{\frac{d}{dx}(v(x)), K\right\}\right\} \text{ for } a.e.\ x \in \mathbb{R}.$$

Consider the modified problem composed by the equation

$$u''(x) = sp(x) - f\left(x, \delta(x, u(x)), \eta\left(\frac{d}{dx}(\delta(x, u(x)))\right)\right) \tag{3.3}$$

and the boundary conditions,

$$\begin{aligned} u(a) &= \delta\begin{pmatrix} a, u(a) + \\ L_0(\delta(\cdot, u), \delta(a, u(a)), u'(a)) \end{pmatrix}, \\ u(b) &= \delta\begin{pmatrix} b, u(b) + \\ L_1(\delta(\cdot, u), \delta(b, u(b)), u'(b)) \end{pmatrix}. \end{aligned} \tag{3.4}$$

The proof will follow several steps:

Step 1 - Every solution u of problem (3.3) – (3.4), satisfies $\alpha(x) \leq u(x) \leq \beta(x)$ *and* $|u'(x)| < K$, *for every* $x \in [a, b]$, *with* $K > 0$ *given in (3.1).*

Let u be a solution of the modified problem (3.3) – (3.4). Assume, by contradiction, that there exists $x \in [a, b]$ such that $\alpha(x) > u(x)$ and let $x_0 \in [a, b]$ be such that

$$\min_{x \in I}(u - \alpha)(x) := (u - \alpha)(x_0) < 0. \tag{3.5}$$

As, by (3.4), $u(a) \geq \alpha(a)$ and $u(b) \geq \alpha(b)$, then $x_0 \in (a, b)$. So, there is $(x_1, x_2) \subset (a, b)$ such that for $x_0 \in (x_1, x_2)$,

$$u(x) < \alpha(x), \forall x \in (x_1, x_2), \quad (u - \alpha)(x_1) = (u - \alpha)(x_2) = 0. \tag{3.6}$$

Therefore, for all $x \in (x_1, x_2)$ it is satisfied that $\delta(x, u) = \alpha(x)$ and $\frac{d}{dx}(\delta(x, u)) = \alpha'(x)$. Therefore we deduce that

$$\begin{aligned} u''(x) &= sp(x) - f\left(x, \delta(x, u(x)), \eta\left(\frac{d}{dx}(\delta(x, u(x)))\right)\right) \\ &= sp(x) - f(x, \alpha(x), \alpha'(x)) \\ &\leq \alpha''(x) \qquad \text{for a. e. } x \in (x_1, x_2). \end{aligned}$$

Hence $(u-\alpha)'$ is nonincreasing on the interval (x_1, x_2). However as $(u-\alpha)'(x_0) = 0$, then $(u-\alpha)$ is nonincreasing on (x_0, x_2), which contradicts (3.5) and (3.6).

The inequality $u(x) \leq \beta(x)$, in $[a,b]$, can be proved in the same way and, so,

$$\alpha(x) \leq u(x) \leq \beta(x), \ \forall\, x \in [a,b]. \tag{3.7}$$

Applying previous bounds in Lemma 2.2, and remarking that by (3.1),

$$\int_k^K \frac{s}{\varphi(s)} ds \geq \int_k^r \frac{s}{\varphi(s)} ds,$$

it is obtained the *a priori* bound $|u'(x)| < K$, for $x \in [a,b]$. For details, see [3, Lemma 2.1].

<u>*Step 2*</u> *- Problem (3.3) – (3.4) has at least one solution.*

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$u''(x) = \lambda \left[sp(x) - f\left(x, \delta(x,u(x)), \eta\left(\frac{d}{dx}(\delta(x,u(x)))\right)\right)\right] \tag{3.8}$$

and the boundary conditions

$$\begin{aligned} u(a) = \ & \lambda\delta\begin{pmatrix} a, u(a) + \\ L_0(\delta(\cdot,u), \delta(a,u(a)), u'(a)) \end{pmatrix} \equiv \lambda L_a, \\ u(b) = \ & \lambda\delta\begin{pmatrix} b, u(b) + \\ L_1(\delta(\cdot,u), \delta(b,u(b)), u'(b)) \end{pmatrix} \equiv \lambda L_b. \end{aligned} \tag{3.9}$$

Define the operators $\mathcal{L} : C([a,b]) \to C([a,b]) \times \mathbb{R}^2$ by $\mathcal{L}u = (u'', u(a), u(b))$ and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C([a,b]) \to C([a,b]) \times \mathbb{R}^2$ by

$$\mathcal{N}_\lambda u = \left(\lambda\left[sp(x) - f\left(x, \delta(x,u(x)), \eta\left(\frac{d}{dx}(\delta(x,u(x)))\right)\right)\right], L_a, L_b\right).$$

Since L_0, L_1 and f are continuous functions, then, from Lemma 3.1, $\mathcal{N}_\lambda$ is continuous. Moreover, as $\mathcal{L}^{-1}$ is compact, it can be defined the completely continuous operator $\mathcal{T}_\lambda : C([a,b]) \to C([a,b])$ by $\mathcal{T}_\lambda u = \mathcal{L}^{-1}\mathcal{N}_\lambda(u)$.

It is obvious that the fixed points of operator $\mathcal{T}_\lambda$ coincide with the solutions of problem (3.8) – (3.9).

Defining in $C([a,b]) \times \mathbb{R}^2$ the norm

$$|(v, v_1, v_2)| = \max\{\|v\|, \max\{|v_1|, |v_2|\}\},$$

by Remark 3.3, $\mathcal{N}_\lambda u$ is uniformly bounded in $C([a,b])$, we have that any solution of the problem *(3.8) – (3.9)*, verifies the following *a priori* bound $\|u\| \leq \|\mathcal{L}^{-1}\| \, |\mathcal{N}_\lambda(u)| \leq \bar{K}$, for some $\bar{K} > 0$ independent of λ.

In the set $\Omega = \left\{u \in C([a,b]) : \|u\| < \bar{K} + 1\right\}$ the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by the invariance under homotopy, $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0)$.

As the equation $x=\mathcal{T}_0(x)$ is equivalent to the problem

$$\begin{cases} u''(x)=0 \\ u(a)=u(b)=0, \end{cases}$$

which has only the trivial solution, then $d(\mathcal{I}-\mathcal{T}_0,\Omega,0)=\pm1$. So by degree theory, the equation $x=\mathcal{T}_1(x)$ has at least one solution, that is, the problem *(3.3) – (3.4)* has at least one solution in Ω.

<u>*Step 3*</u> *- Every solution* u *of problem (3.3) – (3.4) is a solution of (1.1) – (1.2).*

Let u be a solution of the modified problem (3.3) – (3.4). By previous steps, function u fulfills equation (1.1). So, it will be enough to prove the following inequalities:

$$\begin{aligned} \alpha(a)\leq\ & u(a)+L_0(\delta(\cdot,u),\delta(a,u(a)),u'(a)) \leq\beta(a),\\ \alpha(b)\leq\ & u(b)+L_1(\delta(\cdot,u),\delta(b,u(b)),u'(b)) \leq\beta(b).\end{aligned}$$

Assume that

$$u(a)+L_0(\delta(\cdot,u),\delta(a,u(a)),u'(a))>\beta(a). \tag{3.10}$$

Then, by (3.4), $u(a)=\beta(a)$. By (2.6) and previous steps, it is obtained the following contradiction with (3.10):

$$\begin{aligned} u(a)+L_0(\delta(\cdot,u),\delta(a,u(a)),u'(a)) &= \beta(a)+L_0(\beta,\beta(a),\beta'(a))\\ &\leq \beta(a).\end{aligned}$$

Applying similar arguments it can be proved that

$$\alpha(a)\leq u(a)+L_0(\delta(\cdot,u),\delta(a,u(a)),u'(a)).$$

and

$$\alpha(b)\leq u(b)+L_1(\delta(\cdot,u),\delta(b,u(b)),u'(b)) \leq\beta(b).$$

□

4. **Existence and Non-existence results**

For clearness of arguments the dependence of solution on s will be discussed in $[0,1]$, without loss of generality. The obvious modifications must be considered in the corresponding definitions of lower and upper solutions. Some extra hypotheses on the continuous functions L_0, L_1 are required to obtain the existence and nonexistence results:

Theorem 4.1. *Let* $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ *be a continuous function that verifies the assumptions on Theorem 3.2. Moreover if :*

(i)

$$f(x,y_0,y_1) \text{ is nonincreasing on } y_0; \tag{4.1}$$

(ii) *there is* $s_1 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\frac{f(x,0,0)}{p(x)} < s_1 < \frac{f(x,-r,0)}{p(x)}, \tag{4.2}$$

for every $x \in [0,1]$;

(iii) *boundary functions* L_0, L_1 *verify conditions (H1), (H2) and*

(H3) $L_i(z,z,0) \geq 0$, *for every* $z \leq -r$, *and* $L_i(0,0,0) \leq 0$, *for* $i = 0,1$,

then there is $s_0 < s_1$ *(with the possibility that* $s_0 = -\infty$*) such that:*

1) *for* $s < s_0$, *(1.1)-(1.2) has no solution.*
2) *for* $s_0 < s \leq s_1$, *(1.1)-(1.2) has at least one solution.*

Proof. Define

$$s^* = \max_{x \in [0,1]} \frac{f(x,0,0)}{p(x)}. \tag{4.3}$$

By (4.2), there is $x^* \in [0,1]$ such that

$$\frac{f(x,0,0)}{p(x)} \leq s^* = \frac{f(x^*,0,0)}{p(x^*)} < s_1, \ \forall x \in [0,1].$$

For r given by (4.2), $\beta(x) \equiv 0$ is an upper solution of (1.1)-(1.2) for $s = s^*$ and, as by (4.1) and (4.2),

$$0 > s_1 p(x) - f(x,-r,0) \geq sp(x) - f(x,-r,0), \tag{4.4}$$

therefore $\alpha(x) = -r$ is a lower solution of (1.1)-(1.2) for every $s \leq s_1$. So by Theorem 3.2 there exists a solution for problem (1.1)-(1.2) for $s = s^*$.

Suppose that problem (1.1)-(1.2) has a solution $u_\sigma(x)$ for $s = \sigma \leq s_1$. So $u_\sigma(x)$ is an upper solution of (1.1)-(1.2) for $\sigma \leq s \leq s_1$.

Let $R > 0$ sufficiently large such that, for r given by (4.2),

$$r \leq R, \ \max_{x \in [0,1]} u_\sigma(x) \geq -R \tag{4.5}$$

As in (4.4), $\overline{\alpha}(x) = -R$ is a lower solution of (1.1)-(1.2), for s such that $s \leq s_1$. By (4.5) it is obtained that $\overline{\alpha}(x) \leq u_\sigma(x)$, in $[0,1]$. Therefore, by Theorem 3.2, there is a solution to problem (1.1)-(1.2) for $\sigma \leq s \leq s_1$.

Consider the set

$$S = \{s \in \mathbb{R} : \text{ (1.1)-(1.2) has a solution}\}.$$

For s^* given by (4.3), $s^* \in S$ then S is a non-empty set. Let $s_0 = \inf S$. So, for $s < s_0$, problem (1.1)-(1.2) has no solution. By the definition of s_0 and $s_0 \leq s^* < s_1$, thus, (1.1)-(1.2) has a solution for $s \in]s_0, s_1]$.

It is pointed out that if $s_0 = -\infty$ then, every problem (1.1)-(1.2) has a solution for $s \leq s_1$. $\square$

5. **Examples**

In this section we will consider a couple of examples that illustrate conditions (H1) to (H3) and how they relate with previous theorems.

Example 5.1. Let us consider, for $x \in [0,1]$, the problem given by the equation

$$u''(x) + \arctan\left(u(x)^2\right) + (u'(x))^{\frac{2}{3}} = sp(x) \tag{5.1}$$

along with the functional boundary conditions

$$\begin{array}{c} \max\limits_{x\in[0,1]} u(x) + k_1 u(0) = 0 \\ \max\limits_{x\in[0,1]} \int_0^x u(s)\,ds + k_2 u(1) = 0 \end{array}. \tag{5.2}$$

The functions

$$\alpha(x) = -x - 1$$

and

$$\beta(x) = x + 1$$

are, respectively, lower and upper solutions to the problem (5.1)-(5.2), for $k_1 \leq -2$, $k_2 \leq \frac{-3}{2}$ and for

$$\frac{1 + \arctan 1}{\max\limits_{x\in[0,1]} p(x)} \leq s \leq \frac{1 + \arctan 1}{\min\limits_{x\in[0,1]} p(x)}. \tag{5.3}$$

Given

$$\begin{array}{c} f(x, y_0, y_1) = \arctan\left(y_0^2\right) + (y_1)^{\frac{2}{3}} \\ L_0(y_0, y_1, y_2) = \max\limits_{x\in[0,1]} y_0(x) + k_1 y_1 \\ L_1(y_0, y_1, y_2) = \max\limits_{x\in[0,1]} \int_0^x y_0(s)\,ds + k_2 y_1 \end{array},$$

function f verifies conditions (2.1) and (2.2) in

$$E = \left\{(x, y_0, y_1) \in [0,1] \times \mathbb{R}^2 : -x - 1 \leq y_0 \leq x + 1\right\},$$

therefore by Theorem 3.2 there is at least a solution $u(x)$ of the problem (5.1)-(5.2), satisfying

$$-x - 1 \leq u(x) \leq x + 1,\ \forall x \in [0,1].$$

Remark that from (5.3) this solution is not the trivial one.

To obtain existence and nonexistence information for the problem (5.1)-(5.2) extra conditions were required: to apply Theorem 4.1 stronger conditions were imposed both on the function f and on the boundary conditions. In the previous example the function presented does not verify (4.1) and the boundary conditions (5.2) do not verify condition (H3). A new example, with a suitable function f and boundary conditions is presented in the next example to illustrate Theorem 4.1.

Example 5.2. Let us consider, for $x \in [0,1]$, the problem given by the equation

$$u''(x) - u(x)^3 + (u'(x)+1)^{\frac{2}{3}} = sp(x) \tag{5.4}$$

along with the functional boundary conditions

$$\begin{aligned} -u(0)^3 + u'(0) &= 0 \\ \delta u(1) - u'(1) &= 0. \end{aligned} \tag{5.5}$$

The functions

$$\alpha(x) = -x - 1$$

and

$$\beta(x) = x + 1$$

are, respectively, lower and upper solutions to the problem (5.4)-(5.5), for $\delta \leq 0$ and for

$$\frac{-1 + 2^{\frac{2}{3}}}{\max\limits_{x\in[0,1]} p(x)} \leq s \leq \frac{8}{\min\limits_{x\in[0,1]} p(x)}.$$

Given

$$\begin{aligned} f(x, y_0, y_1) &= -y_0^3 + (y_1+1)^{\frac{2}{3}} \\ L_0(y_0, y_1, y_2) &= -(y_1)^3 + y_2 \\ L_1(y_0, y_1, y_2) &= \delta y_1 - y_2 \end{aligned} \quad ,$$

function f verifies conditions (2.1) and (2.2) in

$$E = \left\{ (x, y_0, y_1) \in [0,1] \times \mathbb{R}^2 : -x - 1 \leq y_0 \leq x + 1 \right\}.$$

With

$$\frac{1}{\max\limits_{x\in[0,1]} p(x)} < s_1 < \frac{r^3 + 1}{\min\limits_{x\in[0,1]} p(x)},$$

boundary conditions (5.5) satisfy condition (H3), therefore by Theorem 4.1 there is $s_0 < s_1$ such that:

- for $s < s_0$, the problem (5.4)-(5.5)has no solution
- for $s_0 < s \leq s_1$, the problem (5.4)-(5.5)has at least one solution.

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